

## OPTIMAL TAXATION POLICY MAXIMUM-ENTROPY APPROACH

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**Abstract:** The object of this paper is firstly to present entropic measure of income inequality and secondly to develop maximum entropy approaches for the optimal reduction of income inequality through taxation.

**Keywords:** Income inequality, entropy, optimal taxation.

### 1. INTRODUCTION

Differences between individuals or between groups of individuals are not only normal but also unavoidable phenomena in the biological world. But only within the human species do we find from the dawn of history, inequalities of a different nature - social inequalities, which has little to do with the biological differences [4]. Social conflicts of all times have hinged on economic inequality between social classes and this social difference singles out the human species from others. Thus there are different economic inequalities - income inequalities among individuals of a population, wealth inequalities between developed and developing countries, concentration of industry in the hands of a few individual companies etc. There are different economic programmes aimed at removing economic inequalities between social structures. Economic models of taxation, subsidies, income transfer and financial aids etc. are some of the means adopted to reduce the social inequalities i.e. to reduce the difference between the rich and the poor.

Attempts to introduce quantitative measures of inequality of income or of wealth had started early in this century. In order to evaluate proposed measures it becomes desirable to determine how income or wealth distribution might be compared in order to say that one distribution was 'more equal' than the other. The first attempt along this line was made by Lorenze [11] in introducing what has become as 'Lorenze curve'. 'Lorenze' technique was later discussed and modified by many authors [22, 23, 24]. Later Dalton [2] took a different view point, leading to the principle of transfers. Dalton's work is of significant importance in mathematical economics; it paved the way of introducing a

general measure of inequality, not necessarily of economic system and which led to the notion of entropy-like function much earlier to the works of Shannon [25] in information theory. There are various measures of income inequality introduced by various authors. We are however, interested in the entropic measure of income inequality for which Dalton is the pioneer. In the present paper our first objective is to investigate the process of introducing an entropic measure of income inequality and then to develop a maximum-entropy method for the optimal reduction of income inequality through the process of taxation.

## 2. INCOME INEQUALITY AND ENTROPY

The concept of inequality arises in various contexts and there is considerable interest in its measurement. Besides economics, in Political Science and Sociology also inequalities of voting strength resulting from legislative misapportionment, of tax structure are measures using various indices. The measurement of species diversity in ecology is essentially a problem of measuring equality [19]. Measurement of income inequality is discussed and surveyed by many authors [24, 26]. A rational approach to a general measure of inequality (not necessarily of income or wealth) is first due to Dalton [2]. According to Dalton [2] a function  $\phi$  is said to be a measure of inequality (better an index of inequality) if it satisfies the conditions [12]:

(i) For any two vectors  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$

$$x \prec y \Rightarrow \phi(x) \leq \phi(y)$$

i.e.  $\phi$ -should be Schur-convex.

(ii)  $x \prec y$  and  $x$  is not a permutation of  $y \Rightarrow \phi(x) < \phi(y)$

i.e.  $\phi$ -should be strictly Schur-convex.

The notation  $x \prec y$  implies that the arguments of  $x$  are 'more equal' than those of  $y$ . These conditions were first formulated by Dalton [2] although they are hinted at or are implicit in the works of Lorenze [11] and Pigou [20]. Again if  $\phi$  be a measure of inequality, then the function  $\psi$  defined for all  $x$  such that  $\sum_{i=1}^n x_i \neq 0$  by

$$\psi(x) = \phi \left( \frac{x_1}{\sum_{i=1}^n x_i}, \frac{x_2}{\sum_{i=1}^n x_i}, \dots, \frac{x_n}{\sum_{i=1}^n x_i} \right)$$

is also a measure of inequality satisfying Daltons' conditions. For measure of equality or species diversity in biology, it is desirable that a maximum be achieved when all the arguments are equal, so in (i) and (ii) Schur-concavity should replace Schur-convexity. A common measure of equality of unnormed distribution  $x = (x_1, x_2, \dots, x_n) \geq 0$  (negatively taken measures of inequality of  $x$ ) considered in econometrics by Lorentz [11], Pigou [20], Dalton [2] and others are the functions of the form [17]:

$$\phi(x_1, x_2, \dots, x_n) = H \left( \frac{x_1}{\sum_{i=1}^n x_i}, \frac{x_2}{\sum_{i=1}^n x_i}, \dots, \frac{x_n}{\sum_{i=1}^n x_i} \right)$$

where  $H$  is some suitable entropy function. Different measures of inequality can be obtained for different forms of entropy functions  $H$  [12]. This is a brief account of the interrelations between the concept of entropy and the measure of inequality as developed by Dalton [2] and others and this is valid for all types of the system.

Let us now turn to a specific economic system. It is the income distribution of individuals of a population or society. To determine a suitable measure of income inequality we follow Theil [27]. Let us consider a society consisting of  $n$  income earners with incomes  $q_i$  ( $i = 1, 2, \dots, n$ ). It is assumed that  $q_i$  are non-negative and that at least some

of them are positive, so that both the total personal income  $\sum_{i=1}^n q_i = C$  and per capita

personal income  $\bar{C} = \left( \sum_{i=1}^n q_i \right) / n$  are positive. The income share of  $i$ -th individual is his share of the total personal income:

$$p_i = \frac{q_i}{\sum_{i=1}^n q_i} = \frac{C}{n\bar{C}}. \quad (2.1)$$

His population share is his share of the total population, which is simply  $1/n$  for each individual. Then following Theil [27] we define the measure of income inequality as the expected information of the message which transforms the population shares into the income shares:

$$I = \sum_{i=1}^n p_i \ln \left( \frac{p_i}{1/n} \right) = \frac{1}{n} \sum_{i=1}^n \frac{q_i}{\bar{C}} \ln \left( \frac{q_i}{\bar{C}} \right) \quad (2.2)$$

Now replacing  $\bar{C}$  by  $C/n$ , the expression (2.2) can be reduced to the form,

$$I = \ln n - \left[ \sum_{i=1}^n -\frac{q_i}{C} \ln \left( \frac{q_i}{C} \right) \right]. \quad (2.3)$$

The second term of the r.h.s. of (2.3) is the form of Shannon entropy [25]

$$S = -\sum_{i=1}^n \frac{q_i}{C} \ln \left( \frac{q_i}{C} \right) = -\sum_{i=1}^n p_i \ln p_i. \quad (2.4)$$

The individual share  $p_i = q_i / C$  satisfying the conditions  $p_i \geq 0$ , ( $\forall i = 1, 2, \dots, n$ ) and  $\sum_{i=1}^n p_i = 1$  defines a probability distribution and the Shannon-entropy  $S$  measures the diversity of the probability distribution  $\{p_1, p_2, \dots, p_n\}$ . Maximum is reached when  $p_1 = p_2 = \dots = p_n = 1/n$  i.e. when all the income earners have the same income [9]

$$c_1 = c_2 = \dots = c_n = c/n . \quad (2.5)$$

We can then write  $I$  as [9]

$$I = S_{\max} - S = \ln n - \sum_{i=1}^n \left[ -\frac{q_i}{C} \ln \left( \frac{q_i}{C} \right) \right] \quad (2.6)$$

From (2.6) we see that to reduce the income inequality we have to increase the value of the entropy  $S$ . For the optimal reduction of the income inequality we have to maximize the entropy  $S$  subject to some constraints or policies determined by the Government.

### 3. MAXIMUM-ENTROPY ALGORITHM UNDER INEQUALITY CONSTRAINTS

In this section we shall briefly present the maximum-entropy algorithm under inequality constraints to be employed in the next section for the optimal reduction of income inequality through taxation.

The maximum entropy method of estimation of an unknown probability distribution  $\{p_1, p_2, \dots, p_n\}$  consists of the maximization of the entropy.

$$S = -\sum_{i=1}^n p_i \ln p_i \quad (3.1)$$

subject to the given information or constraints usually expresses in the form of the inequalities.

$$\sum_{k=1}^n g_{ik} p_k = \langle g_i \rangle, \quad i = 1, 2, \dots, n .$$

The expressed values  $\langle g_i \rangle$  are assumed to be known exactly but in practical cases these averages are obtained either from physical measurement or from empirical experiments so that these experimental measures are usually subjected to errors. So, strict equalities in Eq. (3.2) are unrealistic and so we shall discuss maximum entropy algorithm for inequality constraints [6].

Our problem in general is to maximize the entropy

$$S(\bar{p}) = -\sum_{i=1}^n p_i \ln p_i \quad (3.3)$$

subject to constraints

$$g_i(\bar{p}) = \sum_{k=1}^n g_{ik} p_k \leq b_i, \quad i = 1, 2, \dots, u \quad (3.4)$$

$$g_i(\bar{p}) = \sum_{k=1}^n g_{ik} p_k \geq b_i, \quad i = u+1, u+2, u+3, \dots, v \quad (3.5)$$

$$g_i(\bar{p}) = \sum_{k=1}^n g_{ik} p_k = b_i, \quad i = v+1, v+2, \dots, n. \quad (3.6)$$

First of all we convert constraints (3.4) into equations by adding slack-variables, thus obtaining  $g_i(\bar{p}) + p_{si} = b_i, \quad i = 1, 2, 3, \dots, u$ .

Similarly constraints (3.5) can be converted into equations by adding surplus variables which gives  $g_i(\bar{p}) - p_{si} = b_i, \quad i = u+1, u+2, u+3, \dots, v$ .

Thus the original problem is equivalent to:

Maximize  $S(\bar{p})$  subject to constraints:

$$g_i(\bar{p}) + p_{si} = b_i, \quad i = 1, 2, 3, \dots, u \quad (3.7)$$

$$g_i(\bar{p}) - p_{si} = b_i, \quad i = u+1, u+2, u+3, \dots, v \quad (3.8)$$

$$g_i(\bar{p}) = b_i, \quad i = v+1, v+2, v+3, \dots, n \quad (3.9)$$

where  $\bar{p} = (p_1, p_2, \dots, p_n)$ .

Now, either  $p_{si} > 0$  or  $p_{si} = 0$ . If we consider that each  $p_{si} > 0$ , then the Lagrangian

$$L(\bar{p}, \bar{p}_s, \lambda) = S(\bar{p}) + \sum_{i=1}^u \lambda_i [b_i - p_{si} g_i(\bar{p})] + \sum_{i=u+1}^{i=v} \lambda_i [b_i + p_{si} - g_i(\bar{p})] + \sum_{i=v+1}^n \lambda_i [b_i - g_i(\bar{p})] \quad (3.10)$$

For the maximum of  $S(\bar{p})$

$$\frac{\partial L}{\partial p_{si}} = -\lambda_i = 0, \quad i = 1, 2, 3, \dots, u$$

$$\frac{\partial L}{\partial p_{si}} = \lambda_i = 0, \quad i = u+1, u+2, u+3, \dots, v$$

So, we see that if  $p_{si} > 0, \forall i = 1, 2, 3, \dots, v$  then  $\lambda_i = 0, \forall i = 1, 2, 3, \dots, v$ , and so we can ignore the inequality constraints so far the optimality is concerned; in other words the inequality constraints are useless at the point where  $S(\bar{p})$  attains its optimum value. Now, if  $p_{si} = 0$  for some 'i' then the i-th inequality becomes equality and we shall assume the corresponding  $\lambda_i$  to be non-zero.

From the above discussion we can make an algorithm to find the point  $(\bar{p})$  at which  $S(\bar{p})$  has its maximum value subject to the constraints (3.4), (3.5) and (3.6).

First of all we will consider the optimum of  $S(\bar{p})$  ignoring the inequality constraints. If the point so obtained also satisfies the inequality constraints then that point will be the solutions of (3.3). If one or more inequality constraints are not satisfied then we will select one of the inequality constraints as an equality constraint ignoring others and repeat the process. If the point obtained in this step satisfies all the inequality constraints (except the one which became equality) then that point gives the solution of (3.3). Again if the solutions obtained in the second step do not satisfy one or more inequality constraints then we shall make two inequality constraints into equations and repeat the process. In

this way we are to proceed until the optimum is obtained satisfying all the inequality constraints.

#### 4. OPTIMAL TAXATION: MAXIMUM ENTROPY ALGORITHM

In this section we shall consider a method of reduction of the income (or wealth) inequality by taxation and study the role of the technique of maximum-entropy algorithm described in section 2 in determining the optimal taxation policy.

As stated before let  $c_1, c_2, \dots, c_n$  be the income of the  $n$  individuals in a population and  $C = \sum_{i=1}^n c_i$  be the total income of the population. Let  $f(c_i)$  be the taxation function for the certain taxation policy so that the income charged from a person is  $c_i f(c_i)$  whose income is  $c_i$ . We assume that no body is charged more tax than his income and there is no negative taxation or subsidies.

So we have

$$0 \leq f(c_i) \leq 1, \quad i = 1, 2, \dots, n. \quad (4.1a)$$

One way of reducing income inequality is through taxation. However, in order that the income inequality is reduced through taxation we must have  $f(c_i)$  to be an increasing function of  $c_i$  [9]. Let a person whose income is  $c_i$  have the real income  $[c_i - f(c_i)]$  after taxation. We also assume the fair taxation policy:

$$c_1 - c_1 f(c_1) \leq c_2 - c_2 f(c_2) \leq \dots \leq c_n - c_n f(c_n) \quad (4.1b)$$

so that after taxation the richer does not become poorer. Then to minimize the income inequality is to maximize the taxation entropy [9]:

$$S = - \sum_{i=1}^n \frac{c_i - c_i f(c_i)}{\sum_{i=1}^n [c_i - c_i f(c_i)]} \ln \frac{c_i - c_i f(c_i)}{\sum_{i=1}^n [c_i - c_i f(c_i)]} \quad (4.2)$$

or equivalently

$$S = - \sum_{i=1}^n [c_i - c_i f(c_i)] \ln [c_i - c_i f(c_i)] \quad (4.3)$$

subject to constraints

$$0 \leq f(c_i) \leq 1 \quad (\forall i = 1, 2, 3, \dots, n) \quad (4.4)$$

and

$$\sum_{i=1}^n c_i f(c_i) = T, \quad T < C \quad (4.5)$$

the later implying the fixed income tax revenue. So, the problem is

$$\text{to maximize } S = -\sum_{i=1}^n q_i x_i \ln(q_i x_i) \text{ i.e. to maximize } S = -\sum_{i=1}^n q_i \ln q_i \quad (4.6a)$$

where  $x_i = 1 - f(q_i)$ ,  $q_i = q x_i$ ,  $i = 1, 2, \dots, n$   
subject to constraints:

$$0 \leq q_i \leq q \quad (4.6b)$$

and

$$\sum_{i=1}^n q_i = C - T \quad (4.6c)$$

Now, to solve this we shall follow a technique of solving optimization problems under inequality constraints.

Let us first ignore the inequality constraints (4.6b) and consider the Lagrangian

$$L(\bar{q}, \lambda) = -\sum_{i=1}^n q_i \ln q_i - \lambda \left[ \sum_{i=1}^n q_i - (C - T) \right] \quad (4.7)$$

Now  $\frac{\partial L}{\partial q_i} = 0$  gives

$$-[\ln q_i + 1 - \lambda] = 0 \quad (4.8)$$

or  $q_i = e^{\lambda-1} = \mu$  (say). Now, equation (4.6c) gives  $\sum_{i=1}^n \mu = C - T$

$$\begin{aligned} \Rightarrow \mu &= \frac{C - T}{n} \\ \Rightarrow q_i &= \frac{C - T}{n} \quad (i = 1, 2, \dots, n) \end{aligned} \quad (4.9)$$

But this may not satisfy the first constraint (4.6b) unless we allow subsidies.

(i) Now if  $\frac{C - T}{n} \leq q$  ( $i = 1, 2, \dots, n$ ) then  $q_i - q f(q) = \frac{C - T}{n}$ .

So that after paying taxes, everybody has the same income.

(ii) If we see that,  $\frac{C - T}{n} > c_m$  for some  $m$  then we will make the inequality constraints

$q_m \leq c_m$  into an equality i.e.  $q_m = c_m$  so that the Lagrangian in this step will be

$$L(\bar{q}, \lambda) = -\sum_{i=m}^n q_i \ln q_i - \lambda \left[ \sum_{i \neq m} q_i + c_m - (C - T) \right] \quad (4.10)$$

then  $\frac{\partial L}{\partial q_i} = 0$ ,  $i \neq m$  gives  $\ln q_i + 1 - \lambda = 0$

$$\Rightarrow q_i = e^{\lambda-1} = \mu_i \text{ (say), } i \neq m \quad (4.11)$$

$$\begin{aligned}
\text{So, } \sum_{i \neq m} q_i + c_m &= (C - T) \\
\Rightarrow \mu_i (n - 1) + c_m &= C - T \\
\Rightarrow \mu_i &= \frac{C - T - c_m}{n - 1} \quad (4.12)
\end{aligned}$$

So, in this case,

$$q_i = \begin{cases} \frac{C - T - c_m}{n - 1}, & i \neq m \\ c_m, & i = m \end{cases} \quad (4.13)$$

Now, if in this step  $q_i = \frac{C - T - c_m}{n - 1} \leq c_i$ ,  $i \neq m$  then the taxation will be

$$q_i = \begin{cases} \frac{C - T - c_m}{n - 1}, & i \neq m \\ 0, & i = m \end{cases}$$

implying that every person except one will be left with income  $\frac{C - T - c_m}{n - 1}$  each while for the  $m$ -th person it is  $c_m$ .

(iii) Again if in this step we see that, when  $\frac{C - T - c_m}{n - 1} > c_r$  for some  $r$  then, we will make two inequalities  $q_m \leq c_m$  and  $q_r \leq c_r$  into equalities

$$\left. \begin{aligned} q_m &= c_m \\ q_r &= c_r \end{aligned} \right\} \quad (4.14)$$

So, in this step our problem is equivalent to the maximization of  $-\sum_{i=1}^n q_i \ln q_i$  subject to the constraints:

$$q_m = c_m \quad (4.15)$$

$$q_r = c_r \quad (4.16)$$

$$\sum_{i=1}^n q_i = C - T \quad (4.17)$$

This leads to,  $q_i = \frac{C - T - c_m - c_r}{n - 2} \leq c_i$ ,  $i \neq m, r$  if (4.6b) holds, so that after paying taxes the will be left with income  $c_r$ ,  $c_m$ ,  $\frac{C - T - c_m}{n - 1}$ ,  $\frac{C - T - c_r - c_m}{n - 2}$ , ... then the taxation will be



$$q_i f(q_i) = \begin{cases} q_i - \frac{C-T-c_m-c_r}{n-2} \leq q_i, & i \neq m, r \\ 0, & i = r, m \end{cases}$$

but if (4.6b) does not hold for some  $i$  then we are to make three inequalities into equalities and we have to proceed in this way as long as necessary.

## 5. OPTIMAL TAXATION: AN ALTERNATIVE MAXIMUM-ENTROPY APPROACH

We shall now follow an alternative approach to the solution of the optimal taxation problem. In the previous section the optimal problem has been reduced to the maximization of Shannon entropy (4.6a) subject to the constraints (4.6b) and (4.6c). In the present approach we shall decouple the inequality constraint (4.6b) from the equality constraint (4.6c) and modify the entropy (4.6a) to take account of the inequality constraint (4.6b), so that the new  $q_i$  estimated from the maximization of the new entropy subject to the equality constraint (4.6c) will automatically satisfies the constraints (4.6b).

The modified form of entropy taking into account the inequality constraint (4.6b) is given by [9]

$$S' = -\sum_{i=1}^n q_i \ln q_i - \sum_{i=1}^n (q_i - q_i) \ln(q_i - q_i) \quad (5.1)$$

subject to the equality constraint (4.6c)

$$\sum_{i=1}^n q_i = C - T. \quad (5.2)$$

Let us consider the Lagrangian

$$L(q_1, q_2, \dots, q_n; \alpha) = -\sum_{i=1}^n q_i \ln q_i - \sum_{i=1}^n (q_i - q_i) \ln(q_i - q_i) - \alpha \left[ \sum_{i=1}^n q_i - (C - T) \right] \quad (5.3)$$

Then  $\frac{\partial L}{\partial q_i} = 0$  gives  $\ln\left(\frac{q_i}{q_i - q_i}\right) = \alpha$

$$\Rightarrow q_i = \frac{q_i}{1 + e^{-\alpha}} \quad (i = 1, 2, \dots, n) \quad (5.4)$$

where the Lagrangian parameter  $\alpha$  is determined by the inequality constraint:

$$\sum_{i=1}^n q_i = C - T$$

leading to the value  $\frac{1}{1 + e^{-\alpha}} = \frac{C - T}{C}$

then,

$$q_i = q - f(q) = \left( \frac{C-T}{C} \right) q, \quad (i = 1, 2, \dots, n) \quad (5.5)$$

so that after paying taxes the person is left with income  $\left( \frac{C-T}{C} \right) q, (i = 1, 2, \dots, n)$ .

Thus, we see that the incomes of each person are reduced by a fixed fraction  $\frac{C-T}{C}$ . Finally we note that the solution (5.5) satisfies both the constraints (4.6b) and (4.6c).

The above solution is very simple in comparison with the earlier one. The earlier one is a generalization and in fact provides a mathematical foundation of the heuristic approach of Kapur [9]. The income inequality is one of the economic inequalities between the poor and the rich. If one of the objectives of taxation policy is to reduce the income inequality among the individuals of a population, the above two approaches based on maximum entropy principle, in spite of their limitations, provide two effective methods of solution of optimal taxation problems.

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