A SEPARABLE APPROXIMATION DYNAMIC PROGRAMMING ALGORITHM FOR ECONOMIC DISPATCH WITH TRANSMISSION LOSSES

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Abstract: The standard way to solve the static economic dispatch problem with transmission losses is the penalty factor method. The problem is solved iteratively by a Lagrange multiplier method or by dynamic programming, using values obtained at one iteration to compute penalty factors for the next until stability is attained. A new iterative method is proposed for the case where transmission losses are represented by a quadratic formula (i.e., by the traditional B-coefficients). A separable approximation is made at each iteration, which is much closer to the initial problem than the penalty factor approximation. Consequently, lower cost solutions may be obtained in some cases, and convergence is faster.

Keywords: Economic dispatch, transmission losses, B-coefficients, penalty factors, separable approximation, dynamic programming.

1. INTRODUCTION

Due to the enormous costs involved, optimizing the use of equipment for power generation and transmission is a lasting concern. Given several thermal or hydro units, often with different characteristics, one must decide how to distribute the load considered between them. This problem, called *economic dispatch*, has been much studied, both in the static and dynamic cases. It is discussed at length in the book of Wood and Wollenberg [6] *Power generation, operation and control*, an enlarged second edition of which has recently appeared.

The standard way to solve the static case with transmission losses is the *penalty factor* method. The problem is solved iteratively by a Lagrangian multiplier method or by dynamic programming, using values obtained at one iteration to compute penalty factors for the next until stability is attained.

The purpose of this paper is to propose a new iterative method for the case where transmission losses are represented by a quadratic formula (i.e. by the traditional B-coefficients). A separable approximation is made at each iteration, which is much closer to the initial problem than the penalty factor approximation. Moreover, the problem considered at each iteration can be solved by dynamic programming. Convergence is faster than in the scheme, of to Liang and Glover [4], which also uses dynamic programming.

The paper is organized as follows: a mathematical formulation of the problem is given in the next section. Previous solution methods are reviewed in Section 3. The particular case of a separable loss function is studied in Section 4, and illustrated by an example of Wood and Wollenberg [6]. The new method is described in full in Section 5 and applied to examples from [6] and [4]. Conclusions are drawn in Section 6.

2. FORMULATION

Consider N thermal units committed to serve a load of P_R, at minimum cost. Assume that production of each unit is bounded below and above. Assume further that transmission losses in the network are incurred and can be represented by a quadratic formula (with the so-called B-coefficients). This problem can be stated mathematically as

$$\text{minimize } F_T = \sum_{i=1}^{N} F_i(P_i) \tag{1}$$

subject to

$$\sum_{i=1}^{N} P_i = P_R + P_L \tag{2}$$

and

$$P_{i,\min} \leq P_i \leq P_{i,\max} \quad i = 1, 2, ..., N \tag{3}$$

where
- P_i is the output of unit i (in MW),
- $F_i(P_i)$ is the input of unit i, or its cost rate (in $\$/h$),
- P_R is the required load (in MW),
- P_L is the transmission losses (in MW).

Moreover,

$$P_L = \sum_{i=1}^{N} \sum_{j=1}^{N} P_i B_{ij} P_j + \sum_{i=1}^{N} B_{0i} P_0 + B_{00} . \tag{4}$$

Derivation of this formula for losses is explained in [6]. Neglecting transmission losses, i.e., setting $P_L = 0$ in (2), problem (1) - (3) can be solved by a Lagrange multiplier method [6] or by dynamic programming [1, 4, 6].

3. PREVIOUS SOLUTION METHODS

When transmission losses are considered, the standard solution method is the penalty factor approach. Consider the Lagrange function, with a multiplier λ for (2):

$$\text{minimize } \mathcal{L} = \sum_{i=1}^{N} F_i(P_i) + \lambda(P_R + P_L - \sum_{i=1}^{N} P_i) \ . \tag{5}$$

Neglecting bounds on the P_i, the first order conditions are equation (2) above and

$$\frac{1}{1 - \dfrac{dP_L}{dP_i}} \cdot \frac{dF_i}{dP_i} = \lambda, \quad i = 1, 2, \ldots, N \ , \tag{6}$$

where

$$\frac{dP_L}{dP_i} = \sum_{j=1}^{N} 2B_{ij}P_j + B_{0i} \tag{7}$$

are the *incremental losses* and

$$\frac{1}{1 - \dfrac{dP_L}{dP_i}} = \frac{1}{1 - \sum\limits_{j=1}^{N} 2B_{ij}P_j - B_{0i}} = PF_i \ , \tag{8}$$

are the *penalty factors* PF_i for units $i = 1, \ldots, N$. Equations (2) and (6) are called *coordination equations*.

If bounds on the P_i are taken into account the first order conditions become

$$PF_i \cdot \frac{dF_i}{dP_i} \begin{cases} \geq \lambda & \text{if } P_i = P_{i,\min}, \\ = \lambda & \text{if } P_i \in (P_{i,\min}, P_{i,\max}), \\ \leq \lambda & \text{if } P_i = P_{i,\max}. \end{cases} \tag{9}$$

An iterative method to solve (1)-(3) is the following [6]:

1. Solve the coordination equations with unit penalty factors (i.e. neglecting losses) to get an initial solution $P_1^0, P_2^0, \ldots, P_N^0$. Set the iteration counter $k = 1$.

2. Compute penalty factors with values of the last iteration, i.e.,

$$PF_i^{(k)} = \frac{1}{1 - \sum\limits_{j=1}^{n} 2B_{ij}P_j^{(k-1)} - B_{0i}}, \quad i = 1, 2, \ldots, N \ ,$$

3. Solve the coordination equations to get a solution $P_1^{(k)}, P_2^{(k)}, \ldots, P_N^{(k)}$. (This can be done in various ways, e.g. a search technique for the optimal λ , a Newton-Raphson search, etc, see [6]).

4.   Compute the current difference between the production and load plus losses, i.e.,

$$\delta = \sum_{i=1}^{N} P_i^{(k)} - P_R - \sum_{i=1}^{N} \sum_{j=1}^{N} P_i^{(k)} B_{ij} P_j^{(k)} - \sum_{i=1}^{N} B_{0i} P_i^{(k)} - B_{00}$$

If $\delta \leq \varepsilon$ (a given tolerance), stop. Otherwise, increase k by 1 and return to step 2.

While no formal proof of convergence for this method seems to have been published, and values of δ may oscillate, a local and possibly global optimum is usually reached in a fairly small number of iterations.

Modification to the iterative method just described to allow solution by dynamic programming are proposed in [4]. In Step 1 the initial solution is found by the recursion

$$F_i^*(D_i) = \min_{P_i \in R_i} \{F_{i-1}^*(D_i - P_i) + F_i(P_i)\} \tag{10}$$

where R_i is the set of integers in $[P_{i,\min}, P_{i,\max}]$ the range of production for unit i, assuming that approximation to 1 MW and D_i is chosen among the range of possible productions for the i first units, i.e., it is integer and

$$\sum_{k=1}^{i} P_{k,\min} \leq D_i \leq \sum_{k=1}^{i} P_{k,\max} . \tag{11}$$

In Step 3, each cost function F_i is multiplied by the penalty factor P_i^{k-1}, the load P_R is augmented by the losses as estimated and the problem is again solved by the dynamic programming recursion (10).

The Lagrange function for this step is

$$\sum_{i=1}^{N} \left(\frac{1}{1 - \sum_{j=1}^{N} 2 B_{ij} P_j^{(k-1)} - B_{0i}} \right) F_i(P_i)$$

$$+ \lambda \left(P_R + \sum_{i=1}^{N} \sum_{j=1}^{N} P_i^{(k-1)} B_{ij} P_j^{(k-1)} + \sum_{i=1}^{N} B_{0i} P_i^{(k-1)} + B_{00} - \sum_{i=1}^{N} P_i \right) \tag{12}$$

and the first order condition , as the losses are fixed,

$$\sum_{i=1}^{N} \left(\frac{1}{1 - \sum_{j=1}^{N} 2 B_{ij} P_j^{(k-1)} - B_{0i}} \right) \frac{dF_i}{dP_i} \begin{cases} \geq \lambda & \text{if } P_i = P_{i,\min}, \\ = \lambda & \text{if } P_i \in (P_{i,\min}, P_{i,\max}), \\ \leq \lambda & \text{if } P_i = P_{i,\max}. \end{cases} \tag{13}$$

and are the same as (9) except that the values of P_i are fixed in the penalty factors.

4. SEPARABLE LOSSES

Economic dispatch with transmission losses can be optimized by dynamic programming without using penalty factors. The easiest case is when losses are separable in the P_i. Assume also that they are quadratic, a particular case considered in [6]. Then the power P_i produced by unit will be equal to the power P_i' going to the load and the losses. Thus

$$P_i' = P_i - b_{i0} - b_{i1}P_i - b_{i2}P_i^2 \tag{14}$$

where $b_{i0},\ b_{i1}$ and b_{i2} are the known coefficients of the loss function. Hence

$$P_i = \frac{1 - b_{i1} - \sqrt{(1-b_{i1})^2 - 4b_{i2}(P_i' + b_{i0})}}{2b_{i2}} \ . \tag{15}$$

One can associate to P_i' a cost function $F_i'(P_i')$ equal to the cost necessary for unit i to contribute P_i' to the load, i.e., $F_i(P_i)$ where P_i is obtained from P_i' by (14). The correspondence between F_i' and F_i is illustrated on Figure 1.
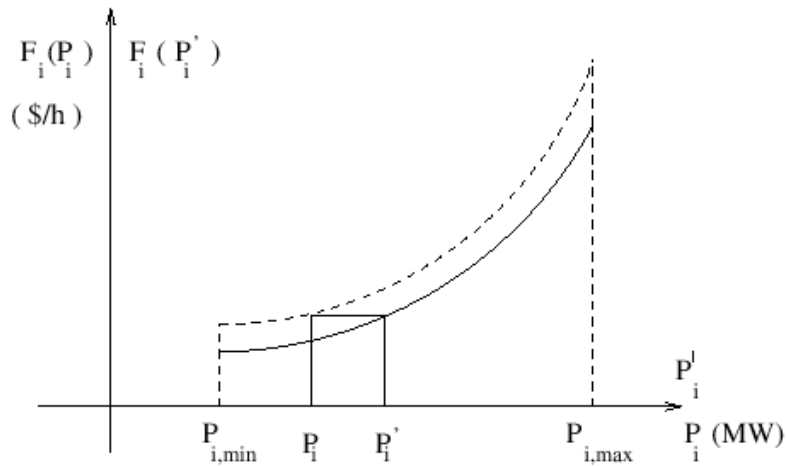
Figure 1: Input-Output functions $F_i(P_i)$ and $F_i(P_i')$

The range of values of P_i', deduced from that of P_i is, assuming integer values, or, in other words an approximation of 1 MW:

$$\begin{aligned}
P_{i,\min}' &= [P_{i,\min} - b_{i0} - b_{i1}P_{i,\min} - b_{i2}P_{i,\min}^2] \le P_i' \\
&\le [P_{i,\max} - b_{i0} - b_{i1}P_{i,\max} - b_{i2}P_{i,\max}^2] = P_{i,\max}'
\end{aligned} \tag{16}$$

where $\lceil a \rceil$ and $\lfloor a \rfloor$ are respectively the smallest integer not smaller than and the largest integer not larger than a ($P_{i,\min}$ and $P_{i,\max}$ are assumed to be integer as well).

The dynamic programming recursion is

$$F_i'^*(D_i) = \min_{P_i' \in R_i} \{F_{i-1}'^*(D_i - P_i') + F_i'(P_i')\} \tag{17}$$

where $R_i = [P_{i,\min}', P_{i,\max}']$, the range of effective production of unit i (i.e. that which is not lost) and

$$D_i \in \left[\sum_{k=1}^{i} P_{k,\min}', \sum_{k=1}^{i} P_{k,\max}' \right] \tag{18}$$

the range of demand which can be satisfied by the first i units. This recursion is used to find the optimal policy $P_1'^*, P_2'^*, ..., P_N'^*$, from which the powers $P_1^*, P_2^*, ..., P_N^*$ to be produced by each unit are obtained by (15). The cost of this policy is $F_N'^*(P_R) = \sum_{i=1}^{N} F_i(P_i^*)$.

Observe that the method proposed gives a globally optimal solution (up to the approximations in the values of the P_i), even if the cost functions $F_i(P_i)$ and/or P_i are non-convex. In contrast, the classical penalty function method may stop at a local optimum in this case.

Example 1. (Wood and Wollenberg [6], p 36). This problem involves three units with input/output functions and production ranges:

$$F_1(P_1) = 561.0 + 7.920P_1 + 0.001562P_1^2,$$
$$P_{1,\min} = 150 \ MW, \ P_{1,\max} = 600 \ MW,$$
$$F_2(P_2) = 310.0 + 7.850P_2 + 0.001940P_2^2,$$
$$P_{2,\min} = 100 \ MW, \ P_{2,\max} = 400 \ MW,$$
$$F_3(P_3) = 78.0 + 7.970P_3 + 0.00482P_3^2,$$
$$P_{3,\min} = 50 \ MW, \ P_{2,\max} = 200 \ MW,$$

$P_R = 850 \ MW$ and losses are given by

$$P_L = 0.00003P_1^2 + 0.00009P_2^2 + 0.00012P_3^2 .$$

Solution. The following solution is obtained by applying dynamic programming to the transformed problem: $P_1 = 434.668, \ P_2 = 300.106, \ P_3 = 131.061$ with losses equal to 15.8350. Note that the same losses are obtained by the penalty factor approach in 4 iterations ([6]).

5. SEPARABLE QUADRATIC APPROXIMATION

An iterative method using a separable quadratic approximation of the loss function is easily obtained by fixing at each iteration one value P_j in each quadratic term involving two such values P_i and P_j. Such an approximation is much more precise than the one obtained by fixing both values in all such terms, discussed above. Then the algorithm of the previous section is as follows:

(a) Obtain an initial solution $P_1^{(0)}, P_2^{(0)}, ..., P_N^{(0)}$ by solving the problem after deleting all quadratic terms $P_i B_{ij} P_j$ with $i \neq j$ in the loss function (or, which is better, by some heuristics, see below). Set the iteration counter $k = 1$.

(b) Compute the following separable quadratic approximation of the loss function

$$\sum_{i=1}^{N} B_{ii} P_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} P_i B_{ij} P_j^{(k-1)} + \sum_{i=1}^{N} B_{0i} P_i + B_{00} \qquad (19)$$

(c) Solve the problem with the approximate loss function (19) to obtain a solution $P_1^{(k)}, P_2^{(k)}, ..., P_N^{(k)}$ (for this purpose, use DP explained in the previous section).

(d) Compute the approximate and exact losses using $P_1^{(k)}, P_2^{(k)}, ..., P_N^{(k)}$ in (19) and (4). If these values differ in absolute value by less than ε, stop. Otherwise increase k by 1 and return to (b).

A better initial approximation to the loss function may be obtained by computing a feasible initial solution by heuristics such as following:

(h1) Set $P_i = P_{i,\min}$ for $i = 1, ..., N$;

(h2) Compute $\delta P = \dfrac{P_R - \sum_{i=1}^{N} P_i}{N}$;

(h3) If $P_i + \delta P > P_{i,\max}$ for some i, set $P_i = P_{i,\max}$ delete that index value, and reduce N by 1. Return to (h2);

(h4) Set $P_i + \delta P$ for all remaining i.

The aim of this heuristics is to obtain quickly a feasible solution in which the productions above the minimum ones of the various units are as close as possible. Therefore, it first attempts to give an equal load to each unit. If some upper bound is not satisfied the largest possible load is assigned to the corresponding unit. The procedure is iterated with the total unassigned load. More sophisticated heuristics could take cost functions into account, but this does not appear to be necessary.

Example 2. (Wood and Wollenberg [6], p 117-120)). The fuel cost curves for the three units in the six-bus network considered are given by

$$F_1(P_1) = 213.1 + 11.669P_1 + 0.00533P_1^2$$
$$F_2(P_2) = 200.0 + 10.333P_2 + 0.00889P_2^2$$
$$F_3(P_3) = 240.0 + 10.833P_3 + 0.00741P_3^2$$

with a total load to be supplied $P_R = 210$ and unit dispatch limits

$$50.0 \le P_1 \le 200$$
$$37.5 \le P_2 \le 150$$
$$45.0 \le P_3 \le 180.$$

The coefficients $B_{00} = 4.04$, $B_{0i} = (-0.07660, -0.00342, 0.01890)$, $(i = 1,2,3)$ and the B matrix is

$$B = \begin{bmatrix} 0.0006760 & 0.0000953 & -0.0000507 \\ 0.0000953 & 0.0005210 & 0.0000901 \\ -0.0000507 & 0.0000901 & 0.0002940 \end{bmatrix}.$$

Solution. The optimal solution is obtained in 3 iterations:

It.	F	P_1	P_2	P_3	$P_R + P_L$	Error
1	3146.28	70.3011	76.1023	71.1801	217.583	0.48846
2	3165.09	71.5868	73.9075	73.5692	219.063	0.01805
3	3164.86	71.5625	73.9348	73.5485	219.046	0.00021

Example 3. (Liang and Glover [4]). The operating costs of the generators are represented by the following polynomials of third order:

$$a_0 + a_1 P_i + a_2 P_i^2 + a_3 P_i^3, \quad i = 1,2,3$$

where coefficients as well as P_{min} and P_{max} are listed below:

Unit	a_0	a_1	a_2	a_3	P_{min}	P_{max}
1	11.200	5.10238	−0.00264290	0.00000333333	100	500
2	−632.000	13.01000	−0.03057140	0.00003333333	100	500
3	147.144	4.28997	0.00030845	−0.00000017677	200	1000

The load is $P_R = 1400$ and the losses are given by

$$P_L = 0.000075P_1^2 + 0.000015P_2^2 + 0.000045P_3^2.$$

The B-matrix is

$$B = \begin{bmatrix} 0.0000750 & 0.000005 & 0.0000075 \\ 0.0000050 & 0.000015 & 0.0000100 \\ 0.0000075 & 0.000010 & 0.0000450 \end{bmatrix}.$$

Solution. If an initial solution is obtained by neglecting the off-diagonal elements of matrix B, the optimal solution is obtained in four iterations:

It.	F	P_1	P_2	P_3	$P_R + P_L$	Error
1	6592.20	359.704	406.478	665.957	1432.14	10.46985
2	6642.32	360.266	406.927	676.145	1443.34	0.075683
3	6642.68	361.361	406.969	675.079	1443.41	0.001587
4	6642.69	360.293	406.967	676.158	1443.42	0.000366

Another way to derive an initial solution is also developed above, in Section 5. In this example, the initial solution obtained is $P_1 = P_2 = 433.333$; $P_3 = 533.333$. The optimal solution is then reached in only two iterations:

It.	F	P_1	P_2	P_3	$P_R + P_L$	Error
1	6641.60	363.129	407.547	672.448	1443.12	0.229736
2	6642.68	359.222	406.959	677.244	1443.42	0.000977

Example 4. In the previous examples convex programming problems are solved, since both the objective function F_T and the transmission losses P_L are convex. Note that convexity of the P_L function follows from the positive definiteness of the B matrix (i.e., the main minors of B are positive in Examples 1, 2 and 3 above). In this example, we change elements b_{12} and b_{21} from Example 3, i.e., we now have $b_{12} = b_{21} = 0.00005$.

Then the minor of the second order is negative $(\det = 0.000075 \cdot 0.000015 - 0.00005^2 < 0)$ and thus, there are more than one local minima. Since the penalty factor method is derived from the first order conditions, it is a local optimization method, and may not reach the global optimum.

Solution. With our separable approximation dynamic programming method, the following locally optimal solution is obtained in six iterations:

It.	F	P_1	P_2	P_3	$P_R + P_L$	Error
1	6699.85	375.187	100.875	999.538	1475.60	8.50537
2	6657.20	419.456	98.993	946.974	1465.42	0.259155
3	6684.72	348.296	401.648	703.157	1453.10	3.97778
4	6703.30	328.416	403.299	725.495	1457.21	0.324585
5	6701.74	327.425	404.014	725.391	1456.83	0.009888
6	6701.69	327.437	403.992	725.391	1456.82	0.000122

We then ran the *penalty factor* method to see what solution it gives for this non-convex example. For that purpose we used software attached to the book [6]. The solution obtained is as follows: $F(P_1, P_2, P_3) = 6940.35$; $P_1 = 428.6$, $P_2 = 100.0$, $P_3 = 1,000.0$, $\lambda = 4.6765$. Therefore, the objective function value and the total losses are 6940.35 and 128.6 respectively (compare those values with 6701.69 and 56.82 obtained by our approach).

We also tried a modification of the penalty factor method: instead of adjusting λ in the inner loop (as suggested in [6] for Step 3 in the algorithm given in Section 3), we simply find the closest solution that satisfies the range constraints (3) as well. In other words, within Step 3, we iteratively project the current solution $(P_1, ..., P_n)$ onto the hyperplane defined by (2) and the hypercube (3) until a feasible solution is reached. Note that both methods are equivalent if the point obtained after solving the coordination equations is feasible. The result, obtained in 5 iterations, is: $F(219.21, 243.79, 1000.00) = 6812.31$. Again, this solution is worse than the dynamic programming one.

6. CONCLUSION

A new method for static termal power units economic dispatch problem with transmission losses is proposed. It uses a separable quadratic approximation to the loss function, and refines it iteratively. Convergence is quicker than with previous methods. Moreover, it may lend to better locally optimal solutions than those methods in case of non-convexity of the loss function.

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