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QUALITATIVE INVESTIGATION OF A MODEL OF ECONOMIC GROWTH*

A.N. GOLOVTSOV

Faculty of Science
People's Friendship University of Russia, Moscow
Russia

Abstract: This work addresses the mathematical aspects of a model of economic growth. The system of general differential equations that describes the double-sectional model of economic growth with interindustrial linkages was qualitatively investigated.

By using averaging over fast oscillations a system of nonlinear differential equations was obtained. Averaging was justified by the interpretation of N.N. Bogolubov's first fundamental theorem. For this system it has been possible to construct a phase pattern.

The modes of balanced growth were also studied.

Keywords: Economic growth, qualitative investigation, averaging, phase pattern.

1. INTRODUCTION

Models of economic growth comprise the main part of economic growth theory that is one of the most important sections of macroeconomics.

Methods developed by this theory make the transition possible to a growing economy that has more ability to satisfy the new public's requirements, the settle domestic and foreign socioeconomic problems. The application of mathematics to the issue of economic growth is rather perspective for its study. The main difficulty is the practical lack of necessary analytical investigations even for the simplest models where the controlling mechanisms of interindustrial proportions are explicitly depicted.

That is why every method that makes such research possible is interesting.

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2. DESCRIPTION OF THE PROBLEM

In this paper Petrov - Pospelov's model of economic growth [5] is considered. A model of this type with one-sided interindustrial linkages was studied by Prof. A.A. Shananin in [7].

The object of the present paper is to qualitatively investigate a model of economic growth with two-sided interindustrial linkages and as it will be shown below, this fact is rather significant.

The model of economic growth that is under investigation can be described by the system of nonlinear differential equations:

$$\frac{dM_1}{dt} = \frac{\Phi + sx_2 M_2 + p_1 a_1 M_2 - p_2 a_2 M_1}{p_1 b_1} - \mu M_1;$$
$$\frac{dM_2}{dt} = \frac{p_2 a_2 M_1 - p_1 a_1 M_2 - sx_2 M_2}{p_1 b_2} - \mu M_2;$$
$$\frac{dQ_1}{dt} = M_1 - \frac{\Phi + sx_1 M_1 + sx_2 M_2 + p_1 a_1 M_2}{p_1};$$
$$\frac{dQ_2}{dt} = M_2 - a_2 M_1;$$
$$\frac{d^2 p_1}{dt^2} = -\alpha_1 \frac{1}{M_1} \frac{dQ_1}{dt} p_1;$$
$$\frac{dp_2}{dt} = -\alpha_2 \frac{1}{M_2} \frac{dQ_2}{dt} p_2;$$
$$\frac{d\Phi}{dt} = \frac{\tilde{k}}{p_1} \Phi.$$

Here we assume the following:

1) Both industries are working at full capacities M_1 and M_2 respectively.

2) The rate of wages s and the mean expenditures of the labour resources for both industries of economic products x_1 and x_2, respectively, remains constant.

3) Φ is the total sum of money that both industries spend on investments.

4) $p_i(t)$ is the uniform price of the i - th $(i = 1,2)$ industrial product at each moment of time t. The working people spend the whole salary $(sx_1 M_1 + sx_2 M_2)$ for the consumption without delay.

5) The first industry produces a product that is used for investments, consumption and also for the manufacture of the second industry's product in the constant proportion a_1 units of first industry product to each unit of second industry product.

6) The second industry produces a product that is used to manufacture the first industry's product in the constant proportion a_2 units of the second industry's product to each unit of the first industry's product.

7) We consider an economy that consists of two industries. Each industry produces homogeneous products. The only kind of primary resources that are used for production are labour resources also homogeneous.

8) The current profit of the first industry is $(\Phi + sx_2 M_2 + p_1 a_1 M_2 - p_2 a_2 M_1)$, and of the second $(p_2 a_2 M_1 - p_1 a_1 M_2 - sx_2 M_2)$. Therefore the total current profit of both industries is Φ.

9) The sum of money invested in each industry is equal to its current profit.

10) The sum of money that is spent for investments changes as follows: the increase in expenses is in proportion to the cost of reinvestments in constant prices.

11) The change of the industries' product reserves Q_1 and Q_2 is determined as the difference between supply and demand.

Furthermore in the system: b_i $(i = 1,2)$ is the rate of capital increase of the i-th industry; μ is the capacity decrease rate; p_i $(i = 1,2)$ is the product price of the i-th industry; $\alpha_i > 0$ $(i = 1,2)$ is a fixed coefficient.

Let us substitute values $\frac{dQ_1}{dt}$ and $\frac{dQ_2}{dt}$ into the price change expressions and then we change the variables for the convenience of analytic investigations:

$$m = \frac{M_2}{a_2 M_1}; \quad Z = \frac{\Phi}{s(x_1 + a_2 x_2) M_1}; \quad P_1 = \frac{p_1}{s(x_1 + a_2 x_2)}; \quad P_2 = \frac{p_2 a_2}{s(x_1 + a_2 x_2)};$$

$$q = \frac{a_2 x_2}{(x_1 + a_2 x_2)}; \quad B = \frac{b_2 a_2}{b_1}; \quad a = a_1 a_2 .$$

Then the system of differential equations that describes our model takes the following form:

$$\frac{dM_1}{dt} = \left(\frac{Z + qm - P_2 + mP_1 a}{P_1 b_1} - \mu \right) M_1; \tag{2.1}$$

$$\frac{dm}{dt} = \frac{1}{b_1} \left[\frac{P_2 - qm - maP_1}{P_1 B} - m \frac{Z + qm - P_2 + mP_1 a}{P_1} \right]; \tag{2.2}$$

$$\frac{d^2 P_1}{dt^2} = -\alpha_1 \left[P_1 (1 - am) - 1 - Z - q(m-1) \right]; \tag{2.3}$$

$$\frac{dP_2}{dt} = -\frac{\alpha_2 s(x_1 + a_2 x_2)}{a_2} \left[1 - \frac{1}{m} \right] P_2; \tag{2.4}$$

$$\frac{dZ}{dt} = \frac{\tilde{k} Z}{P_1 s(x_1 + a_2 x_2)} - Z \left(\frac{Z + qm - P_2 + aP_1 m}{P_1 b_1} - \mu \right). \tag{2.5}$$

The main task is to carry out the qualitative investigation of solving system (2.1) – (2.5).

For this the method of averaging [1] will be widely used.

Averaging is justified by the interpretation of N.N. Bogolubov's first fundamental theorem [4].

Remark. The use of variables Z and m instead of Φ and M_2 leads to a decomposition of the system of differential equations that makes its investigation easier.

Remark. The model under consideration has the modes of balanced growth under the fulfillment of productivity condition: $1-\mu(b_1a_1+b_2a_2)>0$.

The system of differential equations (2.1) – (2.5) has partial solutions as follows:

$$m(t)=1;\ M_1(t)=M_0\exp[t\gamma/(b_1a_1+b_2a_2)];$$
$$P_1(t)=P_0>0,\ P_2(t)=R_0>0,\ Z(t)=Z_0>0,$$

where $P_0=\dfrac{1+k_0}{1-a-\mu_0}$; $Z_0=\dfrac{k_0(1-a)+\mu_0}{1-a-\mu_0}$; $R_0=\dfrac{Z_0}{2}+aP_0+q$; $k_0=\dfrac{k(b_2a_2+b_1)}{s(x_1+a_2x_2)}$; $\mu_0=\mu(b_2a_2+b_1)$.

Hence the double-sectional model's mode of balanced growth converts to a one-sectional model's mode of exponential growth [7].

3. SYSTEM REARRANGEMENT AND AVERAGING

Let us average system (2.1) – (2.5) from Section 2 for its further investigation.

The averaging will be carried out in two steps:

a) Carry out Van der Pol's substitution in order to put the system (2.1) – (2.5) in a form convenient for averaging:

$$P_1=\frac{1}{(1-am)}(1+Z+q(m-1))+A_1\cos(\varphi_1); \tag{3.1}$$

$$\frac{dP_1}{dt}=-\frac{1}{b_1}\sqrt{\alpha_1 b_1^2}\,A_1\sin(\varphi_1). \tag{3.2}$$

Notation: $K=\dfrac{1}{(1-am)}(1+Z+q(m-1))$; $K_1=\left[\left(\dfrac{1+Z+q(m-1)}{(1-am)}\right)^2-A_1^2\right]^{\frac{1}{2}}$.

Then the system (2.1) – (2.5) takes the form (3.3) – (3.8):

$$\frac{dM_1}{dt}=\left(\frac{Z+qm-P_2}{Kb_1}+\frac{ma}{b_1}-\mu\right)M_1; \tag{3.3}$$

$$\frac{dm}{dt}=\frac{1}{b_1}\left[\frac{P_2-qm}{KB}-\frac{ma}{B}-m\left(\frac{Z+qm-P_2}{K}+ma\right)\right]; \tag{3.4}$$

$$\frac{dZ}{dt}=\frac{\tilde{k}Z}{Ks(x_1+ax_2)}-Z\left[\frac{Z+qm-P_2}{Kb_1}+\frac{am}{b_1}-\mu\right]; \tag{3.5}$$

$$\frac{dP_2}{dt}=-\frac{\alpha_2 s(x_1+a_2x_2)}{a_2}\left[1-\frac{1}{m}\right]P_2; \tag{3.6}$$

$$\frac{dA_1}{dt}=-\frac{\sqrt{\alpha_1}}{2}amA_1\sin(2\varphi_1)-\frac{1}{(1-am)}\frac{dZ}{dt}\cos(\varphi_1)-$$
$$-\left(\frac{q}{(1-am)}+\frac{a(1+Z+q(m-1))}{(1-am)^2}\right)\frac{dm}{dt}\cos(\varphi_1); \tag{3.7}$$

$$\frac{d\varphi_1}{dt}=\sqrt{\alpha_1}-\sqrt{\alpha_1}am\cos^2(\varphi_1)+\frac{1}{(1-am)A_1}\frac{dZ}{dt}\sin(\varphi_1)+$$
$$+\frac{1}{A_1}\left(\frac{q}{(1-am)}+\frac{a(1+Z+q(m-1))}{(1-am)^2}\right)\frac{dm}{dt}\sin(\varphi_1). \tag{3.8}$$

Let us assume because P_1 should be positive that

$$\alpha_1 b_1^2>>1, \tag{3.9}$$

$$0\le A_1<\frac{1}{(1-am)}(1+Z+q(m-1))\ . \tag{3.10}$$

From (3.9) we have that variable φ_1 changes much faster than M_1, m, Z, A_1.

Let us average system (3.3) – (3.8) over φ_1 according to the method described in [1]. Then (3.3) - (3.8) takes the following form:

$$\frac{dM_1}{dt}=\left(\frac{Z+qm-P_2}{K_1b_1}\right)M_1; \tag{3.11}$$

$$\frac{dm}{dt}=\frac{1}{b_1}\left[\frac{P_2-qm}{K_1B}-m\left(\frac{Z+qm-P_2}{K_1}\right)\right]; \tag{3.12}$$

$$\frac{dZ}{dt}=\frac{\tilde{k}Z}{K_1s(x_1+ax_2)}-Z\left[\frac{Z+qm-P_2}{K_1b_1}\right]; \tag{3.13}$$

$$\frac{dP_2}{dt}=-\frac{\alpha_2 s(x_1+a_2x_2)}{a_2}\left[1-\frac{1}{m}\right]P_2; \tag{3.14}$$

$$\frac{dA_1}{dt}=\left\{\frac{Z(Z+qm-P_2)}{b_1(1-am)}-\frac{\tilde{k}Z}{s(x_1+a_2x_2)(1-am)}\right\}\frac{1}{A_1}\left[1-\frac{(1+Z+q(m-1))}{(1-am)K_1}\right]-$$
$$-\frac{1}{b_1}\left(\frac{q}{(1-am)}+\frac{a(1+Z+q(m-1))}{(1-am)^2}\right)\left(\frac{P_2-qm}{B}-m(Z+qm-P_2)\right)\times$$
$$\times\frac{1}{A_1}\left[1-\frac{(1+Z+q(m-1))}{(1-am)K_1}\right]; \tag{3.15}$$
$$\frac{d\varphi_1}{dt}=-\sqrt{\alpha_1}\left[1+\frac{am}{2}\right]. \tag{3.16}$$

Let us consider Cauchy's problem for the systems (3.3) – (3.8) and (3.11) – (3.16) with identical initial conditions: $M_1(0)=M_1^0>0,\ m(0)=m^0>0,\ Z(0)=Z^0>0,$

$$0\le A_1(0)=A_1^0<\frac{1}{(1-am)}(1+Z^0+q(m^0-1)), \tag{3.17}$$
$$\varphi_1(0)=\varphi_1^0\ . \tag{3.18}$$

The notation $M_1(t,\alpha_1),\ m(t,\alpha_1),\ Z(t,\alpha_1),\ A_1(t,\alpha_1),\ \varphi_1(t,\alpha_1),$ denotes the solution of Cauchy's problem for (3.3) – (3.8) and $M_1'(t)$, $m'(t)$, $Z'(t)$, $A_1'(t)$ – the solution of Cauchy's problem for (3.11) – (3.16). Then according to the interpretation of N.N. Bogolubov's theorem [1] we have that for $\forall T>0\ \exists$ constant $C(T)$:

$$\max[|\ M_1(t,\alpha_1)-M_1'(t)\,|+|\ m(t,\alpha_1)-m'(t)\,|+|\ Z(t,\alpha_1)-Z'(t)\,|+|\ A_1(t,\alpha_1)-A_1'(t)\,|]\le\frac{C(T)}{\sqrt{\alpha_1}},$$

where $t\in[0,T]$.

In other words, the solution of Cauchy's problem for averaged system (3.11) – (3.16) differs little from the solution of the corresponding Cauchy's problem for initial system (3.3) – (3.8) under assumptions that were made before.

System (3.12) – (3.15) has the following state of equilibrium:

$$m=1;\ \ Z=Z_0=\frac{\tilde{k}b_1}{s(x_1+a_2x_2)}(1+B);\ \ P_2=\frac{B}{1+B}Z_0+q;\ A_1=0$$

Remark. $A_1=0$ because the right part of (3.15) can be continuously determined by zero.

b) In the system (3.11)-(3.15) change from variables m and P_2 to a new pair of variables y and P_2, where $y=\frac{dP_2}{dt}$. Then system (3.11) – (3.15) takes the form (3.19) – (3.23) where $I=s(x_1+a_2x_2)$.

Now carry out Van der Pol's substitution in the system (3.19) – (3.23):

$$P_2 = \frac{B}{1+B} Z + q + A_2 \cos(\varphi_2) \,, \tag{3.24}$$

$$\frac{dP_2}{dt} = y = \sqrt{\alpha_2} A_2 \sin(\varphi_2) \,. \tag{3.25}$$

Then according to [4] we have that system (3.19) – (3.23) takes the form (3.26) – (3.30) that is convenient for averaging over fast variable φ_2.

$$\frac{dM_1}{dt} = \left(Z + \frac{q l \alpha_2 P_2}{\alpha_2 l P_2 + a_2 y} - P_2 \right) \frac{M_1}{K_1 b_1}; \tag{3.19}$$

$$\frac{dy}{dt} = \frac{\alpha_2 l}{a_2} \left[-y + \frac{\alpha_2 l P_2 + a_2 y}{\alpha_2 l P_2} \left(\frac{q}{b_1 K_1 B} + \frac{Z - P_2}{b_1 K_1} + y \right) - \frac{(l \alpha_2 P_2 + a_2 y)^2}{P_2 \alpha_2^2 l^2 b_1 K_1 B} + \frac{q}{b_1 K_1} \right]; \tag{3.20}$$

$$\frac{dZ}{dt} = \frac{\tilde{k} Z}{K_1 l} - Z \left[\frac{Z - P_2}{K_1 b_1} + \frac{q \alpha_2 l P_2}{K_1 b_1 (\alpha_2 P_2 l + a_2 y)} \right]; \tag{3.21}$$

$$\frac{dA_1}{dt} = \frac{\alpha_2 P_2 l + a_2 y}{(1-a)\alpha_2 l P_2 + a_2 y} \left\{ \frac{Z(Z + q\alpha_2 l P_2 (\alpha_2 l P_2 + a_2 y)^{-1} - P_2)}{b_1} - \frac{\tilde{k} Z}{s(x_1 + a_2 x_2)} - \right.$$

$$- \frac{1}{b_1} \left(\frac{P_2(\alpha_2 l P_2 + a_2 y) - q \alpha_2 l P_2 - B \alpha_2 l P_2 (Z + q \alpha_2 P_2 l (\alpha_2 l P_2 + a_2 y)^{-1} - P_2)}{B(\alpha_2 l P_2 + a_2 y)} \right) \times$$

$$\left. \times \left(q + \frac{a((1+Z)(\alpha_2 l P_2 + a_2 y) - q y a_2)}{(1-a)\alpha_2 l P_2 + a_2 y} \right) \right\} \frac{1}{A_1} \left[1 - \frac{((1+Z)(\alpha_2 l P_2 + a_2 y) - q a_2 y)}{((1-a)\alpha_2 l P_2 + a_2 y) K_1} \right]; \tag{3.22}$$

$$\frac{dP_2}{dt} = y. \tag{3.23}$$

After substitution (3.24), (3.25) we have (3.26) – (3.30):

$$\frac{dM_1}{dt} = \frac{(Z(1+B)^{-1} - A_2 \cos(\varphi_2))}{K_1 b_1} M_1; \tag{3.26}$$

$$\frac{dZ}{dt} = \frac{\tilde{k} Z}{K_1 l} - Z \left[\frac{Z(1+B)^{-1} - A_2 \cos(\varphi_2)}{K_1 b_1} \right]; \tag{3.27}$$

$$\frac{dA_1}{dt} = \frac{1}{A_1} \left(1 - \frac{(1+Z)}{(1-a) K_1} \right) \left[\frac{Z(Z(1+B)^{-1} - A_2 \cos(\varphi_2))}{b_1 (1-a)} - \frac{\tilde{k} Z}{s(x_1 + a_2 x_2)(1-a)} - \right.$$

$$\left. - \frac{1}{b_1} \left(\frac{q}{(1-a)} + \frac{a(1+Z)}{(1-a)^2} \right) \left(\frac{BZ(1+B)^{-1} + A_2 \cos(\varphi_2)}{B} - \frac{Z}{1+B} + A_2 \cos(\varphi_2) \right) \right] \tag{3.28}$$

$$\frac{dA_2}{dt} = \frac{\sqrt{\alpha_2}}{2} A_2 \sin(2\varphi_2) - \frac{A_2(1+B)}{2B} \frac{\sqrt{\alpha_2} I}{a_2 b_1 K_1} \sin(2\varphi_2) -$$

$$- \frac{B\cos(\varphi_2)}{(1+B)} \left(\frac{\tilde{k}Z}{K_1 I} - Z \left[\frac{Z(1+B)^{-1} - A_2\cos(\varphi_2)}{K_1 b_1} \right] \right); \tag{3.29}$$

$$\frac{d\varphi_2}{dt} = -\frac{(1+B)}{B} \frac{\sqrt{\alpha_2} I}{a_2 b_1 K_1} + \frac{dA_2}{dt} \frac{\tan(\varphi_2)}{A_2}. \tag{3.30}$$

Where $I = s(x_1 + a_2 x_2)$; $K_1 = \sqrt{\left[\left(\frac{1+Z}{1-a} \right)^2 - A_1^2 \right]}$.

Average system (3.26) – (3.30) over φ_2. Then the system (3.26) – (3.30) takes the form (3.31) – (3.34):

$$\frac{dM_1}{dt} = \frac{ZM_1}{K_1 b_1 (1+B)}; \tag{3.31}$$

$$\frac{dZ}{dt} = \left(\frac{\tilde{k}}{s(x_1 + a_2 x_2)} - \frac{Z}{b_1(1+B)} \right) \frac{Z}{K_1}; \tag{3.32}$$

$$\frac{dA_1}{dt} = \frac{1}{A_1} \left(1 - \frac{(1+Z)}{(1-a) K_1} \right) \left(\frac{Z}{b_1(1+B)} - \frac{\tilde{k}}{s(x_1 + a_2 x_2)} \right) \frac{Z}{(1-a)}; \tag{3.33}$$

$$\frac{dA_2}{dt} = -\frac{ZA_2 B}{2(1+B) K_1 b_1}. \tag{3.34}$$

In a similar way as in a) we concluded that the solution of Cauchy's problem for averaged system (3.31) – (3.34) differs little from the solution of the corresponding Cauchy's problem for initial system (3.26) – (3.30).

Taking into account the substitutions (3.1), (3.2), (3.24), (3.25) we have that system (2.1) – (2.5) with variables M_1, Z, A_1, A_2, φ_1, φ_2 takes the form (I).

In (I):

$$\frac{dm}{dt} = \frac{1}{b_1} \left\{ \frac{[B(1+B)^{-1} Z + q + A_2 \cos(\varphi_2)] - qm}{KB} - \frac{ma}{B} - \right.$$

$$\left. - m \left(\frac{Z + qm - \left[B(1+B)^{-1} Z + q + A_2 \cos(\varphi_2) \right]}{K} + ma \right) \right\}$$

$$m = \frac{\alpha_2 I [B(1+B)^{-1} Z + q + A_2 \cos(\varphi_2)]}{[B(1+B)^{-1} Z + q + A_2 \cos(\varphi_2)] \alpha_2 I + \sqrt{\alpha_2} a_2 A_2 \sin(\varphi_2)}\ ; \ I = s(x_1 + a_2 x_2)\ .$$

So from the conclusions given before and from N.N. Bogolubov's first fundamental theorem we obtain the following result:

Theorem. Assume that $\tilde{M}_1(t,\alpha_1,\alpha_2)$, $\tilde{Z}(t,\alpha_1,\alpha_2)$, $\tilde{A}_1(t,\alpha_1,\alpha_2)$, $\tilde{A}_2(t,\alpha_1,\alpha_2)$ and $M_1(t)$, $Z(t)$, $A_1(t)$, $A_2(t)$ are the solutions of system (I) and averaged system (3.31) – (3.34) correspondingly with identical conditions in initial instant of time $t=0$.

Then it is possible to correlate any as small as wished ρ,η and as big as wished $L>0$ with ε_0: if $\left(\frac{1}{\sqrt{\alpha_1}},\frac{1}{\sqrt{\alpha_2}}\right)<\varepsilon_0$ then in the interval $0<t<L/\varepsilon$, $0<\varepsilon<\varepsilon_0$ the following inequalities are valid:

$$\left|\tilde{M}_1(t,\alpha_1,\alpha_2)-M_1(t)\right|<\eta, \qquad \left|\tilde{Z}(t,\alpha_1,\alpha_2)-Z(t)\right|<\eta,$$

$$\left|\tilde{A}_1(t,\alpha_1,\alpha_2)-A_1(t)\right|<\eta, \qquad \left|\tilde{A}_2(t,\alpha_1,\alpha_2)-A_2(t)\right|<\eta.$$

System (I):

$$(\mathrm{I})\ \begin{cases}
\dfrac{dM_1}{dt}=\left(\dfrac{Z+qm-\left[B(1+B)^{-1}Z+q+A_2\cos(\varphi_2)\right]}{Kb_1}+\dfrac{ma}{b_1}-\mu\right)M_1; \\
\dfrac{dZ}{dt}=\dfrac{\tilde{k}Z}{Kl}-Z\left[\dfrac{Z+qm-\left[B(1+B)^{-1}Z+q+A_2\cos(\varphi_2)\right]}{Kb_1}+\dfrac{am}{b_1}-\mu\right]; \\
\dfrac{dA_1}{dt}=-\dfrac{\sqrt{\alpha_1}}{2}amA_1\sin(2\varphi_1)-\dfrac{1}{(1-am)}\dfrac{dZ}{dt}\cos(\varphi_1)- \\
\qquad -\left(\dfrac{q}{(1-am)}+\dfrac{a(1+Z+q(m-1))}{(1-am)^2}\right)\dfrac{dm}{dt}\cos(\varphi_1); \\
\dfrac{dA_2}{dt}=\left(\sqrt{\alpha_2}A_2\sin(\varphi_2)-B(1+B)^{-1}\dfrac{dZ}{dt}\right)\cos(\varphi_2)-\dfrac{\sqrt{\alpha_2}l}{a_2}\left(\left[1-\dfrac{1}{m}\right]\sqrt{\alpha_2}A_2\sin(\varphi_2)+\right. \\
\qquad \left.+\left[B(1+B)^{-1}Z+q+A_2\cos(\varphi_2)\right]\dfrac{1}{m^2}\dfrac{dm}{dt}\right)\sin(\varphi_2); \\
\dfrac{d\varphi_1}{dt}=\sqrt{\alpha_1}-\sqrt{\alpha_1}am\cos^2(\varphi_1)+\dfrac{1}{(1-am)A_1}\dfrac{dZ}{dt}\sin(\varphi_1)+ \\
\qquad +\dfrac{1}{A_1}\left(\dfrac{q}{(1-am)}+\dfrac{a(1+Z+q(m-1))}{(1-am)^2}\right)\dfrac{dm}{dt}\sin(\varphi_1); \\
\dfrac{d\varphi_2}{dt}=\dfrac{1}{A_2}\left\{-\dfrac{\sqrt{\alpha_2}l}{a_2}\left(\left(1-\dfrac{1}{m}\right)\sqrt{\alpha_2}A_2\sin(\varphi_2)+\left[B(1+B)^{-1}Z+q+A_2\cos(\varphi_2)\right]\times\right.\right. \\
\qquad \left.\left.\times\dfrac{1}{m^2}\dfrac{dm}{dt}\right)\cos(\varphi_2)-\left(\sqrt{\alpha_2}A_2\sin(\varphi_2)-B(1+B)^{-1}\dfrac{dZ}{dt}\right)\sin(\varphi_2)\right\}.
\end{cases}$$

4. INVESTIGATION OF A DOUBLE-SECTIONAL MODEL OF ECONOMIC GROWTH WITH INTERINDUSTRIAL LINKAGES

4.1. Stability investigation

Consider system (3.31) – (3.34) from Section 3.

Theorem. System's (3.32)-(3.34) state of equilibrium $A_1^* = 0,\ A_2^* = 0,\ Z^* = \kappa b_1(1+B)$, where $\kappa = \dfrac{\tilde{k}}{s(x_1 + a_2 x_2)}$ is stable by Lyapunov; the system also has asymptotic stability by variables A_2, Z.

Proof: We consider the subsystem (3.32)-(3.34):

$$\begin{cases} \dfrac{dZ}{dt} = \left(\kappa - \dfrac{Z}{b_1(1+B)}\right) \cdot \dfrac{Z}{K_1}; \\ \dfrac{dA_1}{dt} = \dfrac{1}{A_1}\left(1 - \dfrac{1+Z}{(1-a)K_1}\right) \cdot \left(\dfrac{Z}{b_1(1+B)} - \kappa\right) \cdot \dfrac{Z}{1-a}; \\ \dfrac{dA_2}{dt} = -\dfrac{ZA_2B}{2(1+B)K_1 b_1}. \end{cases}$$

Substitution: $d\tau = \dfrac{dt}{K_1}$. Let us denote $\tilde{A}_1 = A_1^2$. Then:

$$\begin{cases} \dfrac{dZ}{d\tau} = \left(\kappa - \dfrac{Z}{b_1(1+B)}\right) Z; & (4.1) \\ \dfrac{d\tilde{A}_1}{d\tau} = 2\left(K_1 - \dfrac{1+Z}{1-a}\right) \cdot \left(\dfrac{Z}{b_1(1+B)} - \kappa\right) \cdot \dfrac{Z}{1-a}; & (4.2) \\ \dfrac{dA_2}{d\tau} = -\dfrac{ZA_2B}{2(1+B)b_1}. & (4.3) \end{cases}$$

where $K_1 = \left[\left(\dfrac{1+Z}{1-a}\right)^2 - \tilde{A}_1\right]^{\frac{1}{2}}$.

Remark. Substitution $d\tau = \dfrac{dt}{K_1}$ leads to the decomposition of the system (3.32)-(3.34) that is very important because we have the opportunity to investigate the equilibrium position's stability for the final system (4.1) – (4.3) through an investigation of the stability of the corresponding equilibrium positions of its subsystems (4.1), (4.3) and (4.1), (4.2).

System (4.1) – (4.3) has the following equilibrium position:

$$\tilde{A}_1^* = 0, \quad A_2^* = 0, \quad Z^* = \kappa b_1(1+B) . \tag{4.4}$$

a) Consider the subsystem (4.1), (4.3) of the system (4.1) – (4.3):

$$\begin{cases} \dfrac{dZ}{d\tau} = \left(\kappa - \dfrac{Z}{b_1(1+B)}\right)Z; \\ \dfrac{dA_2}{d\tau} = -\dfrac{ZA_2B}{2(1+B)b_1}. \end{cases}$$

Lemma 1. The subsystem (4.1), (4.3) of the system (4.1) – (4.3) has asymptotically stable equilibrium state:

$$A_2^* = 0, \quad Z^* = \kappa b_1(1+B) . \tag{4.5}$$

Proof: Now we investigate stability using linear approximation.

Let us linearize system (4.1), (4.3) so we have:

$$\begin{cases} \dfrac{dZ}{d\tau} = \kappa Z \\ \dfrac{dA_2}{d\tau} = -\kappa \dfrac{B}{2} A_2 \end{cases}$$

Both roots of the characteristic polynomial are negative: $\lambda_1 = a_{11} = -\kappa < 0$, $\lambda_2 = a_{22} = -\kappa \dfrac{B}{2} < 0$.

Hence the equilibrium state (4.5) is asymptotically stable by Lyapunov.

b) Consider the subsystem (4.1), (4.2) of the system (4.1) – (4.3):

$$\begin{cases} \dfrac{dZ}{d\tau} = \left(\kappa - \dfrac{Z}{b_1(1+B)}\right)Z; \\ \dfrac{d\tilde{A}_1}{d\tau} = 2\left(K_1 - \dfrac{1+Z}{1-a}\right)\cdot\left(\dfrac{Z}{b_1(1+B)} - \kappa\right)\cdot\dfrac{Z}{1-a}. \end{cases}$$

Now let us investigate the stability of equilibrium position:

$$\tilde{A}_1^* = 0, \quad Z^* = \kappa b_1(1+B) . \tag{4.6}$$

Lemma 2. The quantity $\tilde{A}_1$ can be clearly represented by Z in the subsystem (4.1), (4.2) of system (4.1) – (4.3):

$$\tilde{A}_1(Z) = 2C_1^*\left(\frac{1+Z}{1-a}\right) - C_1^{*2}\text{, where } C_1^* = C_1^*(Z(t_0), A_1(t_0)).$$

Proof: From (4.2): $\dfrac{d\tilde{A}_1}{d\tau} = 2\left(\sqrt{\left(\dfrac{1+Z}{1-a}\right)^2 - \tilde{A}_1} - \dfrac{1+Z}{1-a}\right)\cdot\left(\dfrac{Z}{b_1(1+B)} - \kappa\right)\cdot\dfrac{Z}{(1-a)}$.

Substitute (4.1):

$$\frac{d\tilde{A}_1}{d\tau} = -2\left(\sqrt{\left(\frac{1+Z}{1-a}\right)^2 - \tilde{A}_1} - \frac{1+Z}{1-a}\right)\cdot\frac{dZ}{d\tau}\cdot\frac{1}{(1-a)}.$$

From this we obtain that $\dfrac{d\tilde{A}_1}{dZ} = -\dfrac{2}{(1-a)}\left(\sqrt{\left(\dfrac{1+Z}{1-a}\right)^2 - \tilde{A}_1} - \dfrac{1+Z}{1-a}\right)$.

Now we make the substitution:

$$\tilde{Z} = \frac{1+Z}{1-a}. \tag{4.7}$$

So we have: $\dfrac{d\tilde{A}_1}{d\tilde{Z}} = -2\left(\sqrt{\tilde{Z}^2 - \tilde{A}_1} - \tilde{Z}\right) = 2\left(1 - \sqrt{1 - \dfrac{\tilde{A}_1}{\tilde{Z}^2}}\right)\tilde{Z}$.

Let us make the substitution:

$$\hat{Z} = \tilde{Z}^2. \tag{4.8}$$

So we have: $\dfrac{d\tilde{A}_1}{d\hat{Z}} = \left(1 - \sqrt{1 - \dfrac{\tilde{A}_1}{\hat{Z}}}\right)$.

Here we substitute:

$$\tilde{A}_1 = X\hat{Z}. \tag{4.9}$$

Then $\dfrac{d\tilde{A}_1}{d\hat{Z}} = \dfrac{dx}{d\hat{Z}}\hat{Z} + X = (1 - \sqrt{1-X})$. From this we have that $\dfrac{dZ}{d\hat{Z}} = \dfrac{dX}{(1-X) - \sqrt{1-X}}$.

And finally we substitute:

$$t = \sqrt{1-X}. \tag{4.10}$$

Then $\dfrac{dZ}{d\hat{Z}} = -2\dfrac{d(1-t)}{(1-t)}$. From this $\ln|C\hat{Z}(1-t^2)| = 0$, where $C = \text{const}$. Then $\hat{Z} = \dfrac{1}{C(1-t)^2}$.

Now taking into account the substitutions (4.7)-(4.10) we have:

$$\tilde{A}_1 = 2C_1 \frac{1+Z}{1-a} - C_1^{\,2}\text{, where } C_1 = \text{const}. \qquad (4.11)$$

Consider Cauchy's problem for the system (4.1), (4.2) with initial conditions $\tilde{A}(t_0)$, $Z(t_0)$.

Now we identify C_1 depending on the initial conditions.

From (4.11): $\tilde{A}_1(t_0) = 2C_1 \frac{1+Z(t_0)}{1-a} - C_1^{\,2}$ then $C_1^{\,2} - 2C_1 \frac{1+Z(t_0)}{1-a} + \tilde{A}_1(t_0) = 0$ and finally

$$C_1^{*} = \frac{1+Z(t_0)}{1-a} \pm \sqrt{\left(\frac{1+Z(t_0)}{1-a}\right)^2 - \tilde{A}_1(t_0)}\,, \qquad (4.12)$$

because we can always choose $\delta(\varepsilon)$ such that C_1^{*} exists.

Now the substitution of C_1^{*} in (4.11): $\tilde{A}_1(Z) = 2C_1^{*}\left(\frac{1+Z}{1-a}\right) - C_1^{*2}$. The Lemma has been proved.

Consequence. The equilibrium position (4.6) of the system (4.1), (4.2) is stable by Lyapunov. In fact the stability by Lyapunov of equilibrium position (4.6) means that for $\forall \varepsilon > 0, \exists \delta > 0$: for $\forall T > 0$ when $|A(t_0)| < \delta(\varepsilon)$ and $|z_0 - z(t_0)| < \delta(\varepsilon)$ the $|\tilde{A}_1(Z)| < \varepsilon$ is true.

Consider $0 < \tilde{A}_1(Z) < \varepsilon$ then $2C_2^{*}\left(\frac{1+Z}{1-a}\right) + \varepsilon > 0$.

From (4.12): $C_2^{*2} = 2C_2^{*}\left(\frac{1+Z(t_0)}{1-a}\right) - \tilde{A}_1(t_0)$ then $2C_2^{*}(Z(t_0) - Z)\frac{1}{1-a} + \varepsilon - \tilde{A}_1(t_0) > 0$.

In greater detail:

$$2\left(\frac{1+Z(t_0)}{(1-a)^2} \pm \frac{1}{(1-a)^2}\sqrt{(1+Z(t_0))^2 - \tilde{A}_1(t_0)(1-a)}\right)(Z(t_0) - Z) + \varepsilon - \tilde{A}_1(t_0) > 0 \text{ then}$$

$$2(1+Z(t_0) \pm \sqrt{(1+Z(t_0))^2 - \tilde{A}_1(t_0)(1-a)})(Z(t_0) - Z) + (\varepsilon - \tilde{A}_1(t_0))(1-a)^2 > 0.$$

Consider the vicinity of equilibrium position: $Z = Z_0 + h$. We have the following because of the asymptotic stability of equilibrium state (4.5) of the system (4.1), (4.3). Z approaches Z_0 when t tends to infinity if and only if h approaches zero under the same condition. Then

$$2(1+Z(t_0)\pm\sqrt{(1+Z(t_0))^2-\tilde{A}_1(t_0)(1-a)})(Z(t_0)-Z_0-h)+(\varepsilon-\tilde{A}_1(t_0))(1-a)^2>0\,, \quad (4.13)$$

$|Z(t_0)-Z_0|<\delta(\varepsilon)$.

Hence $\exists\ \delta(\varepsilon)$: h approaches zero when t tends to infinity and (4.13) assert.

So the equilibrium state (4.6) of the system (4.1), (4.2) is not asymptotically stable by Lyapunov.

So from Lemma 1, Lemma 2 and its implication we have the theorem's statement. The theorem has been proved.

According to Lemma 1 the subsystem (4.1), (4.3) of system (4.1) – (4.3) has asymptotically stable equilibrium state (4.5).

The further investigation of subsystem (4.1), (4.3) is rather interesting.

For this we are going to use methods of the qualitative theory of general differential equations [2].

4.2. The phase pattern

Consider the subsystem (4.1), (4.3) of the system (4.1) – (4.3):

$$\begin{cases} \dfrac{dZ}{d\tau}=\left(\kappa-\dfrac{Z}{b_1(1+B)}\right)Z; \\ \dfrac{dA_2}{d\tau}=-\dfrac{ZA_2B}{2(1+B)b_1}. \end{cases}$$

Let us introduce a new variable Y $\left(0\le Y\le\dfrac{\pi}{2}\right)$ instead of A_2 according to the formula: $A_2=\left(\dfrac{B}{1+B}Z+q\right)\sin(Y)$. So we have:

$$\begin{cases} \dfrac{dZ}{d\tau}=\left(\kappa-\dfrac{Z}{b_1(1+B)}\right)Z; & (4.14) \\ \dfrac{dY}{d\tau}=-((B^2-2)Z+(1+B)(Bq+2\kappa b_1))\dfrac{Z}{2b_1(1+B)^2\left(\dfrac{B}{1+B}Z+q\right)}\mathrm{tg}(Y). & (4.15) \end{cases}$$

Now let us investigate the qualitative (topological) structure of trajectory partitioning using the elements of Poincare-Bendixon's theory [2].

Trajectory partitioning will be carried out in the domain $U = \left\{ Z \in [0,\infty), Y \in [0,\frac{\pi}{2}) \right\}$ because only U has economic sense. It will be useful to change variables in the system by introducing a new independent variable τ_1 [2]:

$$d\tau_1 = \frac{Z d\tau}{\left(\frac{B}{1+B} Z + q\right)(1+B)\cos(Y)\cos\left(\frac{Y}{2}\right)} \quad \text{then}$$

$$\begin{cases} \dfrac{dZ}{d\tau_1} = (\kappa b_1(1+B) - Z)\left(\dfrac{B}{1+B} Z + q\right)\cos(Y)\cos\left(\dfrac{Y}{2}\right) & (4.16) \\ \dfrac{dY}{d\tau_1} = -((B^2-2)Z + \Theta)\dfrac{1}{2(1+B)}\sin(Y)\cos\left(\dfrac{Y}{2}\right) & (4.17) \end{cases}$$

where $\Theta = (1+B)(Bq + 2\kappa b_1)$.

According to Poincare-Bendixon's theory for the construction of trajectory partitioning schemes [2] of the system (4.16), (4.17) we need to:

1) Investigate the properties of its equilibrium position;

2) Study the trajectory's behaviour while it approaches infinity;

3) Establish the absence of limiting cycles;

4) Find out from an analysis of principal isoclines the trajectories' behaviour while $\tau_1 \to +\infty$.

Remark. In the domain U the system has only equilibrium states that correspond to the one-sectional model's modes of exponential growth [7].

Recall that in our designations $B = \frac{b_2 a_2}{b_1}$ where b_1, b_2 - are the rates of capital increase.

From equation (4.17) we have that system's solution behaviour and therefore its phase pattern will change in accordance with the initial values of b_1, b_2 and a_2.

Remark. The sign of $B^2 - 2$ in (4.17) is very significant.

Let us consider two cases:

1. $B^2 > 2$.

Then system (4.16), (4.17) has only one equilibrium state:

$$Z_0 = \kappa b_1(1+B), \; Y = 0. \qquad (4.18)$$

The roots of the secular equation of the system (4.16), (4.17) that was linearized in the equilibrium position (4.18) are:

$$\lambda_1 = -(B\kappa b_1 + q) < 0\ ,\ \ \lambda_2 = -((B^2-2)\kappa b_1(1+B)+\Theta)\frac{1}{2(1+B)} < 0$$

thus in (4.18) we have a "stable node".

Let us investigate when $Z \to \infty$. Substitution: $u = \frac{1}{Z}$, $d\tau_1 = u\,d\tau_2$. Then (4.16), (4.17) takes the form:

$$\begin{cases} \dfrac{du}{d\tau_2} = -(u\kappa b_1(1+B)-1)\left(\dfrac{B}{1+B}+qu\right)u\cos(Y)\cos\left(\dfrac{Y}{2}\right) & (4.19)\\ \dfrac{dY}{d\tau_2} = -((B^2-2)+u\Theta)\dfrac{1}{2\ \ (1+B)}\sin(Y)\cos\left(\dfrac{Y}{2}\right) & (4.20)\end{cases}$$

Remark. Tending to infinity by coordinate z in the new variables means that $u \to +0$.

From $\frac{du}{d\tau_2} = 0$ when $u = 0$ we have that segment $\left\{u-0, 0 \le Y \le \frac{\pi}{2}\right\}$ consists of this system's trajectories. Equilibrium state:

$$u = 0\ ,\ Y_\infty = 0 \qquad (4.21)$$

The roots of the secular equation of system (4.19), (4.20) that was linearized in (4.21) are:

$$\lambda_1 = \frac{B}{1+B} > 0\ ,\ \lambda_2 = \frac{2-B^2}{2(1+B)} < 0\,. \qquad (4.22)$$

That means that (4.21) is an "unstable saddle".

Then for the first case (look at Fig.1);

$\frac{dZ}{d\tau_1} = 0$ on the straight line $Y = \frac{\pi}{2}$ (principal isocline) $\frac{dY}{d\tau_1} < 0$ for $\forall Y \in \left(0,\ \frac{\pi}{2}\right]$.

Hence the system trajectories and straight line $Y = \frac{\pi}{2}$ intersect at a right angle within the domain

$\frac{dZ}{d\tau_1} > 0$ when $Z < Z_0 = \kappa b_1(1+B)$; $\frac{dZ}{d\tau_1} < 0$ when $Z > Z_0$; $\frac{dZ}{d\tau_1} = 0$ when $Z = Z_0$.

2. $B^2 < 2$.

Then system (4.16), (4.17) in addition to (4.18) has one more equilibrium position:

$$Z_1 = \frac{\Theta}{2 - B^2}, \ Y = \frac{\pi}{2}. \tag{4.23}$$

The roots of the secular equation of the system (4.16), (4.17) that was linearized in the equilibrium position (4.23) are:

$$\lambda^2 = \frac{\sqrt{2}}{2}\left(\kappa b_1(1+B) - \frac{\Theta}{2-B^2}\right)\left(\frac{B}{1+B}\frac{\Theta}{2-B^2} + q\right)\frac{\sqrt{2}}{2}\cdot\frac{B^2-2}{2\ (1+B)} = \xi^2 > 0 \ \text{ because } \ Z_1 > Z_0.$$

Hence $\lambda_{1,\,2} = \pm\xi$ where $\xi > 0$. So in (4.23) we have an "unstable saddle".

From the first case in (4.18) we have as before a "stable node".

For (4.21) (in infinity) (4.22) looks like this: $\lambda_1 = \dfrac{B}{1+B} > 0$, $\lambda_2 = \dfrac{2-B^2}{2\ (1+B)} > 0$ - "unstable node".

In the domain D_1: $Z > Z_1$ and $Y \in \left[0,\ \dfrac{\pi}{2}\right]$, $\dfrac{dz}{d\tau_1} < 0$, $\dfrac{dY}{d\tau_1} > 0$.

On the straight line $Z = Z_1$: $\dfrac{dY}{d\tau_1} = 0$ and trajectories intersect the straight line at a right angle.

In the domain D_2: $Z_0 < Z < Z_1 \quad Y \in \left[0,\ \dfrac{\pi}{2}\right]$, $\dfrac{dZ}{d\tau_1} < 0$, $\dfrac{dY}{d\tau_1} < 0$.

In the domain D_3: $0 < Z < Z_0 \quad Y \in \left[0,\ \dfrac{\pi}{2}\right]$, $\dfrac{dZ}{d\tau_1} > 0$, $\dfrac{dZ}{d\tau_2} < 0$.

Figure 2 illustrates the phase pattern for the second case.

Let us describe the separatrix.

Now we linearize system (4.16), (4.17) in the equilibrium position (4.23).

The linearized system takes the form:

$$\begin{cases} \dfrac{dZ}{d\tau_1} = a_{12}Y & (4.24) \\ \dfrac{dY}{d\tau_1} = a_{21}Z & (4.25) \end{cases}$$

where $a_{12} = -\dfrac{\sqrt{2}}{2}\left(\kappa b_1(1+B) - \dfrac{\Theta}{2-B^2}\right)\left(\dfrac{B}{1+B}\dfrac{\Theta}{2-B^2} + q\right) > 0$, $a_{21} = -\dfrac{\sqrt{2}}{4}\dfrac{B^2-2}{1+B} > 0$.

From (4.24), (4.25): $\frac{dZ}{dY} = \frac{a_{12}}{a_{21}}\frac{Y}{Z}$. In the vicinity of equilibrium position (4.23) the separatrix takes the form of the straight line $Z = \sigma Y$. Then $\sigma = \frac{Z}{Y}$.

From the other hand, $\frac{dZ}{dY} = \sigma$. Hence $\frac{dZ}{dY} = \frac{Z}{Y} = \sigma$. So we have: $\sigma = \frac{a_{12}}{a_{21}\sigma}$.
From this we have $\sigma^2 = \frac{a_{12}}{a_{21}}$ then $\sigma = \pm\sqrt{\frac{a_{12}}{a_{21}}}$.

Then the separatrix corresponding to $\sigma < 0$ in the vicinity of equilibrium position (4.23) has the tangent: $Z - Z_1 = -\sqrt{\frac{a_{12}}{a_{21}}}\left(Y - \frac{\pi}{2}\right)$ [9].

The separatrix l_1 is determined by the solution of the differential equation from (4.16), (4.17):

$$\frac{dZ}{dY} = \frac{2(\kappa b_1(1+B) - Z)(BZ + q(1+B))}{((B^2-2)Z + \Theta)}\operatorname{ctg}(Y) \text{ where}$$

$$\Theta = (1+B)(Bq + 2\kappa b_1)\,;\ Y = \arcsin\left[A_2\left(\frac{B}{(1+B)}Z + q\right)^{-1}\right].$$

So we examined both cases and finally have the following:

1) Subsystem (4.1), (4.3) of system (4.1) – (4.3) has asymptotically stable equilibrium position $A^*{}_2 = 0,\ Z^* = \kappa b_1(1+B)$ when $B^2 > 2$ and unstable equilibrium position when $t \to \infty$ (saddle).

2) Subsystem (4.1), (4.3) of system (4.1) – (4.3) has asymptotically stable equilibrium position $A^*{}_2 = 0,\ Z^* = \kappa b_1(1+B)$ when $B^2 < 2$ and unstable equilibrium position in $A_2^* = \left(\frac{B}{(1+B)}Z_1^* + q\right)$; $Z_1^* = \frac{\Theta}{(2-B^2)}$ - saddle and "unstable node" when $t \to \infty$.

System trajectories situated lower than separatrix l_1 $(Z < l_1)$ when $\tau_1 \to \infty$ (hence when $t \to \infty$) approach stable equilibrium position (4.18) and higher trajectories are outside the bounds of the domain U .

In the problem under consideration there are no boundary cycles because all equilibrium positions are located on the straight lines $Y = 0$ and $Y = \frac{\pi}{2}$.

Remark. Outside of the domains we discussed the amplitude of price oscillations increases. This leads to a contradiction since prices should be positive.

5. CONCLUSION

In this work a double-sectional model of economic growth with interindustrial linkages was qualitatively investigated. As it was said before, every method that makes it possible to carry out such research is interesting.

This paper presents a method with two-stage averaging that is justified by the interpretation of N.N. Bogolubov's first fundamental theorem [4]. The application of this method to an averaged system leads to its decomposition into two subsystems. One of them has an asymptotically stable by Lyapunov equilibrium position. For this subsystem the phase pattern was constructed and it was established that it changes in accordance with the initial values of the rates of capital increase b_1, b_2 and interindustrial proportion a_2 that shows its importance.

The other interesting fact is that the ratio between the industries' capacities m is equal to 1. This means that capacities tend to the balanced state.

Comparing this paper's results with the results that were obtained in [7] it should be noted that in [7] the averaged system was stable by Lyapunov (not asymptotically) and one-stage averaging was used for its investigation, also the ratio between the industries' capacities was a parameter.

What both models have in common is that corresponding systems of differential equations have the modes of balanced growth that convert to the corresponding one-sectional model's modes of exponential growth.

Hence, the method presented in this work can be rather useful for investigation of the models of economic growth.

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APPENDIX

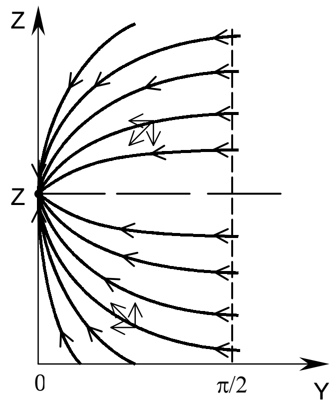

Figure 1: Phase pattern for the case when $B^2 > 2$

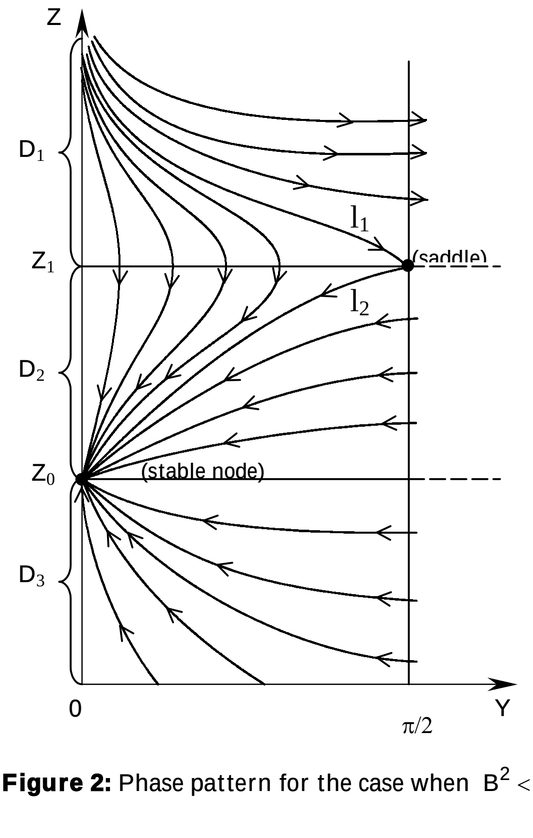

Figure 2: Phase pattern for the case when $B^2 < 2$