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NORMALISATION AFFECTS THE RESULTS OF MADM METHODS

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Abstract: This paper deals with the effects that the popular normalisation procedures: Simple (SN), Linear (LN), and Vector (VN) could have on the results of the Multiple Attribute Decision Making (MADM) methods (Simple Additive Weighting-SAW, TOPSIS and ELECTRE). It shows that the deformations of empirical data, caused by the usage of normalisations, could affect the final choices. If the MADM methods are based on normalised ratings of the SN or VN type, then their results could depend on: 1) the measurement units used for quantitative attributes that are measurable by interval scales, and 2) the type of Likert scale used for measuring qualitative attributes. It also shows that the MADM methods violate certain conditions of consistent choice and that this violation could be attributed to SN, LN and VN normalisation procedures.

Keywords: Multiple attribute decision making, normalisation procedures.

1. INTRODUCTION

With MADM methods we make choices among m alternatives $A_i = (x_{i1}, x_{i2}, ..., x_{ij}, ..., x_{in})$ $(i = 1,2,...,m)$ on the basis of n attributes X_j $(j = 1,2,...,n)$, where x_{ij} is the value of the j-th attribute for the i-th alternative. Normalisation is a process by which the values of all attributes x_{ij}, which are measured by different measurement scales, are transformed into normalised ratings r_{ij}, i.e. for every attribute $X_j = (x_{1j}, x_{2j}, ..., x_{ij}, ..., x_{mj})$ $(j = 1,2,...,n)$ they are separately mapped onto a scale [0,1], or onto one of its sub-segments $(0 < r_{ij} < 1)$. Normalisation gives way to the comparison of all attributes on a common scale, thus enabling the evaluation of each alternative by a single value, and subsequently the choice of the best alternative according to the measure obtained by the applied MADM method.

In this paper we shall show that the final choices of the MADM methods, based on Simple (SN) or Vector (VN) normalisation procedures, could depend on: 1) the units used for measuring quantitative attributes that are measurable by interval scales, and 2) the Likert scale used for measuring qualitative attributes. We shall also show that the MADM methods, based on VN normalisations, violate two conditions of consistent choice concerning both the inclusion and exclusion of some alternatives from the set of observable alternatives, $S = \{A_1, A_2, ..., A_i, ..., A_m\}$. The MADM methods, based on SN or Linear (LN) ratings, also violate these conditions, and we will observe special cases in which the violation could be attributed solely to the applied normalisation. The effects of SN and LN normalisations will be illustrated by using the SAW (Simple Additive Weighting) method (see, for example, [8]), while the effects of VN normalisation will be observed by using two methods, TOPSIS ([1]) and ELECTRE ([5], [6]). For the sake of simplicity, we shall presume that in the following examples all attributes have equal weights. This assumption, however, has no implications on the conclusions whatsoever.

2. NORMALISATION PROCEDURES

Table 1 contains the formulae of the chosen normalisation techniques for benefit and cost attributes.

Table 1: Formulae for calculating normalised ratings of Simple, Linear and Vector type, for benefit and cost attributes

Type of attribute	Normalisation		
	Simple (SN)	Linear (LN)	Vector (VN)
Benefit, X_j	$r_{ij}^S = \frac{x_{ij}}{x_j^*}$, $x_{ij} > 0$	$r_{ij}^L = 1 - \frac{x_j^* - x_{ij}}{x_j^* - x_j^-}$	$r_{ij}^V = x_{ij} / \sqrt{\sum_{i=1}^{m} x_{ij}^2}$
Cost X_j	$r_{ij}^S = \frac{x_j^-}{x_{ij}}$, $x_{ij} > 0$	$r_{ij}^L = 1 - \frac{x_{ij} - x_j^-}{x_j^* - x_j^-}$	$r_{ij}^V = \frac{1}{x_{ij}} / \sqrt{\sum_{i=1}^{m} \left(\frac{1}{x_{ij}}\right)^2}$, $x_{ij} > 0$

$x_j^* = \max_i x_{ij}$, and $x_j^- = \min_i x_{ij}$.

In this paper we shall analyse the effects of certain transformations of attributes' values on their normalised ratings. Here we emphasise that we shall observe a *benefit* attribute X_j only, all its possible values being greater than 0, i.e. $x_{ij} > 0$. The following Table describes some characteristics of the normalised ratings obtained by these three procedures, which are important for our further analysis.

Table 2: Some characteristics of normalised ratings of the SN, LN and VN type

		Benefit attribute, X_j, and its transformation, Y_j	Normalisation		
			Simple (SN)	Linear (LN)	Vector (VN)
1	Domain	$x_{\min} > 0$, $y_{\min} > 0$	$(0 < r_{\min}, 1]$	$[0,1]$	$(0 < r_{\min}, r_{\max} < 1)$
2	Linear Transformation	$y_{ij} = ax_{ij}, a > 0 \Rightarrow r(y_{ij}) = r(x_{ij})$	✓	✓	✓
3	Positive affine Transformation	$y_{ij} = ax_{ij} + b, a > 0, b \neq 0 \Rightarrow r(y_{ij}) = r(x_{ij})$		✓	

1. First, it should be noted that the three types of normalised ratings differ in their domain. In the case of SN, the domain spans the interval between $r_{\min}$ (which is in this case greater than 0) and 1. For LN, it covers the whole interval [0,1]. Finally, in VN it is restricted to a sub-segment placed strictly inside the unit interval. The length of the domain of SN and VN ratings depends on the domain of X_j. However, it does not depend on its absolute length (i.e. the difference $x_{j\max} - x_{j\min}$), but on the ratio $x_{j\max} / x_{j\min}$ ($x_{j\min} > 0$) instead. So, as this ratio decreases (i.e. $(x_{j\max} / x_{j\min}) \to 1$), the length of the domain of normalised ratings of the SN and VN type decreases as well (i.e. $(r_{j\max} - r_{j\min}) \to 0$). This means that the values r_{ij}^S concentrate into a small interval close to 1, while the values r_{ij}^V cover a small sub-segment of (0,1) (which could be placed on different parts of the interval). Analogously, as the ratio increases (i.e. $(x_{j\max} - x_{j\min}) \to \infty$) so does the length of the domain of normalised ratings of the SN and VN type (i.e. $(r_{j\max} - r_{j\min}) \to 1$); the values of r_{ij}^S tend to cover the whole interval (0,1], while the values of r_{ij}^V cover most of the interval (0, 1). Unlike SN and LN ratings, the domain of VN ratings also depends on the distribution of the attribute X_j inside the interval $(x_{j\min}, x_{j\max})$ with fixed boundaries. As the positive skew of the distribution of X_j increases, the length of the domain of VN ratings increases as well. Similarly, as the negative skew of the distribution of X_j increases, the length of the domain of VN ratings decreases. Also, the domain of VN ratings depends on the number of observed alternatives (m): For the same distributions of X_j inside the interval with fixed boundaries, as the number of alternatives increases, the length of the domain decreases. ([4])

2. Linear transformations of data do not change the values of normalised ratings of all three types:

$$\text{Simple: } r(y_{ij}) = y_{ij} / y_j^* = ax_{ij} / ax_j^* = x_{ij} / x_j^* = r(x_{ij}),\ \ y_j^* = \max_j y_{ij} \tag{1}$$

$$\text{Linear: } r(y_{ij}) = 1 - (y_j^* - y_{ij}) / (y_j^* - y_j^-) = 1 - (ax_j^* - ax_{ij}) / (ax_j^* - ax_j^-) = \\ = 1 - (x_j^* - x_{ij}) / (x_j^* - x_j^-) = r(x_{ij}),\ \ y_j^- = \min_j y_{ij} \tag{2}$$

Vector: $r(y_{ij}) = y_{ij} / \sqrt{\sum_{i=1}^{m} y_{ij}^2} = ax_{ij} / \sqrt{a^2 \sum_{i=1}^{m} x_{ij}^2} = x_{ij} / \sqrt{\sum_{i=1}^{m} x_{ij}^2} = r(x_{ij})$ (3)

3. However, with the positive affine transformations of data, only LN ratings remain unchanged, while the other two, SN and VN ratings, are affected by the transformation:

Simple: $r(y_{ij}) = y_{ij} / y_j^* = (ax_{ij} + b)/(ax_j^* + b) \neq x_{ij} / x_j^* = r(x_{ij})$ (4)

Linear: $r(y_{ij}) = 1 - (y_j^* - y_{ij})/(y_j^* - y_j^-) =$

$= 1 - ((ax_j^* + b) - (ax_{ij} + b))/((ax_j^* + b) - (ax_j^- + b)) =$

$= 1 - (x_j^* - x_{ij})/(x_j^* - x_j^-) = r(x_{ij})$ (5)

Vector: $r(y_{ij}) = y_{ij} / \sqrt{\sum_{i=1}^{m} y_{ij}^2} = (ax_{ij} + b) / \sqrt{\sum_{i=1}^{m} (ax_{ij} + b)^2} \neq x_{ij} / \sqrt{\sum_{i=1}^{m} x_{ij}^2} = r(x_{ij})$ (6)

In the next sections we shall show that these characteristics of normalised ratings could essentially determine the rankings of alternatives (or the choice set) obtained by the MADM methods. This could happen in the "aggregation stage" of every MADM method, where normalised ratings of all attributes are somehow combined and a single value of the measure is calculated for every alternative, making possible their evaluation, comparison and the final choice.

3. MEASUREMENT UNITS AND NORMALISED RATINGS

3.1. Quantitative attributes

It was shown that normalised ratings of the SN and VN type are robust in linear transformations (formulae (1) and (3)), but vulnerable to positive affine transformations of the data (formulae (4) and (6)), which are important characteristics concerning quantitative attributes. Let us recall that positive affine transformations are permissible transformations of all attributes that are measurable by interval scales ([2]). One of them is *temperature*, which is most frequently measured by the Celsius scale (^0C) or by the Fahrenheit scale (^0F). The choice of the scale is determined by the custom of the country the decision maker (DM) lives in, where the conversion from one scale into another is given by: $^0F=(9/5)\cdot{}^0C+32$. Therefore, for the rankings of alternatives (the choice set) obtained by the MADM methods it should be completely irrelevant which temperature scale is used. Unfortunately, if we apply any of the MADM methods that are based on SN or VN normalisation it is not necessarily so.

Example 1: Let us observe the choice between three tourist tours on the basis of three equally important criteria: X_1 - average temperature (the warmer the better),

X_2 - duration of the tour (the longer the better) and X_3 - price (the lower the better). We shall suppose that the Tourist Agency has offices both in the USA and Germany, where it offers its clients the same tours at the same prices. Since all the necessary information, which is contained in the brochures, is expressed in traditionally used measurement units and currencies, we have two normatively equivalent presentations of the same choice problem (Table 3): in Germany the temperature is given in degrees of Celsius (^0C) and the price in Deutsch marks (DM), while in the USA the same values are measured in degrees of Fahrenheit (^{0}F) and in American dollars ($US), respectively. We shall suppose that the exchange rate is 1$US=2DM.

Table 3: Two normatively equivalent presentations of the choice problem in example 1

Alt.	GERMANY			UNITED STATES		
	Temperature in ^0C (X_1)	Number of days (X_2)	Price in DM ($^-X_3$)	Temperature in ^0F (X_1)	Number of days (X_2)	Price in $US ($^-X_3$)
A_1	20	10	1150	68	10	575
A_2	24	12	1600	75.2	12	800
A_3	28	7	1350	82.4	7	675

$^0F = (9/5)\,^0C + 32$; 1$US = 2DM

We expect every MADM method to provide the same results irrespective of the measurement units in which the attributes are measured. In other words, while using the same MADM method the clients from both countries should choose the same destination.

Let us apply the three chosen methods on both presentations. With SAW we will use SN ratings, while with TOPSIS and ELECTRE, VN ratings will be used. Tables 3a and 3b contain the values of measures of SAW and TOPSIS and the rankings of alternatives based on them, as well as the *aggregate dominance matrices* (ADM) of ELECTRE and the alternatives contained in *kernels*.

Table 3a: The results obtained in Germany

Alt.	SAW		TOPSIS		ELECTRE	
	$V(A_i)$ (SN)	Rankings	C^+	Rankings	ADM	Kernel
A_1	.849	A_2	.535	A_2	0 0 0	∅
A_2	.859*	A_1	.593*	A_1	0 0 0	
A_3	.812	A_3	.417	A_3	0 0 0	

Table 3b: The results obtained in the United States

Alt.	SAW		TOPSIS		ELECTRE	
	$V(A_i)$ (SN)	Rankings	C^+	Rankings	ADM	Kernel
A_1	.886*	A_1	.616*	A_1	0 0 1	A_1
A_2	.877	A_2	.602	A_2	0 0 0	
A_3	.812	A_3	.333	A_3	0 0 0	

As can be seen, by using the SAW or the TOPSIS method clients from Germany will choose tour A_2, while in the USA the same methods will suggest tour A_1. Similarly, by using ELECTRE tourists living in Germany would not be able to decide, while in the USA the same method will propose the choice of A_1.

Different results for two normatively equivalent presentations of the problem are owing to the difference in the temperature scales used, and not to different currencies in which the prices are expressed. Namely, the currencies are convertible by the linear transformation: $y = 2x$, and linear transformations of data have no effect on their normalised ratings. On the other hand, temperatures expressed in degrees of Celsius could be converted into degrees of Fahrenheit by the positive affine transformation: $y = (9/5)x + 32$. This transformation of data, however, causes changes of the normalised ratings of the SN and VN type. We can conclude that if the MADM methods are based on SN or VN ratings, then the rankings of alternatives (the choice set) obtained by them could depend on the measurement units used for all quantitative attributes that are measurable by interval scales (formulae (4) and (6)). However, changes in measurement units used for attributes that are measurable by ratio scales have no effect on the results (formulae (1)-(3)).

3.2. Qualitative attributes

In the MADM methods qualitative attributes are often measured by so-called Likert-type scales. The five-point scale is frequently used and it is marked either with numbers from 1 (for the lowest level of the attribute) to 5 (for the highest level), or from 1 to 9 (1,3,5,7,9), where numbers correspond to the same levels of a qualitative attribute. In the MADM literature Likert scales are mostly treated as ordinal scales, although in some rare cases they are considered to be interval scales, mutually connected by the positive affine transformation: $y = 2x - 1$ (see, for instance, [8]). However, irrespective of the treatment of the Likert scales, their usage with the MADM methods, as well as the arbitrary choice between them, was being widely accepted. Therefore, the results of the MADM methods should not depend on the type chosen. The following example shows that just the opposite could be the case.

Example 2: Suppose that we have to choose between two office spaces, on the basis of the following four equally important attributes: the location, the technical equipment that already exists, the spaciousness, and the rent arrangements expressed in the number of instalments (which is a cost attribute) (Table 4).

Table 4: The choice problem in example 2

Alt.	Location X_1	Technical equipment X_2	Size (in m^2) X_3	Number of instalments X_4
A_1	Unfavourable	Poor	70	9
A_2	Favourable	Very poor	100	12

The first two attributes are qualitative ones, and if we decide to measure them on a five-point scale we can arbitrarily apply either the scale 1,2,3,4,5, or the scale 1,3,5,7,9. Table 4a contains the modalities of *location* and *technical equipment* attributes, as well as the numbers of the two scales that are attached to them.

Table 4a: Likert scales I and II and different modalities of attributes X_1 and X_2

Likert scale I	Likert scale II	Location	Technical equipment
1	1	Very unfavourable	*Very poor*
2	3	*Unfavourable*	Poor
3	5	Average	Average
4	7	*Favourable*	Good
5	9	Very favourable	Very good

It appears that the problem can be presented in two normatively equivalent ways (Tables 4b and 4c).

Table 4b. Problem presentation by using Likert scale I

Alt.	X_1	X_2	X_3	$^-X_4$
A_1	**2**	**2**	70	9
A_2	**4**	**1**	100	12

Table 4c. Problem presentation by using Likert scale II

Alt.	X_1	X_2	X_3	$^-X_4$
A_1	**3**	**3**	70	9
A_2	**7**	**1**	100	12

However, as can be seen from Table 4d, the MADM methods, which are based on SN or VN ratings, will not provide us with the same rankings (or the same choice set) of alternatives for both presentations of the problem. If we choose the scale 1,2,3,4,5, and apply SAW based on SN data, we will select alternative A_2, while by using the scale 1,3,5,7,9, we will choose A_1. The same results would be obtained by applying the TOPSIS method. Similarly, by using ELECTRE in the first case (Ls I) both alternatives will be proposed as acceptable (both are in the kernel), while in the second case (Ls II) alternative A_1 will be proclaimed the better one.

Table 4d: The results of SAW, TOPSIS and ELECTRE, based on two normatively equivalent presentations of the problem shown in example 2

SAW				TOPSIS				ELECTRE			
Ls I		Ls II		Ls I		Ls II		Ls I		Ls II	
$V(A_i)$ (SN)	Rankings	$V(A_i)$ (SN)	Rankings	C^+	Rankings	C^+	Rankings	ADM	Kernel	ADM	Kernel
.800	$\mathbf{A_2}$	.782	$\mathbf{A_1}$	.489	$\mathbf{A_2}$	.534	$\mathbf{A_1}$	0 1	$\mathbf{A_1, A_2}$	0 1	$\mathbf{A_1}$
.812	A_1	.771	A_2	.510	A_1	.466	A_2	1 0		0 0	

Again, different results for two normatively equivalent presentations of the problem can be attributed to the positive affine transformation of the data. Since the two Likert scales are mutually connected by the transformation $y = 2x - 1$, the normalised ratings of the SN and VN type for the same levels of qualitative attributes differ between themselves, and this difference affected the final choices. In other words, this example shows that if the MADM methods are based on SN or VN normalised ratings, then the choices that are proposed by these methods could depend on the type of Likert scale used.

Therefore, with examples 1 and 2 we have shown that the rankings of alternatives (the choice sets) obtained by the MADM methods, based on SN or VN type of data: 1) are vulnerable to changes in the measurement units used for quantitative attributes that are measurable by interval scales, and 2) depend on the type of Likert scale used for measuring qualitative attributes. The inconsistency of the results is caused by the instability of SN and VN normalised ratings on positive affine transformations of the data (formulae (4) and (6)).

3.3. Weights of attributes and measurement units

In the MADM methods it has been widely accepted that any change in the domain of attribute values should be followed by some change in their weights. Since all our examples have used fixed attribute weights, they might seem responsible for the above results.

Let us recall that the weights reflect the relative importance of attributes for the DM in the evaluation of the observed alternatives. Apart from that, they are partially determined by the data, i.e. they depend on the extent to which the domains of attributes of the observed alternatives differ from their potential domains. If the values of an attribute cover its entire potential domain, while the values of some other attribute cover a narrow sub-segment of its potential domain, then some corrections of the weights should be made. For example, if the alternatives differ slightly on a very important attribute, then its large weight should be reduced. By this correction, we reduce the possibility of the final choice being influenced by negligible differences on an important attribute to a greater degree than it would be influenced by sound differences on a less important attribute (whose weight would be consequently enlarged). However, it does not mean that any change in the domain of data should cause an instant change in the weight of that attribute, and especially not if that change is caused by a change in the measurement units used.

As far as MADM methods are concerned, their results seem to support our opinion, at least partially. Namely, if attributes are measurable on ratio scales, then the results of the MADM methods are robust to changes in the measurement units used (formulae (1)-(3)). However, if attributes are measurable on interval scales, then the results will be robust only if the MADM methods are based on the LN transformation of data (formula (5)). It is hard to find a rational explanation as to why in the first case the weights should not be changed, while in the second case they should (but only if SN or VN normalisations are used).

The same logic applies to qualitative attributes, which could also be measured on different scales. Although the Likert scales should not affect the preferences of the DM, it is well known that they can influence attributes' weights. However, the problem of the robustness of the DM's preferences when different measurement scales are used for qualitative attributes, in general, is not of our concern at the moment. We shall here observe only two Likert scales that have been used in this paper (1,2,3,4,5, and 1,3,5,7,9), which are both five-points scales, i.e. scales with equal precision. Therefore, although at first glance it might appear that the domain of an attribute has been extended (from (1, 5) into (1, 9)), it can be easily seen that only the numerical expressions of the qualitative modalities of the attributes have changed, while the "domains" of those attributes have remained the same (namely, the verbal expressions of the extreme modalities of every attribute have not altered and, consequently, the "interval" of their qualitative values has not widened).

Therefore, different robustness of the results of the MADM methods to: 1) changes in units used for quantitative attributes that are measurable on ratio and interval scales, and to 2) the Likert scale used to measure qualitative attributes, clearly shows that the inconsistent results, which are presented in this paper, should not be attributed to the assumption of fixed attributes' weights, but to the effects of the normalisations used.

4. EFFECTS OF INCLUSION AND EXCLUSION OF AN ALTERNATIVE

Let us point to some inconsistent choices that might occur due to the effects that the inclusion/exclusion of some alternatives into/from the set of observable alternatives could have on the normalised ratings. Adding new alternatives to the set or eliminating some of the existing ones from the set of observable alternatives, could change the domains of attributes (or at least, it could change the distribution of attribute values inside the intervals). This change could subsequently produce a change in their normalised ratings (Table 2, comment 1), and through them, a change in the final results. In order to preserve the weights of the attributes, we shall consider only cases in which the elimination of alternative(s) from the set S does not change (or at least not substantially) the domain of any attribute, and therefore we shall use the same weights as before.

Let us first define two criteria of consistent choice concerning the stability of the results of MADM methods to changes in the set S of observed alternatives. We shall adopt condition α, defined by A. Sen in the social choice theory ([7], p. 17.), and we define a new one, condition β^*.

Condition α: *If alternative A is the best in the set S, then it has to be the best in every subset E to which it belongs* $(A \in E,\ E \subset S)$.

Condition β^*: *If alternative A is by binary comparisons better than any other alternative from the set S, then it has to be the best in the whole set S.*

Since the MADM methods are used to support rational choices, we think that the fulfillment of conditions α and β^* should be considered desirable characteristics of every MADM method.

However, if the methods are based on normalised ratings of the VN type, then they violate these conditions. VN ratings are not only sensitive to changes in the domain of X_j, but also to changes in the distribution of data inside the fixed interval. This means that the exclusion/inclusion of some alternatives, even if it causes no changes in the domains of the attributes (preserving that way the weights), will cause a change in the distribution of data. Consequently, the VN ratings will change, and subsequently they could affect the final choice. We shall illustrate this with the following example.

Example 3: Let us observe the problem shown in Table 5, where we choose from the set $S = \{A_1, A_2, A_3, A_4\}$ on the basis of four equally important benefit attributes, and let us apply the ELECTRE method.

Table 5. The choice from the set $S = \{A_1, A_2, A_3, A_4\}$ by ELECTRE

	Attributes					
Alt	X_1	X_2	X_3	X_4	ADM	Kernel
A_1	1000	1000	300	9.5	0 0 0 0	
A_2	860	780	410	11.0	0 0 0 0	A_3
A_3	900	840	330	10.3	0 0 0 1	
A_4	900	880	330	9.7	0 0 0 0	

Table 5a. The choice from the subset $E^1 = \{A_1, A_2, A_3\}$ by ELECTRE

	Attributes					
Alt	X_1	X_2	X_3	X_4	ADM	Kernel
A_1	1000	1000	300	9.5	0 0 1	
A_2	860	780	410	11.0	0 0 1	A_1, A_2
A_3	900	840	330	10.3	0 0 0	

Since A_3 is the only alternative in the kernel (it over-ranks alternative A_4), it will be selected as the best. If we eliminate A_4 from the set S, condition α requires A_3 to be chosen from the subset $E^1 = \{A_1, A_2, A_3\}$, as well. However, as can be seen from Table 5a, ELECTRE proposes alternatives A_1 and A_2, which shows that condition α is violated.

Note that in this case we have excluded A_4, i.e., the alternative in comparison to which A_3 was proclaimed the best. Therefore, since on the basis of ADM (Table 5) we could not make the complete rankings of alternatives, we might agree with some changes in the kernel for E^1 (Table 5a). We could, perhaps, accept it to be empty, or that apart from alternative A_3 it now contains some other alternative(s), as well. However, surprisingly enough, the best alternative in the set S is now proclaimed the worst, i.e., it is the only one not contained in the kernel. (Note that the exclusion of A_4 has caused no changes in the domains of the attributes.)

Table 5b: The choice-problem from all binary subsets of the set S, and the solutions proposed by the ELECTRE method

Set	Alt.	Attributes				ELECTRE	
		X_1	X_2	X_3	X_4	ADM	Kernel
E_1	A_1	1000	1000	300	9.5	0 0	A_2
	A_2	860	780	410	11.0	1 0	
E_2	A_1	1000	1000	300	9.5	0 1	A_1
	A_3	900	840	330	10.3	0 0	
E_3	A_1	1000	1000	300	9.5	0 1	A_1
	A_4	900	880	330	9.7	0 0	
E_4	A_2	860	780	410	11.0	0 1	A_2
	A_3	900	840	330	10.3	0 0	
E_5	A_2	860	780	410	11.0	0 1	A_2
	A_4	900	880	330	9.7	0 0	
E_6	A_3	900	840	330	10.3	0 1	A_3
	A_4	900	880	330	9.7	0 0	

If we look at the results shown in Table 5b, we see that condition β^* is also violated. On the basis of the ADMs for all binary comparisons made between alternatives in the subsets E_1-E_6 of the set S, we can proclaim alternative A_2 the best. Even more, by binary comparisons the complete rankings of alternatives could be made: A_2, A_1, A_3, A_4. Therefore, we expect alternative A_2 to be chosen from the whole set S. However, the method proposes the choice of A_3, which proves that condition β^* is violated.

As far as binary comparisons are concerned, they are made with the usage of constant attribute weights, which are used for the whole set S. We are aware of the fact that if we were choosing between any two alternatives separately, we would make some corrections to the weights (because of the differences in the domains of the attributes), and consequently, the contents of the kernels might differ from those obtained. Therefore, one might disagree that the above results support the conclusion that the method violates condition β^*. However, binary comparisons enable us to determine the position of any alternative in relation to every other alternative in the set S, on the basis of the attributes and their weights that would be actually used in the final decision. These results raise an interesting question: If we use the same weights, should we get the same results when we apply the method to the whole set S, as when we choose on the basis of the rankings of alternatives (which is obtained by applying the same method on every pair of alternatives), if the rankings exists? We think so.

Example 4: Let us now observe the choice-problem (shown in Table 6) where we choose between four alternatives on the basis of four equally important benefit attributes, and let us apply the TOPSIS method.

Table 6: The choice from the set $S = \{A_1, A_2, A_3, A_4\}$ by TOPSIS

Alt.	Attributes X_1	X_2	X_3	X_4	C^+	Rankings
A_1	1020	1000	300	9.7	.4984	A_2
A_2	800	780	410	11.0	.5016	A_1
A_3	980	840	330	10.3	.4265	A_4
A_4	980	880	330	9.7	.4414	A_3

Although A_2 is proclaimed the best in the set S, from the results shown in Table 6a it can be seen that in none of the subsets E^1, E^2 and E_1, is alternative A_2 ranked first. This means that the TOPSIS method violates condition α. (In subsets E^1, E^2 and E_1 the domains of all the attributes are the same as in the whole set S.)

Table 6a: The choice-problem from the subsets $E^1 = \{A_1, A_2, A_3\}, E^2 = \{A_1, A_2, A_4\}$, $E^3 = \{A_1, A_3, A_4\}$, and from all binary subsets of the set S, and the solutions proposed by TOPSIS

Set	Alt	Attributes X_1	X_2	X_3	X_4	TOPSIS C^+	Rankings
E^1	A_1	1020	1000	300	9.7	.5033	A_1
	A_2	800	780	410	11.0	.4967	A_2
	A_3	980	840	330	10.3	.4298	A_3
E^2	A_1	1020	1000	300	9.7	.5007	A_1
	A_2	800	780	410	11.0	.4993	A_2
	A_4	980	880	330	9.7	.4445	A_4
E^3	A_1	1020	1000	300	9.7	.6181	A_1
	A_3	980	840	330	10.3	.3819	A_4
	A_4	980	880	330	9.7	.4071	A_3
E_1	A_1	1020	1000	300	9.7	.5092	A_1
	A_2	800	780	410	11.0	.4908	A_2
E_2	A_1	1020	1000	300	9.7	.6126	A_1
	A_3	980	840	330	10.3	.3874	A_3
E_3	A_1	1020	1000	300	9.7	.5840	A_1
	A_4	980	880	330	9.7	.4160	A_4
E_4	A_2	800	780	410	11.0	.5118	A_2
	A_3	980	840	330	10.3	.4882	A_3
E_5	A_2	800	780	410	11.0	.5149	A_2
	A_4	980	880	330	9.7	.4851	A_4
E_6	A_3	980	840	330	10.3	.5633	A_3
	A_4	980	880	330	9.7	.4367	A_4

The results also show that by binary comparisons the complete rankings of alternatives is made: A_1, A_2, A_3, A_4. In addition, alternative A_1 is the best in every subset of the set S to which it belongs, i.e., in E^1- E^3 and E_1 - E_3. Therefore, although we expected A_1 to be chosen from the whole set S, the TOPSIS method selected alternative A_2, which proves that it violates condition β^*, as well.

Therefore, we have shown that MADM methods, which are based on the VN ratings, violate conditions α and β^*.

Let us recall that SN and LN ratings depend on the maximum values of the attributes (SN), and on both minimum and maximum values of the attributes (LN) (Table 1). Therefore, they are sensitive only to changes in the attributes' domain (Table 2, comment (1)). Since a change in the domains of the attributes could be followed by a change in their weights, the violation of conditions α and β^* by the MADM methods that are based on SN or LN ratings cannot be attributed solely to the normalisation used, unless a change in the set S leaves the weights unchanged. Namely, by the inclusion/exclusion of an alternative into/from the set S, the subsequent change in the domains of some attributes might be negligible from the point of view of the DM (in which case he/she shall use previously determined weights), and at the same time considerable enough to affect the final result of the MADM method based on SN or LN ratings.

Example 5: Let us choose between three alternatives on the basis of four equally important benefit attributes (Table 7). In this example we shall apply the SAW method on normalised ratings of the SN type only.

Table 7: The choice from the set $S = \{A_1, A_2, A_3\}$ by the SAW(SN) method

	Attributes					
Alt.	X_1	X_2	X_3	X_4	V(A_i)	Rankings
A_1	3	15	90	92	.706	A_2
A_2	6	14	95	70	.778	A_3
A_3	3	40	60	90	.777	A_1

As can be seen from Table 7, the method proposes the choice of alternative A_2, and if we exclude A_1 (as it is ranked last), we expect to get the same rankings of the remaining alternatives as before. However, the result for the subset $E_3 = \{A_2, A_3\}$ (Table 7a) shows the opposite, which means that condition α is violated.

Table 7a: The choice-problem from all binary subsets of the set S, and the solutions proposed by SAW(SN)

		Attributes				SAW(SN)	
Subset	Alt	X_1	X_2	X_3	X_4	V(A_i)	Rankings
E_1	A_1	3	15	90	92	.645	A_2
	A_2	6	14	95	70	.983	A_1
E_2	A_1	3	15	90	92	.844	A_3
	A_3	3	40	60	90	.911	A_1
E_3	A_2	6	14	95	70	.782	A_3
	A_3	3	40	60	90	.783	A_2

Also, by observing the results of binary comparisons it can be seen that A_3 is the best one, and that the obtained rankings of alternatives: A_3, A_2, A_1, differs from the one previously formed for the whole set S. Therefore, we may conclude that SAW, when based on SN ratings, violates condition β^*, as well.

Should the DM consider the domain reduction of attribute X_4 from (70, 92) to (70, 90), which is caused by the exclusion of A_1, as relevant or not? If the potential domain of X_4 is substantially wider than the one of observed data, then the DM will probably regard the above domain reduction as negligible, and decide to use the previously determined attributes' weights. In that case the violation of conditions α and β^* could be exclusively attributed to SN normalisation.

We can conclude that the MADM methods, which are based on VN data, violate conditions α and β^*. On the other hand, the MADM methods based on SN or LN ratings also violate these conditions, but the violation could be attributed solely to SN or LN normalisation only if the changes in the domain of some attributes do not affect their weights.

The fact that MADM methods violate condition α is already well known, and it is discussed under the name of *the rank-reversal phenomenon*. However, as far as we know, rank-reversals are not related to any common cause, but to the specificities of the observed methods (see, for instance, [9]). We have shown that they can be attributed, at least partially, to deformations caused by normalisation procedures.

5. CONCLUSION

In this paper we have analysed the effects of three normalisation procedures: Simple, Linear and Vector on the results of the MADM methods, i.e., on the rankings of alternatives (or the content of the choice set) obtained by them. Let us repeat the main conclusions of this paper:

1. The results of MADM methods (based on SN or VN data) could depend on the measurement units used for quantitative attributes that are measurable by interval scales.

2. The results of MADM methods (based on SN or VN data) could depend on the type of Likert scale used to measure qualitative attributes.

3. MADM methods, based on VN data, violate conditions α and β^* due to the normalisation used. The same conclusion holds for the MADM methods that are based on SN or LN data. However, with those normalisations the violation of the conditions could be attributed exclusively to the chosen type of normalisation only if the changes in the domain of some attributes do not affect their weights.

We pinpoint normalisation procedures as a cause of the inconsistent choices of the MADM methods. Since normalised ratings represent mathematical transformations of data, at first glance it appears that they are an unbiased basis for rational choices.

Therefore, they are accepted as a fine replacement for the utilities (subjectively determined by the DM), which are used with the MAUD methods. That explains why the sensitivity analysis of the results of the MADM methods has dealt predominantly with the effects of weights on the final choices, whereas the possible effects of normalisation procedures have been almost neglected. Even more, it seems that so far deformations caused by normalisations have generally been ascribed to the attribute weighting procedures in which the DM's subjectivism could not be avoided. These results (see also [3] and [4]) call for a reconsideration of the use of some normalisation procedures in MADM (such as VN), and for the improvement of the normalisation techniques.

It is important to stress that we have not criticized any of the methods used in our examples; we have neither analysed nor compared the specificity of their measures. The reason why TOPSIS and ELECTRE have both been used with VN normalisation is to emphasize its negative effects on the results of two MADM methods, which are based on completely different logical grounds.

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