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FEJER METHODS FOR SOLVING INFINITE SYSTEMS OF CONVEX INEQUALITIES*

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Abstract: The paper is devoted to the mathematical tools of fejer maps for the construction of iterative procedures, which are applied to solving infinite systems of convex inequalities in $\mathbf{R}^n$.

Keywords: Convex inequalities, fejer maps.

1. INTRODUCTION

The investigations of Motzkin [4] and Agmon [1] were the primary source of the fejer method's approach. In the fundamental research of I.I. Eremin [3], the general theory of fejer maps was developed. The main properties of fejer maps were determined, as well as the properties of the sequences, which are recurrently induced by such mappings. The basic constructions of fejer maps in their application to solving finite systems of convex inequalities and to solving linear or convex programming problems were introduced. The basic constructions were realized in the form of sequential relaxation, suspended relaxation and extreme relaxation.

In the case of a finite system of linear inequalities with the set of solutions $M \neq \varnothing$

$$l_j(\mathbf{x}) := (a_j, \mathbf{x}) - b_j \leq 0, \quad j = 1,\dots,m, \tag{1.1}$$

the realizations are the following. Let

$$\varphi_j(\mathbf{x}) = \mathbf{x} - \lambda_j \frac{l_j^+(\mathbf{x})}{\| a_j \|^2} a_j, \quad \lambda_j \in (0,2), \quad l_j^+(\mathbf{x}) = \max\{0, l_j(\mathbf{x})\} ;$$

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$$d(\mathbf{x}) = \max_{(j)} l_j^+(\mathbf{x}) \ \ (= l_{j_\mathbf{x}}^+(\mathbf{x}))\,.$$

Consider the mappings

$$\varphi^{(1)}(\mathbf{x}) = \varphi_1(\varphi_2(...(\varphi_m(\mathbf{x}))...))\,; \tag{1.2}$$

$$\varphi^{(2)}(\mathbf{x}) = \sum_{j=1}^{m} \alpha_j \varphi_j(\mathbf{x}), \quad \alpha_j > 0, \quad \sum_{j=1}^{m} \alpha_j = 1\,; \tag{1.3}$$

$$\varphi^{(3)}(\mathbf{x}) = \mathbf{x} - \lambda \frac{d(\mathbf{x})}{\| a_{j_\mathbf{x}} \|^2} a_{j_\mathbf{x}}, \quad \lambda \in (0,2)\,. \tag{1.4}$$

Each of these maps is M-fejer and realizes the appropriate base construction. Sequences which are inductively generated by them converge to some solution of the system (1.1) (for arbitrary initial $\mathbf{x}_0 \in \mathbf{R}^n$).

Several implementations of fejer methods based on the constructions of fejer maps (1.2) and (1.3) with application to countable systems of convex inequalities were considered in the papers [3, Theorem 3.2.7, page 114] and [5].

The fejer methods for solving convex inequalities of countable and continuum power are developed and justified in the current investigation. The relaxation of the (1.4) type is taken as the basic construction for the mentioned considerations.

2. FEJER MAPS: MAIN CONCEPTS AND PROPERTIES

The main definitions and properties of fejer mappings and recurrently induced sequences are considered [3]. We affect only those properties which are necessary for the substantiation of convergence of the fejer processes here considered.

Definition 2.1. The mapping $\varphi \in \{\mathbf{R}^n \to \mathbf{R}^n\}$ is called M-fejer, if

$$\varphi(\mathbf{y}) = \mathbf{y}, \ \ \|\varphi(\mathbf{x}) - \mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|, \ \ \forall \mathbf{y} \in M, \ \forall \mathbf{x} \notin M\,.$$

It follows from the definition that such mapping transfers a point which does not lie in set M into another point such that the distance from it to each point of the set M decreases.

Definition 2.2. Multi-valued mapping $\varphi \in \{\mathbf{R}^n \to 2^{\mathbf{R}^n}\}$ is called M-fejer, if

$$\varphi(\mathbf{y}) = \mathbf{y}, \ \ \|\mathbf{z} - \mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|, \ \ \forall \mathbf{y} \in M, \ \forall \mathbf{x} \notin M, \ \forall \mathbf{z} \in \varphi(\mathbf{x})\,.$$

According to the definition, the points of set M and only they are the points of immovability of the map φ.

Denote by $\mathbf{F}_M$ some class of M-fejer mappings (both single and multi-valued).

Property 2.1. If $\mathbf{F}_M \neq \varnothing$, then M is a convex closed set (see, for example, [2, 39.4]).

Definition 2.3. Sequence $\{\mathbf{x}_k\} \subset \mathbf{R}^n$, $\{\mathbf{x}_k\} \cap M = \varnothing$ is called M-fejer, if for any $\mathbf{y} \in M$:

$$\|\mathbf{x}_{k+1} - \mathbf{y}\| < \|\mathbf{x}_k - \mathbf{y}\|, \quad \forall k.$$

Now, let us recall some facts, which are used further on. Their proofs can be found in [3, Chapter II]. Let $\{\mathbf{x}_k\}'$ be a set of limit points for a sequence $\{\mathbf{x}_k\}$.

Lemma 2.1. Let $\varphi \in \mathbf{F}_M$ and the sequence $\{\mathbf{x}_k\}$ be recurrently induced by relation $\mathbf{x}_{k+1} \in \varphi(\mathbf{x}_k)$ with arbitrary initial $\mathbf{x}_0$. If $\{\mathbf{x}_k\} \cap M = \varnothing$, then $\{\mathbf{x}_k\}$ is M-fejer.

Lemma 2.2. If $\mathbf{x}'$, $\mathbf{x}''$ are two different limit points of an M-fejer sequence $\{\mathbf{x}_k\}$, then any point $\mathbf{y} \in M$ (therefore, and set M) lies in a hyperplane being a geometrical place of points, equidistant from $\mathbf{x}'$ and $\mathbf{x}''$.

Proof: From the definition of the M-fejer sequence we get that $\forall \mathbf{y} \in M : \inf_{(k)} \|\mathbf{x}_k - \mathbf{y}\| = \|\mathbf{x}' - \mathbf{y}\| = \|\mathbf{x}'' - \mathbf{y}\|$, i.e., $(\mathbf{x}' - \mathbf{y}, \mathbf{x}' - \mathbf{y}) = (\mathbf{x}'' - \mathbf{y}, \mathbf{x}'' - \mathbf{y})$. The transformation of that equality gives us $(\mathbf{x}'' - \mathbf{x}', \mathbf{y}) = \|\mathbf{x}''\|^2 - \|\mathbf{x}'\|^2$, so every $\mathbf{y} \in M$ belongs to the hyperplane $(\mathbf{x}'' - \mathbf{x}', \mathbf{x}) = \gamma := \|\mathbf{x}''\|^2 - \|\mathbf{x}'\|^2$, which is obtained as a geometrical place of points equidistant from $\mathbf{x}'$ and $\mathbf{x}''$.

Corollary. If $\varphi \in \mathbf{F}_M$ and M is a solid set, then $\{\varphi^k(\mathbf{x}_0)\}_k \to \mathbf{x}' \in \mathbf{R}^n$.

Lemma 2.3. If $\{\mathbf{x}_k\}$ is the M-fejer sequence and $\{\mathbf{x}_k\}' \cap M \neq \varnothing$, then $\{\mathbf{x}_k\} \to \mathbf{x}' \in M$.

Proof: Assume, on the contrary, that there exists a limit point $\mathbf{x}'' \neq \mathbf{x}'$. According to Lemma 2.2. we come to the conclusion, that every $\mathbf{y} \in M$ is equidistant from $\mathbf{x}'$ and $\mathbf{x}''$. The consideration of $\mathbf{y} := \mathbf{x}'$ gives us inconsistency.

Lemma 2.4. The mapping $\mathbf{Pr}_M(\mathbf{x})$, that executes the projecting of a point $\mathbf{x}$ in the convex closed set $M \subset \mathbf{R}^n$, is continuous M-fejer. (The proof can be found in [2, Lemma 40.2].)

Definition 2.4. The mapping $\varphi \in \{\mathbf{R}^n \to 2^{\mathbf{R}^n}\}$ is called closed, if from the fact that $\{\mathbf{x}_k\} \to \mathbf{x}'$, $\{\mathbf{y}_k\} \to \mathbf{y}'$, $\mathbf{y}_k \in \varphi(\mathbf{x}_k)$, $\forall k$, it follows $\mathbf{y}' \in \varphi(\mathbf{x}')$.

Denote by $\overline{\mathbf{F}}_M$ the class of closed M-fejer mappings.

Example of a closed mapping [**3, Chapter II**]. Let $d(\mathbf{x})$ be a convex function and $\{\mathbf{x} | d(\mathbf{x}) \le 0\} = M \ne \varnothing$. An important representative of the class $\overline{\mathbf{F}}_M$ is the mapping φ :

$$\mathbf{x} \xrightarrow{\varphi} \left\{ \mathbf{x} - \lambda \frac{d^+(\mathbf{x})}{\|h\|^2} h \mid h \in \partial d(\mathbf{x}) \right\}, \tag{2.1}$$

where $\lambda \in (0,2)$, and $\partial d(\mathbf{x})$ means a subdifferential of function $d(\mathbf{x})$. If $h = 0$, then $\mathbf{x}$ is a point where the minimum of the function $d(\mathbf{x})$ is reached, so $d^+(\mathbf{x}) = 0$. We assume that $\varphi(\mathbf{x}) = \mathbf{x}$ in the case of $h = 0$.

Note that in the case of the differentiability of $d(\mathbf{x})$ the relation (2.1) can be transformed as follows:

$$\varphi(\mathbf{x}) = \mathbf{x} - \lambda \frac{d^+(\mathbf{x})}{\|\nabla d(\mathbf{x})\|^2} \nabla d(\mathbf{x}) . \tag{2.2}$$

If $d(\mathbf{x})$ is a linear function, i.e., $d(\mathbf{x}) = (a, \mathbf{x}) - \alpha,\ a \ne 0,$ then we have

$$\varphi(\mathbf{x}) = \mathbf{x} - \lambda \frac{[(a, \mathbf{x}) - \alpha]^+}{\|a\|^2} a . \tag{2.3}$$

If M is the set of the solutions of the compatible system (1.1), then the construction (1.4) is a particular case of (2.1).

Lemma 2.5. If a mapping $\varphi \in \mathbf{F}_M$ is closed, then the sequence $\{\mathbf{x}_k\}$, induced recurrently by inclusion $\mathbf{x}_{k+1} \in \varphi(\mathbf{x}_k)$ with arbitrary $\mathbf{x}_0 \in \mathbf{R}^n$, converges to $\mathbf{x}' \in M$.

Proof: If $\{\mathbf{x}_k\} \cap M \ne \varnothing$, then the conclusion follows from Definition 2.2 of an M-fejer map.

In the case of $\{\mathbf{x}_k\} \cap M = \varnothing$, we shall show that the M-fejer sequence $\{\mathbf{x}_k\}$ converges to the element $\mathbf{x}' \in M$. The sequence $\{\mathbf{x}_k\}$ is bounded, and we can allocate the subsequence $\{\mathbf{x}_{k_j}\} \to \mathbf{x}'$ so, that $\{\mathbf{x}_{k_j+1}\} \to \mathbf{x}''$. The points $\mathbf{x}'$, $\mathbf{x}''$ evidently are the limit points of the sequence $\mathbf{x}_k$, so, if $\mathbf{x}' \in M$, then according to Lemma 2.3, we have $\mathbf{x}_k \to \mathbf{x}'$.

If $\mathbf{x}' \notin M$, then the closure of φ gives us $\mathbf{x}'' \in \varphi(\mathbf{x}')$, i.e., $\forall \mathbf{y} \in M$ $|\mathbf{x}'' - \mathbf{y}| < |\mathbf{x}' - \mathbf{y}|$. The last inequality contradicts the fact that the points of M are equidistant from $\mathbf{x}'$, $\mathbf{x}''$ (Lemma 2.2). The proof is complete.

Lemma 2.6. If a mapping $\varphi \in \mathbf{F}_M$ and S is a bounded set, then $\bigcup_{\mathbf{x} \in S} \varphi(\mathbf{x})$ is also a bounded set.

We note, that if $T \subset \mathbf{R}^n$ and $\varphi \in \mathbf{R}^n \to 2^{\mathbf{R}^n}$, then $\varphi(T)$ means $\bigcup_{\mathbf{x} \in T} \varphi(\mathbf{x})$.

Theorem 2.1. If mappings $\varphi_j(\mathbf{x}) \in \bar{\mathbf{F}}_{M_j}$, $j = 1, \ldots, m$, and $\sum_{j=1}^{m} \alpha_j = 1$, $\alpha_j > 0$, then

1) $\sum_{j=1}^{m} \alpha_j \varphi_j(\mathbf{x}) \in \bar{\mathbf{F}}_{\bigcap_{j=1}^{m} M_j}$,

2) $\varphi_1(\ldots(\varphi_m(\mathbf{x})\ldots) \in \bar{\mathbf{F}}_{\bigcap_{j=1}^{m} M_j}$.

Proof: The proof of the fejer property of the constructed mappings is elementary. Let us establish their closure.

1) Denote by $\varphi(\mathbf{x}) = \sum_{j=1}^{m} \alpha_j \varphi_j(\mathbf{x})$. Consider $\{\mathbf{x}_k\} \to \mathbf{x}'$, $\{\mathbf{y}_k\} \to \mathbf{y}'$, $\mathbf{y}_k \in \varphi(\mathbf{x}_k)$, i.e. $\mathbf{y}_k = \sum_{j=1}^{m} \alpha_j \mathbf{y}_k^j$, $\mathbf{y}_k^j \in \varphi_j(\mathbf{x}_k)$. Taking into account the boundedness of sequences $\{\mathbf{y}_k^j\}$ $\forall j$, one can allocate a subsequence $\mathbf{y}_{k_l}^j$ that $\forall j: \mathbf{y}_{k_l}^j \to \bar{\mathbf{y}}^j, \mathbf{y}_{k_l}^j \in \varphi_j(\mathbf{x}_{k_l})$. We have $\mathbf{y}_{k_l} = \sum_{j=1}^{m} \alpha_j \mathbf{y}_{k_l}^j$. Passing to the limit for $l \to \infty$, we get $\mathbf{y}' = \sum_{j=1}^{m} \alpha_j \bar{\mathbf{y}}^j$. In addition to the closure of the mapping $\varphi_j(\mathbf{x})$ that means $\sum_{j=1}^{m} \alpha_j \bar{\mathbf{y}}^j \in \sum_{j=1}^{m} \alpha_j \varphi_j(\mathbf{x}')$, i.e., $\mathbf{y}' \in \varphi(\mathbf{x}')$.

2) It is sufficient to check the validity of the statement for $m = 2$. Let $\{\mathbf{x}_k\} \to \mathbf{x}'$, $\{\mathbf{y}_k\} \to \mathbf{y}'$, $\mathbf{y}_k \in \varphi_1(\varphi_2(\mathbf{x}_k))$. Let us show $\mathbf{y}' \in \varphi_1(\varphi_2(\mathbf{x}'))$. Inclusion $\mathbf{y}_k \in \varphi_1(\varphi_2(\mathbf{x}_k))$ may be presented as $\mathbf{y}_k \in \varphi_1(\mathbf{y}_k^1)$, $\mathbf{y}_k^1 \in \varphi_2(\mathbf{x}_k)$. Thus by virtue of Lemma 2.6, it is possible to assume $\mathbf{y}_k^1 \to \mathbf{y}^1$ and $\mathbf{y}^1 \in \varphi_2(\mathbf{x}')$, and as a result $\mathbf{y}' \in \varphi_1(\mathbf{y}^1) \in \varphi_1(\varphi_2(\mathbf{x}'))$.

Theorem 2.1 is proved.

Corollary 2.1. From $\{\varphi(\mathbf{x}) \in \bar{\mathbf{F}}_M$, N is a closed convex set from $\mathbf{R}^n$, $M \cap N \neq \varnothing\}$ it follows from Lemma 2.4 and Theorem 2.1 that $\mathbf{Pr}_N(\varphi(\mathbf{x})) = \bigcup_{\mathbf{y} \in \varphi(\mathbf{x})} \mathbf{Pr}_N(\mathbf{y}) \in \bar{\mathbf{F}}_{M \cap N}$.

3. FEJER PROCESS OF COUNTABLE SYSTEMS OF CONVEX INEQUALITIES

The method of constructing a converging fejer process for a countable system of convex inequalities $f_j(\mathbf{x}) \le 0$, $j = 1, 2, \ldots, m, \ldots$ is stated below. The construction of the type (1.4) is founded on the basis of the mentioned process with the following difference: in each iteration of the process, instead of residual function $d(\mathbf{x}) = \sup_{(j)} f_j(\mathbf{x})$ the function $d_k(\mathbf{x}) = \max_{j \in \overline{1,k}} f_j(\mathbf{x})$ is used.

The necessary result about convergence of the circumscribed process to the solution of a countable system will be obtained as a particular case of a more general situation. Let us describe it.

Let $d(\mathbf{x})$ be a convex function defined on $\mathbf{R}^n$, and $M := \{\mathbf{x} \mid d(\mathbf{x}) \le 0\} \ne \varnothing$. Let us assume

$$\mu_h(\mathbf{x}) = \begin{cases} 0, & d(\mathbf{x}) \le 0; \\ \dfrac{d(\mathbf{x})}{\| h \|^2} h, & d(\mathbf{x}) > 0. \end{cases}$$

Here, $h \in \partial d(\mathbf{x})$. In the second alternative $(d(\mathbf{x}) > 0)$ we get automatically $h \ne 0$. Actually, if $h \in \partial d(\bar{\mathbf{x}}),\ d(\bar{\mathbf{x}}) > 0$, then the inequality $(h, \mathbf{x} - \bar{\mathbf{x}}) \le d(\mathbf{x}) - d(\bar{\mathbf{x}})$ (which is identical on $\mathbf{x}$) gives us $0 < d(\bar{\mathbf{x}}) \le d(\tilde{\mathbf{x}}) \le 0$ (at $h = 0$ and $\mathbf{x} = \tilde{\mathbf{x}} \in M$), which is a contradiction.

The magnitude $\mu_h(\mathbf{x})$ can be written in the form

$$\mu_h(\mathbf{x}) = \frac{d^+(\mathbf{x})}{\| h \|^2} h,$$

assuming $\mu_h(\mathbf{x}) = 0$, if $d^+(\mathbf{x}) = 0$, i.e., $d(\mathbf{x}) \le 0$.

Introduce the following notations:

$$\mu(\mathbf{x}) = \{\mu_h(\mathbf{x}) \mid h \in \partial d(\mathbf{x})\}, \tag{3.1}$$

$$\varphi(\mathbf{x}) = \mathbf{x} - \lambda \mu(\mathbf{x}), \tag{3.2}$$

where $\lambda \in (0,2)$.

The mapping $\varphi(\mathbf{x})$ is multi-valued. If h is fixed, then we accept the representation

$$\varphi(\mathbf{x}) = \mathbf{x} - \lambda \frac{d^+(\mathbf{x})}{\| h \|^2} h, \tag{3.3}$$

where $\lambda \in (0,2),\ h \in \partial d(\mathbf{x})$.

Lemma 3.1. For a mapping $\varphi(\mathbf{x})$ from (3.2) the following inequality takes place:

$$\| \mathbf{z} - \mathbf{y} \| < \| \mathbf{x} - \mathbf{y} \|, \quad \forall \mathbf{x} \notin M, \ \forall \mathbf{z} \in \varphi(\mathbf{x}), \ \forall \mathbf{y} \in M \ , \tag{3.4}$$

i.e., $\varphi \in \mathbf{F}_M$. This statement corresponds to the above-mentioned example (2.1).

Let us link the following M-fejer sequence $\{\mathbf{x}_k\}$ with $\varphi(\mathbf{x})$:

$$\| \mathbf{x}_{k+1} - \mathbf{y} \| < \| \mathbf{x}_k - \mathbf{y} \|, \quad \forall k, \ \forall \mathbf{y} \in M \ , \tag{3.5}$$

generated recurrently by relation

$$\mathbf{x}_{k+1} \in \varphi(\mathbf{x}_k), \quad k = 0,1,\dots , \tag{3.6}$$

i.e., we suppose that $\{\mathbf{x}_k\} \cap M = \varnothing$. If for some $\bar{k} : \mathbf{x}_{\bar{k}} \in M$, then the process is stabilized on element $\mathbf{x}_{\bar{k}}$ from M , so the convergence is settled.

For the purpose of developing fejer processes for infinite systems of convex inequalities, we shall introduce the concept of non-stationary fejer mapping. Let $\{d_k(\mathbf{x})\}$ be a sequence of convex functions converging to $d(\mathbf{x})$ for every $\mathbf{x}$ and

$$d_k(\mathbf{x}) \le d_{k+1}(\mathbf{x}), \quad \forall k \ . \tag{3.7}$$

We consider the mapping

$$\varphi_k(\mathbf{x}) = \mathbf{x} - \lambda_k \mu_k(\mathbf{x}) \ , \tag{3.8}$$

where

$$\lambda_k \in [\delta, 2-\delta] \subset (0,2), \quad \delta > 0 \ ,$$

$$\mu_k(\mathbf{x}) = \frac{d_k^+(\mathbf{x})}{\| h \|^2} h, \quad h \in \partial d_k(\mathbf{x}) \ .$$

Theorem 3.1. Under the suppositions made about functions $\{d(\mathbf{x}), d_k(\mathbf{x})\}_k$, the process

$$\mathbf{x}_{k+1} \in \varphi_k(\mathbf{x}_k), \quad k = 0,1,\dots \tag{3.9}$$

converges to the point $\mathbf{x}' \in M$ $(= \{\mathbf{x} \mid d(\mathbf{x}) \le 0\} \ne \varnothing)$ (for arbitrary initial $\mathbf{x}_0 \in \mathbf{R}^n$).

Example showing a situation of the non-stationary process (3.9). Let

$$f_j(\mathbf{x}) \le 0, \quad j = 1,2,\dots \tag{3.10}$$

be a countable system of convex inequalities defining the set of solutions $M \ne \varnothing$. Let us assume $d_k(\mathbf{x}) = \max_{j \in \overline{1,k}} f_j(\mathbf{x})$ and suppose

$$d(\mathbf{x}) := \sup_{(j)} f_j(\mathbf{x}) < +\infty, \quad \forall \mathbf{x} \in \mathbf{R}^n . \tag{3.11}$$

The functions $\{d(\mathbf{x}), d_k(\mathbf{x})\}_k$ satisfy all the conditions of Theorem 3.1 (this theorem is to be proved).

We shall presuppose a series of lemmas to prove Theorem 3.1.

Lemma 3.2. Let $\{d_k(\mathbf{x})\}_k$ be a sequence of convex functions converging to a convex function $d(\mathbf{x})$ for every $\mathbf{x}$ and the condition of monotonicity (3.7) holds. Then from the fact $\{\mathbf{x}_k\} \to \bar{\mathbf{x}}$ it follows $d_k(\mathbf{x}_k) \to d(\bar{\mathbf{x}})$.

Proof: The continuity of $d(\mathbf{x})$ gives us an opportunity to select $\bar{k}$ for $\varepsilon > 0$, so that

$$| d(\mathbf{x}_k) - d(\bar{\mathbf{x}}) | < \frac{\varepsilon}{2}, \quad \forall k \ge \bar{k} . \tag{3.12}$$

Further, as $d_s(\bar{\mathbf{x}}) \xrightarrow{(s)} d(\bar{\mathbf{x}})$, then for sufficiently large s, let $s \ge \bar{k}$, the following inequality takes place:

$$| d_s(\bar{\mathbf{x}}) - d(\bar{\mathbf{x}}) | < \frac{\varepsilon}{4} .$$

According to (3.7) it means that

$$0 \le d(\bar{\mathbf{x}}) - d_s(\bar{\mathbf{x}}) < \frac{\varepsilon}{4} .$$

Let $s = \bar{k}$, then

$$0 \le d(\bar{\mathbf{x}}) - d_{\bar{k}}(\bar{\mathbf{x}}) < \frac{\varepsilon}{4} .$$

From here and continuity of $d(\mathbf{x})$ and $d_{\bar{k}}(\mathbf{x})$, the relation implies

$$0 \le d(\mathbf{x}_k) - d_{\bar{k}}(\mathbf{x}_k) < \frac{\varepsilon}{2}$$

for sufficiently large k, i.e. $k \ge \bar{\bar{k}} \ge \bar{k}$. But in this case the difference $d(\mathbf{x}_k) - d_s(\mathbf{x}_k)$ will decrease when $s = \bar{k}, \bar{k}+1, \ldots$, which gives us

$$0 \le d(\mathbf{x}_k) - d_s(\mathbf{x}_k) < \frac{\varepsilon}{2}, \quad s \ge \bar{k}, \ k \ge \bar{\bar{k}} . \tag{3.13}$$

Write out an evident inequality

$$| d_s(\mathbf{x}_k) - d(\bar{\mathbf{x}}) | \le | d(\mathbf{x}_k) - d(\bar{\mathbf{x}}) | + [d(\mathbf{x}_k) - d_s(\mathbf{x}_k)] .$$

From here in accordance with (3.12) and (3.13), we get the necessary result.

Lemma 3.3. Let the assumptions of Lemma 3.2. hold. If $\{\mathbf{x}_k\} \to \mathbf{x}'$ and $h_k \in \partial d_k(\mathbf{x}_k)$, then $\sup_{(k)} \| h_k \| < +\infty$.

Proof: By the definition of h_k we have

$$(h_k, \mathbf{x} - \mathbf{x}_k) \leq d_k(\mathbf{x}) - d_k(\mathbf{x}_k), \ \forall \mathbf{x}.$$

Assuming $\mathbf{x} = \mathbf{x}_k + \frac{h_k}{\| h_k \|}$, we get

$$\| h_k \| \leq d_k(\mathbf{x}_k + \frac{h_k}{\| h_k \|}) - d_k(\mathbf{x}_k).$$

The boundedness of $d_k(\mathbf{x}_k + \frac{h_k}{\| h_k \|})$ and $d_k(\mathbf{x}_k)$ leads to the boundedness of $\{\| h_k \|\}$.

We pass to the proof of Theorem 3.1.

First of all, we allocate the situation when $\exists \bar{k}, \forall k \geq \bar{k} : d_k^+(\mathbf{x}_k) = 0$ (i.e. $d_k(\mathbf{x}_k) \leq 0$). It corresponds to the case $\mathbf{x}_{\bar{k}} = \mathbf{x}_{\bar{k}+1} = \cdots =: \mathbf{x}' \in M$. If we eliminate this case and remove possible repetitions in $\{\mathbf{x}_k\}$, then the sequence $\{\mathbf{x}_k\}$ will be M-fejer. The repetitions may appear if $d_{k'}^+(\mathbf{x}_{k'}) = 0$ and there exists $k'' > k' : d_{k''}(\mathbf{x}_{k''}) > 0$. In this case we have a repetition: $\mathbf{x}_{k'} = \mathbf{x}_{k'+1}$.

Examine in turn two cases

$$\underline{\lim}\, d_k^+(\mathbf{x}_k) = 0 \text{ and } \underline{\lim}\, d_k^+(\mathbf{x}_k) =: \gamma > 0. \tag{3.14}$$

1. The first case. Let us allocate a converging subsequence $\{\mathbf{x}_{k_j}\} \to \mathbf{x}'$ such that $\{d_{k_j}(\mathbf{x}_{k_j})\} \to 0$. According to Lemma 3.2 $\{d_{k_j}^+(\mathbf{x}_{k_j})\} \to d^+(\mathbf{x}') = 0$, i.e., $\mathbf{x}' \in M$.

2. Let $\underline{\lim}\, d_k^+(\mathbf{x}_k) = \gamma > 0$ (we can remove "+" from $d_k(\mathbf{x}_k)$ in this case). Then $\exists \bar{\delta} > 0 \ \exists \bar{k} \ \forall k \geq \bar{k} : d_k(\mathbf{x}_k) > \bar{\delta} > 0$. We use the following notations

$$d_k^0(\mathbf{x}_k) := d_k(\mathbf{x}_k) - \delta_0, \quad \lambda_k^0 := \lambda_k \frac{d_k(\mathbf{x}_k)}{d_k(\mathbf{x}_k) - \delta_0}.$$

If $\delta_0 < \bar{\delta}, \ \delta_0 > 0$ is rather small, then the relations are

$$\lambda_k^0 \in [\delta_0, 2 - \delta_0] \subset (0, 2), \quad \forall k \geq \bar{k}.$$

The recurrent relation

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \lambda_k \frac{d_k^+(\mathbf{x}_k)}{\| h_k \|^2} h_k , \tag{3.15}$$

corresponding to the process (3.9), can now be written as

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \lambda_k^0 \frac{d_k^0(\mathbf{x}_k)}{\| h_k \|^2} h_k , \tag{3.16}$$

and the former subgradient $h_k \in \partial d_k(\mathbf{x}_k)$ will be a subgradient for $d_k^0(\mathbf{x}_k) = = d_k(\mathbf{x}_k) - \delta_0$. The process (3.16) now is a fejer process for the system

$$d_k(\mathbf{x}) - \delta_0 \le 0, \quad k = 1, 2, ..., \tag{3.17}$$

assigning a solid set of solutions. According to the corollary of Lemma 2.2, the sequence $\{\mathbf{x}_k\}$ converges to an element $\mathbf{x}'$. Let us prove the inclusion $\mathbf{x}' \in M$. Applying the relation (3.15), we get

$$d_k(\mathbf{x}_k) = \frac{1}{\lambda_k} \| \mathbf{x}_{k+1} - \mathbf{x}_k \| \cdot \| h_k \| . \tag{3.18}$$

By virtue of Lemma 3.3 $\{\| h_k \|\}_k$ are bounded, so if we pass to the limit in (3.18), we obtain $d(\mathbf{x}') = 0$, i.e., $\mathbf{x}' \in M$. The theorem is completely proved.

Theorem 3.2. Let the system of convex inequalities (3.10) be compatible and the condition (3.11) holds. Let us assume $d_k(\mathbf{x}) = \max_{j \in \overline{1,k}} f_j(\mathbf{x})$ and $\varphi_k(\mathbf{x})$ are according to (3.8). Then the process (3.9) converges to a solution of the system (3.10).

The result follows from Theorem 3.1.

Remark. The conditions which provide the convergence of the iterative process (3.9) being applied to a compatible countable system of convex inequalities are reduced to a single condition, in particular,

$$\sup_{(j)} f_j(\mathbf{x}) < +\infty, \quad \forall \mathbf{x} \in \mathbf{R}^n .$$

The condition can be fulfilled in the case of linear system

$$l_j(\mathbf{x}) := (a_j, \mathbf{x}) - b_j \le 0, \quad j = 1, 2, ...$$

If we multiply each inequality of the system by $\varepsilon_j > 0$, so that $\varepsilon_j | a_j | < \delta$, $\varepsilon_j | b_j | < \delta$, $\delta > 0$, then the system

$$\bar{l}_j(\mathbf{x}) := \varepsilon_j l_j(\mathbf{x}) \le 0, \quad j = 1, 2, ...$$

will satisfy the condition

$$\sup_{(j)} \overline{l}_j(\mathbf{x}) < +\infty, \quad \forall \mathbf{x} \in \mathbf{R}^n .$$

In fact,

$$\sup_{(j)} \varepsilon_j[(a_j, \mathbf{x}) - b_j] \le \sup_{(j)} \varepsilon_j[| a_j \| \mathbf{x} | + b_j] \le \delta(| \mathbf{x} | + 1) .$$

4. FEJER PROCESS FOR A CONTINUUM SYSTEM OF CONVEX INEQUALITIES (BASIC TYPE W)

Consider the system of convex inequalities of the kind

$$f_\alpha(\mathbf{x}) \le 0, \quad \forall \alpha \in N \tag{4.1}$$

with the set of solutions $M \neq \varnothing$. The following limitations are superimposed on this system:

1) N is a compact set from $\mathbf{R}^k$;
2) function $\overline{f}(\mathbf{z}) := f_\alpha(\mathbf{x})$ is convex in the variable $\mathbf{x}$ for every $\alpha \in N$ and continuous in the variable $\mathbf{z} = [\mathbf{x}, \alpha] \in \mathbf{R}^n \times \mathbf{R}^k$;
3) the mapping $[\mathbf{x}, \alpha] \to \partial_{\mathbf{x}} f_\alpha(\mathbf{x})$ is closed in the variable $\mathbf{z} = [\mathbf{x}, \alpha]$, i.e., from $\{[\mathbf{x}_k, \alpha_k]\}_k \to [\mathbf{x}', \alpha']$ and $\{h_k\}_k \to h'$, $h_k \in \partial_{\mathbf{x}} f_{\alpha_k}(\mathbf{x}_k)$ it follows $h' \in \partial_{\mathbf{x}} f_{\alpha'}(\mathbf{x}')$;
4) $\forall \alpha \in N \;\; \exists p_\alpha : f_\alpha(p_\alpha) < 0$.

Let us establish the residual function for (4.1)

$$d(\mathbf{x}) = \max_{\alpha \in N} f_\alpha^+(\mathbf{x}) \;\; (= f_{\alpha_{\mathbf{x}}}^+(\mathbf{x})) .$$

This function is continuous.

Let us note the following fact. The operation of determining the maximum of $f_\alpha^+(\mathbf{x})$ in the variable $\alpha \in N$ only identifies index $\alpha_{\mathbf{x}}$, at which this maximum is reached. It is possible to speak about the complexity or simplicity of realizating such an operation only in concrete situations. Consideration of its complexity in general is senseless.

Let us assume $J(\mathbf{x}) = \{\alpha_{\mathbf{x}} \mid d(\mathbf{x}) = f_{\alpha_{\mathbf{x}}}^+(\mathbf{x})\}$. We construct a map $\varphi(\mathbf{x})$:

$$\varphi(\mathbf{x}) = \begin{cases} \mathbf{x}, & \text{if } d(\mathbf{x}) = 0 \;\; (\mathbf{x} \in M); \\ \{\mathbf{x} - \lambda \dfrac{d(\mathbf{x})}{\| h \|^2} h \mid h \in \partial_{\mathbf{x}} f_{\alpha_{\mathbf{x}}}(\mathbf{x}),\; \alpha_{\mathbf{x}} \in J(\mathbf{x})\}, & \text{if } d(\mathbf{x}) > 0, \end{cases} \tag{4.2}$$

where $\lambda \in (0,2)$. The subgradient h from (4.2) is not equal to zero for any $\bar{\mathbf{x}}$, if $d(\bar{\mathbf{x}}) > 0$. Actually, if $h = 0$, then from $0 = (h, \mathbf{x} - \bar{\mathbf{x}}) \le f_{\alpha_{\bar{\mathbf{x}}}}(\mathbf{x}) - f_{\alpha_{\bar{\mathbf{x}}}}(\bar{\mathbf{x}})$ (the last relation takes place for all $\mathbf{x}$ due to convexity of the function $f_{\alpha_{\bar{\mathbf{x}}}}(\mathbf{x})$) it follows $\bar{\mathbf{x}} \in \operatorname{Arg}\min_{(\mathbf{x})} f_{\alpha_{\bar{\mathbf{x}}}}(\mathbf{x})$. This is a contradiction because $f_{\alpha_{\bar{\mathbf{x}}}}(\bar{\mathbf{x}}) = d(\bar{\mathbf{x}}) > 0$, and at the same time there exists $\mathbf{x} := \mathbf{x}' \in M : f_{\alpha_{\bar{\mathbf{x}}}}(\mathbf{x}') \le 0$.

Theorem 3.1. Let system (4.1) satisfy assumptions 1)-4). Then the mapping $\varphi(\mathbf{x})$ assigned by (4.1) is a closed M-fejer mapping.

Proof: The fact that (4.2) is a fejer mapping with respect to set M follows from the example (see Definition 2.4). It is necessary to establish the closure of this map. Let $\{\mathbf{x}_k\} \to \mathbf{x}'$, $\{\mathbf{y}_k\} \to \mathbf{y}'$, $\mathbf{y}_k \in \varphi(\mathbf{x}_k)$. Due to the definition of mapping closure, we are to show $\mathbf{y}' \in \varphi(\mathbf{x}')$. Inclusion $\mathbf{y}_k \in \varphi(\mathbf{x}_k)$ means, that

$$\mathbf{y}_k = \mathbf{x}_k - \lambda \frac{d(\mathbf{x}_k)}{\| h_k \|^2} h_k, \tag{4.3}$$

where h_k is selected according to (4.2), i.e., $h_k \in \partial_{\mathbf{x}} f_{\alpha_k}(\mathbf{x}_k)$, and α_k is a shortening for $\alpha_{\mathbf{x}}$ at $\mathbf{x} = \mathbf{x}_k$. Bearing in mind the compactness of set N, one can assume $\{\alpha_k\} \to \alpha' \in N$ (otherwise it is possible to allocate subsequences from $\{\mathbf{x}_k\}$, $\{\mathbf{y}_k\}$ and to obtain the necessary convergence of $\{\alpha_k\}$).

Two cases are possible.

1. $d(\mathbf{x}') > 0$.

Let us prove the boundedness of the sequence $\{h_k\}$. The inequality $(h_k, \mathbf{x} - \mathbf{x}_k) \le f_{\alpha_k}(\mathbf{x}) - f_{\alpha_k}(\mathbf{x}_k)$ takes place for any $\mathbf{x}$, so if we assume $\mathbf{x} = \mathbf{x}_k + \frac{h_k}{\| h_k \|}$, we get (by analogy to the proof of Lemma 3.3): $\| h_k \| \le f_{\alpha_k}(\mathbf{x}_k + \frac{h_k}{\| h_k \|}) - f_{\alpha_k}(\mathbf{x}_k)$. Due to the second condition this fact completes the proof of the boundedness of $\{h_k\}$. Now, we are able to reckon that $\{h_k\} \to h'$, and the following inclusion is valid due to the third condition: $h' \in \partial_{\mathbf{x}} f_{\alpha'}(\mathbf{x}')$. Besides, $h' \ne 0$. In fact, if $h' = 0$, then an inequality takes place for all $\mathbf{x} : (h', \mathbf{x} - \mathbf{x}') \le f_{\alpha'}(\mathbf{x}) - f_{\alpha'}(\mathbf{x}')$; that is why $f_{\alpha'}(\mathbf{x}') \le f_{\alpha'}(\mathbf{x})$.

But $d(\mathbf{x}') = f_{\alpha'}(\mathbf{x}') > 0$ and according to the fourth condition $\exists p_{\alpha'} : f_{\alpha'}(p_{\alpha'}) < 0$, so if $\mathbf{x} = p_{\alpha'}$ then we get a contradiction. We have proved that $h' \ne 0$. Passing to the limit in (4.3), we obtain

$$\mathbf{y}' = \mathbf{x}' - \lambda \frac{d(\mathbf{x}')}{\| h' \|^2} h', \tag{4.4}$$

i.e. $\mathbf{y}' \in \varphi(\mathbf{x}')$, which means closure of the mapping φ.

2. $d(\mathbf{x}') = 0$.

Examine in turn two subcases.

2.1. $\exists\{\mathbf{x}_{k_j}\} \subset \{\mathbf{x}_k\} : d(\mathbf{x}_{k_j}) = 0$. This corresponds to the case when sequence $\{\mathbf{x}_{k_j}\}$ lies in the set M. Taking into account that in this case $\varphi(\mathbf{x}_{k_j}) = \mathbf{x}_{k_j}$, we obtain $\mathbf{y}_{k_j} = \varphi(\mathbf{x}_{k_j}) = \mathbf{x}_{k_j} \to \mathbf{x}'$, i.e., $\mathbf{y}' = \varphi(\mathbf{x}') = \mathbf{x}'$, i.e., $\mathbf{y}' \in \varphi(\mathbf{x}')$.

2.2. $\exists \bar{k} : d\{\mathbf{x}_k\} > 0, \ \forall k > \bar{k}$. As above, we show that $h' \in \partial_{\mathbf{x}} f_{\alpha'}(\mathbf{x}') \Rightarrow h' \neq 0$. Because of the correctness of the following inequality for all $\mathbf{x}$ $(h', \mathbf{x} - \mathbf{x}') \leq$ $\leq f_{\alpha'}(\mathbf{x}) - f_{\alpha'}(\mathbf{x}')$, in the case $h' = 0$, we get $f_{\alpha'}(\mathbf{x}') \leq f_{\alpha'}(\mathbf{x})\ \forall \mathbf{x}$. But $d(\mathbf{x}') = f_{\alpha'}(\mathbf{x}') = 0$, i.e., $f_{\alpha'}(\mathbf{x}) \geq 0\ \forall \mathbf{x}$. At the same time the conditions of the theorem dictate $\exists p_{\alpha'} : f_{\alpha'}(p_{\alpha'}) < 0$. We have a contradiction at $\mathbf{x} = p_{\alpha'}$. Thus in this case, (4.4) is valid also, i.e., $\mathbf{y}' \in \varphi(\mathbf{x}')$.

The theorem is completely proved.

Corollary. Sequence $\{\mathbf{x}_k\} \subset \mathbf{R}^n$ initiated recurrently by inclusion $\mathbf{x}_{k+1} \in \varphi(\mathbf{x}_k)$ with arbitrary initial $\mathbf{x}_0$ converges to the solution of the system (4.1) (see Lemma 2.5).

Consider the system (4.1) with the additional requirement $\mathbf{x} \in S$, i.e.,

$$f_\alpha(\mathbf{x}) \leq 0, \ \ \forall \alpha \in N, \ \ \mathbf{x} \subset S, \tag{4.5}$$

where S is a convex closed set, $S \in \mathbf{R}^n$.

Consider an analog of the mapping (4.2)

$$\psi(\mathbf{x}) := \mathbf{Pr}_S(\varphi(\mathbf{x})). \tag{4.6}$$

Theorem 4.2. Let the assumptions of Theorem 4.1. hold and the system (4.5) be compatible. Then sequence $\{\mathbf{x}_k\}$, generated by relation $\mathbf{x}_{k+1} \in \psi(\mathbf{x}_k)$ with arbitrary initial $\mathbf{x}_0 \in \mathbf{R}^n$, converges to a solution of the system (4.5).

The proof of the theorem follows from the facts of the closure of mapping $\varphi(\cdot)$ and the continuity of projecting operation $\mathbf{Pr}_S(\cdot)$. These facts mean that superposition $\mathbf{Pr}_S(\varphi(\mathbf{x}))$ $(= \bigcup_{\mathbf{y} \in \varphi(\mathbf{x})} \mathbf{Pr}_S(\mathbf{y}))$ realizes a closed mapping.

5. CASE OF A SYSTEM INTEGRATING A FINITE NUMBER OF SUBSYSTEMS OF THE TYPE W

The offered method to solve system (4.1) can be upgraded to finding the solution of the system of inequalities of the following kind:

$$f_j(\mathbf{x},\mathbf{v}_j) \le 0, \quad \forall \mathbf{v}_j \in V_j, \quad \mathbf{x} \in S_j \subset \mathbf{R}^n, \quad j = 1,\dots,m; \tag{5.1}$$

where $\{f_j(\mathbf{x},\mathbf{v}_j)\}$ are functions satisfying conditions 1) - 4) from the previous section with α replaced by $\mathbf{v}_j$; S_j are convex closed sets.

We shall upgrade the iterative process discussed in the previous section to receive a solution of the system (5.1). Let us put into consideration the residual functions for (5.1):

$$d_j(\mathbf{x}) := \max_{\mathbf{v}_j \in V_j} f_j^+(\mathbf{x},\mathbf{v}_j) = f_j^+(\mathbf{x},\mathbf{v}_j(\mathbf{x})).$$

As before, we shall generate mappings $\psi_j(\mathbf{x})$, $j = 1,\dots,m$, by analogy to (4.6), i.e.,

$$\psi_j(\mathbf{x}) := \mathbf{Pr}_{S_j}(\varphi_j(\mathbf{x})),$$

$$\varphi_j(\mathbf{x}) = \begin{cases} \mathbf{x}, & \text{if } d_j(\mathbf{x}) = 0; \\ \{\mathbf{x} - \lambda_j \dfrac{d_j(\mathbf{x})}{\| h_j \|^2} h_j \mid h_j \in \partial_{\mathbf{x}} f_j(\mathbf{x},\mathbf{v}_j(\mathbf{x})),\ \mathbf{v}_j(\mathbf{x}) \in J_j(\mathbf{x})\}, & \text{if } d_j(\mathbf{x}) > 0. \end{cases}$$

Here, $\lambda_j \in (0,2)$, $j = 1,\dots,m$; $J_j(\mathbf{x}) := \{\mathbf{v}_j(\mathbf{x}) \mid d_j(\mathbf{x}) = f_j(\mathbf{x},\mathbf{v}_j(\mathbf{x}))\}$.

Since the subsystem corresponding to the index j in the system (5.1) is a subsystem of the type (4.5), Theorem 4.2 (naturally, under the assumptions of Theorem 4.1) is valid. Namely, the sequence $\{\mathbf{x}_k\}$, given recurrently by iterative mapping $\varphi_j(\mathbf{x})$ with arbitrary initial $\mathbf{x}_0 \in \mathbf{R}^n$, will converge to $\mathbf{x}' \in S_j \cap M_j$, where $M_j := \{\mathbf{x} \mid f_j(\mathbf{x},\mathbf{v}_j) \le 0,\ \forall \mathbf{v}_j \in V_j\}$.

From here, using property $\varphi_j(\cdot) \in \overline{\mathbf{F}}_{M_j}$, $j = 1,\dots,m$, and Theorem 2.1, by Lemma 2.5, we get the following statement.

Theorem 5.1. If the assumptions on the system of inequalities (5.1) hold, then the sequence $\{\mathbf{x}_k\}_0^{+\infty}$ obtained by relation

$$\mathbf{x}_{k+1} \in \sum_{i=1}^{m} \alpha_i \varphi_i(\mathbf{x}_k), \quad \sum_{i=1}^{m} \alpha_i = 1, \quad \alpha_i > 0, \quad i = 1,\dots,m$$

converges to a solution of the system (5.1) with arbitrary initial $\mathbf{x}_0$.

6. SOLUTION OF CONCAVE-CONVEX GAMES ON THE BASIS OF REDUCTION TO SYSTEMS OF CONVEX INEQUALITIES OF CONTINUUM POWER

Consider a game Γ of two persons with a zero sum. The game is uniquely determined by two sets of strategies M and N of the players and by payoff function $F(\mathbf{x},\mathbf{y})$ [6].

The function $F(\mathbf{x},\mathbf{y})$ is interpreted as a scoring of the first player (penalty of the second player). The principle of guaranteed result, applied to the considered game, leads to problems of searching

$$\max_{\mathbf{x}\in M}\min_{\mathbf{y}\in N} F(\mathbf{x},\mathbf{y}) =: v$$

and

$$\min_{\mathbf{y}\in N}\max_{\mathbf{x}\in M} F(\mathbf{x},\mathbf{y}) =: v^* .$$

If $\bar{t} := v = v^* = F(\bar{\mathbf{x}},\bar{\mathbf{y}}),\ \bar{\mathbf{x}}\in M,\ \bar{\mathbf{y}}\in N$, then the common value of v and v^*, i.e., $\bar{t}$ is called the value of the game, $\bar{\mathbf{x}},\ \bar{\mathbf{y}}$ are the optimal strategies of the players and $\{\bar{\mathbf{x}},\bar{\mathbf{y}},\bar{t}\}$ is a solution of the game.

The solution of the game Γ reduces to the solution of a continuum system of convex inequalities [6]:

$$F(\mathbf{x},\mathbf{v}) \ge t,\ \ \forall \mathbf{v}\in N,\ \mathbf{x}\in M,\ \ F(\mathbf{w},\mathbf{y}) \le t;\ \ \forall \mathbf{w}\in M,\ \mathbf{y}\in N . \tag{6.1}$$

The connection between the game Γ and the system (6.1) is the following: $\bar{\mathbf{z}} = [\bar{\mathbf{x}},\bar{\mathbf{y}},\bar{t}]$ is solution of the game Γ if and only if $\bar{\mathbf{z}}$ is a solution of the system (6.1).

Making renames in (6.1)

$$F^{(1)}_{\mathbf{v}}(\mathbf{x},t) := -F(\mathbf{x},\mathbf{v}) + t,\quad F^{(2)}_{\mathbf{w}}(\mathbf{y},t) := F(\mathbf{w},\mathbf{y}) - t ,$$

we rewrite (6.1) as

$$F^{(1)}_{\mathbf{v}}(\mathbf{x},t) \le 0,\ \ \forall \mathbf{v}\in N,\ \mathbf{x}\in M,\ \ F^{(2)}_{\mathbf{w}}(\mathbf{y},t) \le 0,\ \ \forall \mathbf{w}\in M,\ \mathbf{y}\in N. \tag{6.2}$$

Before we construct the mapping $\tilde{\varphi}(\cdot)\in \bar{F}_{\tilde{M}}$ for the set of solutions $\tilde{M}$ of the system (6.2), let us consider a more general construction of formulating a fejer process with the help of the system of partial fejer maps $\{\varphi_i(\cdot)\}$ having distinct spaces of their images.

In particular, let the union of M_i-fejer mappings be given: $\{\varphi_i(\mathbf{x}_i,\mathbf{y})\}_1^m$, where $\mathbf{x}_i\in \mathbf{R}^{n_i}$, $y\in \mathbf{R}^s$, $\varphi_i : Z_i \to 2^{Z_i}$, $Z_i := \mathbf{R}^{n_i}\times \mathbf{R}^s$, $M_i \subset Z_i$, $i = 1,\dots,m$.

The dot $\tilde{z}=[\tilde{\mathbf{x}}_1,...,\tilde{\mathbf{x}}_m;\tilde{\mathbf{y}}]\in Z:=\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_m}\times\mathbf{R}^s$ is called the dot of immovability for $\{\varphi_i\}_1^m$, if $\forall i:\varphi_i(\tilde{\mathbf{x}}_i,\tilde{\mathbf{y}})=[\tilde{\mathbf{x}}_i,\tilde{\mathbf{y}}]$. The set of immovability dots is denoted by $\tilde{M}$. Let us assume $\tilde{M}\neq\varnothing$.

Let $z_i\in\varphi_i(\mathbf{x}_i,\mathbf{y})$ and

$$z_i=[\bar{\mathbf{x}}_i,\bar{\mathbf{y}}_i],\quad i=1,...,m, \tag{6.3}$$

where $\bar{\mathbf{x}}_i$ is a trace (algebraic projection) of an element z_i in subspace $\mathbf{R}^{n_i}$ of the space Z_i, $\bar{\mathbf{y}}_i$ is a trace of an element z_i in $\mathbf{R}^s\subset Z_i$.

Put $z=[\mathbf{x}_1,...,\mathbf{x}_m;\mathbf{y}]\in Z$. Consider the mapping

$$\varphi(z):=\{[\bar{\mathbf{x}}_1,...,\bar{\mathbf{x}}_m;\frac{1}{m}\sum_{i=1}^m\bar{\mathbf{y}}_i]\,|\,(6.3),\ z_i\in\varphi_i(\mathbf{x}_i,\mathbf{y}),\ i=1,...,m\},$$

where $\{\bar{\mathbf{x}}_1,...,\bar{\mathbf{x}}_m;\bar{\mathbf{y}}_1,...,\bar{\mathbf{y}}_m\}$ is defined by (6.3).

Theorem 6.1. If $\varphi_i(\cdot)\in\bar{F}_{M_i}$, then $\varphi(\cdot)\in\bar{F}_{\tilde{M}}$. Any sequence $\{z^k\}\subset Z$, generated recurrently by mapping $\varphi(z)$, i.e. $z^{k+1}\in\varphi(z^k)$ with arbitrary initial z_0 converges to the dot $\tilde{z}$ from $\tilde{M}$.

Proof: The property $\tilde{z}\in\tilde{M}\Rightarrow\varphi(\tilde{z})=\tilde{z}$ is evident. It is necessary to prove

$$\{\bar{z}\in\varphi(z),\ z\notin\tilde{M},\ \tilde{z}\in\tilde{M}\}\Rightarrow\|\bar{z}-\tilde{z}\|<\|z-\tilde{z}\|. \tag{6.4}$$

We have $\bar{z}=[\bar{\mathbf{x}}_1,...,\bar{\mathbf{x}}_m;\frac{1}{m}\sum_{i=1}^m\bar{\mathbf{y}}_i]$, where $\bar{\mathbf{x}}_i$ is a trace of the dot $z_i\in\varphi_i(\mathbf{x}_i,\mathbf{y})$ in $\mathbf{R}^{n_i}$, $\bar{\mathbf{y}}_i$ is a trace of z_i in $\mathbf{R}^s$, $\tilde{z}=[\tilde{\mathbf{x}}_1,...,\tilde{\mathbf{x}}_m;\tilde{\mathbf{y}}]$, and at the same time $\forall i:\varphi_i(\tilde{\mathbf{x}}_i,\tilde{\mathbf{y}})=[\tilde{\mathbf{x}}_i,\tilde{\mathbf{y}}]$. Due to $\tilde{z}\in\tilde{M}$, there exists an i for which the following inequality is valid:

$$\|[\bar{\mathbf{x}}_i,\bar{\mathbf{y}}_i]-[\tilde{\mathbf{x}}_i,\tilde{\mathbf{y}}]\|^2<\|[\mathbf{x}_i,\mathbf{y}]-[\tilde{\mathbf{x}}_i,\tilde{\mathbf{y}}]\|^2. \tag{6.5}$$

Let us turn to proving ratio (6.4):

$$\|\bar{z}-\tilde{z}\|^2=\|[\bar{\mathbf{x}}_1,...,\bar{\mathbf{x}}_m;\frac{1}{m}\sum_{i=1}^m\bar{\mathbf{y}}_i]-[\tilde{\mathbf{x}}_1,...,\tilde{\mathbf{x}}_m;\frac{1}{m}\sum_{i=1}^m\tilde{\mathbf{y}}]\|^2=$$

$$=\|[\bar{\mathbf{x}}_1-\tilde{\mathbf{x}}_1,...,\bar{\mathbf{x}}_m-\tilde{\mathbf{x}}_m;\frac{1}{m}\sum_{i=1}^m(\bar{\mathbf{y}}_i-\tilde{\mathbf{y}})]\|^2=$$

$$= \sum_{i=1}^{m} \| \bar{\mathbf{x}}_i - \tilde{\mathbf{x}}_i \|^2 + \frac{1}{m^2} \sum_{i=1}^{m} \| \bar{\mathbf{y}}_i - \tilde{\mathbf{y}} \|^2 =$$
$$= \sum_{i=1}^{m} (\| \bar{\mathbf{x}}_i - \tilde{\mathbf{x}}_i \|^2 + \| \bar{\mathbf{y}}_i - \tilde{\mathbf{y}} \|^2) - \frac{m^2 - 1}{m^2} \sum_{i=1}^{m} \| \bar{\mathbf{y}}_i - \tilde{\mathbf{y}} \|^2 \le$$
$$\le \sum_{i=1}^{m} (\| \bar{\mathbf{x}}_i - \tilde{\mathbf{x}}_i \|^2 + \| \bar{\mathbf{y}}_i - \tilde{\mathbf{y}} \|^2) < \sum_{i=1}^{m} \| [\mathbf{x}_i, \mathbf{y}] - [\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}] \|^2 = \| z - \tilde{z} \|^2,$$

as it was required. Thus, the inclusion $\varphi(\cdot) \in \bar{F}_{\tilde{M}}$ is stated. The inclusion $\varphi(\cdot) \in \bar{F}_{\tilde{M}}$ easily follows from the closure of mapping φ (due to the closure of mappings $\{\varphi_i(\cdot)\}$).

Let us return to the system (6.2). As mentioned above, a solution of the game Γ reduces to a solution of this system. We rewrite it in the form

$$\begin{cases} F_{\mathbf{v}}^{(1)}(\mathbf{x}, t) \le 0, \quad \mathbf{x} \in M, \quad \forall \mathbf{v} \in N; \qquad (6.6)_1; \\ F_{\mathbf{w}}^{(2)}(\mathbf{y}, t) \le 0, \quad \mathbf{y} \in N, \quad \forall \mathbf{w} \in M. \qquad (6.6)_2. \end{cases} \tag{6.6}$$

Also we allocate subsystems

$$F_{\mathbf{v}}^{(1)}(\mathbf{x}, t) \le 0, \quad \forall \mathbf{v} \in N; \qquad (6.6)_1^0;$$
$$F_{\mathbf{w}}^{(2)}(\mathbf{y}, t) \le 0, \quad \forall \mathbf{w} \in M. \qquad (6.6)_2^0.$$

These are the systems $(6.6)_1$ and $(6.6)_2$ without requirements of $\mathbf{x} \in M$ and $\mathbf{y} \in N$.

Let us take into consideration residual functions $d_1(\mathbf{x}, t)$ and $d_2(\mathbf{y}, t)$ for subsystems $(6.6)_1^0$ and $(6.6)_2^0$:

$$d_1(\mathbf{x}, t) = \max_{\mathbf{v} \in N} [F_{\mathbf{v}}^{(1)}(\mathbf{x}, t)]^+ \quad (= [F_{\mathbf{v}(\mathbf{x},t)}^{(1)}(\mathbf{x}, t)]^+), \tag{6.7}$$

$$d_2(\mathbf{y}, t) = \max_{\mathbf{w} \in M} [F_{\mathbf{w}}^{(2)}(\mathbf{y}, t)]^+ \quad (= [F_{\mathbf{w}(\mathbf{y},t)}^{(2)}(\mathbf{y}, t)]^+). \tag{6.8}$$

Denote

$$J_1(\mathbf{x}, t) = \{\mathbf{v}(\mathbf{x}, t) \,|\, (6.7)\}, \quad J_2(\mathbf{y}, t) = \{\mathbf{w}(\mathbf{y}, t) \,|\, (6.8)\}.$$

Assume that

$$\varphi_1(\mathbf{x}, t) := \{[\mathbf{x}, t] - \lambda_1 \frac{d_1(\mathbf{x}, t)}{\| h_{\mathbf{v}}^{(1)} \|^2 + 1} [-h_{\mathbf{v}}^{(1)}, 1] \,|\, \mathbf{v} \in J_1(\mathbf{x}, t),\ h_{\mathbf{v}}^{(1)} \in \partial_{\mathbf{x}} F(\mathbf{x}, \mathbf{v})\}, \tag{6.9}$$

$$\varphi_2(\mathbf{y}, t) := \{[\mathbf{y}, t] - \lambda_2 \frac{d_2(\mathbf{y}, t)}{\| h_{\mathbf{w}}^{(2)} \|^2 + 1} [h_{\mathbf{w}}^{(2)}, -1] \,|\, \mathbf{w} \in J_2(\mathbf{y}, t),\ h_{\mathbf{w}}^{(2)} \in \partial_{\mathbf{y}} F(\mathbf{w}, \mathbf{y})\}; \tag{6.10}$$

where $\lambda_1 \in (0,2),\ \lambda_2 \in (0,2)$. Denote by $\alpha_{\mathbf{v}}(\mathbf{x},t)$ and $\beta_{\mathbf{w}}(\mathbf{y},t)$ the coefficient before $[-h_{\mathbf{v}}^{(1)},1]$ and $[h_{\mathbf{w}}^{(2)},-1]$. Then $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ can be rewritten as

$$\varphi_1(\mathbf{x},t) = \{[\underbrace{\mathbf{x} + \alpha_{\mathbf{v}}(\mathbf{x},t)h_{\mathbf{v}}^{(1)}}_{\bar{\mathbf{x}}}, \underbrace{t - \alpha_{\mathbf{v}}(\mathbf{x},t)}_{t'}] \mid \mathbf{v} \in J_1(\mathbf{x},t),\ h_{\mathbf{v}}^{(1)} \in \partial_{\mathbf{x}} F(\mathbf{x},\mathbf{v})\}, \tag{6.11}$$

$$\varphi_2(\mathbf{y},t) := \{[\underbrace{\mathbf{y} + \beta_{\mathbf{w}}(\mathbf{y},t)h_{\mathbf{w}}^{(2)}}_{\bar{\mathbf{y}}}, \underbrace{t + \beta_{\mathbf{w}}(\mathbf{y},t)}_{t''}] \mid \mathbf{w} \in J_2(\mathbf{y},t),\ h_{\mathbf{w}}^{(2)} \in \partial_{\mathbf{y}} F(\mathbf{w},\mathbf{y})\}. \tag{6.12}$$

Let us define a purpose mapping

$$\tilde{\varphi}(\mathbf{x},\mathbf{y},t) := \{[\mathbf{Pr}_M(\bar{\mathbf{x}}), \mathbf{Pr}_N(\bar{\mathbf{y}}), \frac{t'+t''}{2}]\}, \tag{6.13}$$

where $\bar{\mathbf{x}}$ is the first vector fragment in (6.11), $\bar{\mathbf{y}}$ is the first vector fragment in (6.12) and t', t'' are scalar fragments in (6.11) and (6.12), respectively.

Let us take together all constraints on the game Γ, which provide the converging of the iterative process generated recurrently by mapping $\tilde{\varphi}(\mathbf{x},\mathbf{y},t)$, to a solution of the game:

1. M and N are convex compact sets, $M \subset \mathbf{R}^n$, $N \subset \mathbf{R}$;
2. $F(\mathbf{x},\mathbf{y})$ is continuous in $z = [\mathbf{x},\mathbf{y}] \in \mathbf{R}^n \times \mathbf{R}^m$, concave in $\mathbf{x}$ and convex in $\mathbf{y}$;
3. Mappings $[\mathbf{x},\mathbf{y}] \to \partial_{\mathbf{x}} F(\mathbf{x},\mathbf{y}),\ [\mathbf{x},\mathbf{y}] \to \partial_{\mathbf{y}} F(\mathbf{x},\mathbf{y})$ are closed.

Theorem 6.2. If suppositions 1.-3. hold, then

$$\tilde{\varphi}(\mathbf{x},\mathbf{y},t) \in \bar{\bar{F}}_{\tilde{M}},$$

where $\tilde{M}$ is a set of vectors $[\bar{\mathbf{x}},\bar{\mathbf{y}},t] \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$, which are solutions of the game Γ. From here it follows that any sequence $\{z^k\}_0^{+\infty}$, generated recurrently by ratio $z^{k+1} \in \tilde{\varphi}(z_k)$ (with arbitrary initial $z^0 = [\mathbf{x}^0,\mathbf{y}^0,t]$) converges to a solution of the game Γ.

We shall divide the substantiation of the formulated statement essentially prepared by the previous theorems into a series of items.

1) Resolvability of the game Γ is provided by conditions 1 and 2 (it is a known result, see, for example, [6]).

2) Mappings $\varphi_1(\mathbf{x},t)$ and $\varphi_2(\mathbf{y},t)$, i.e., (6.9) and (6.10) corresponding to the systems $(6.6)_1^0$ and $(6.6)_2^0$, are constructed just as mapping (4.2) for the system (4.1)

(the differences are only in the notations). The closure of mapping (4.2) is valid due to conditions 1)-4) (Theorem 4.1). With application to systems $(6.6)_1^0$ and $(6.6)_2^0$ these conditions take place due to suppositions 1.-3.

Explanations are required only for condition 4), which can be written for the system $(6.6)_1^0$ in the following form: $\forall \bar{\mathbf{v}} \in N\ \exists [\bar{\mathbf{x}}, \bar{t}]: F_{\bar{\mathbf{v}}}^{(1)}(\bar{\mathbf{x}}, \bar{t}) < 0$, i.e. $-F(\bar{\mathbf{x}}, \bar{\mathbf{v}}) + \bar{t} < 0$. It is evident that for any $\bar{\mathbf{x}}$ and $\bar{\mathbf{v}}$, the corresponding $\bar{t}$ is selected trivially. The system $(6.6)_2^0$ is treated by analogy to $(6.6)_1^0$. All these facts provide (with respect to Theorem 4.1) the closure of mappings $\varphi_1(\mathbf{x}, t)$ and $\varphi_2(\mathbf{y}, t)$.

3) We have proved that

$$\varphi_1(\cdot) \in \bar{F}_{\mathrm{Fix}\varphi_1(\cdot)}, \quad \varphi_2(\cdot) \in \bar{F}_{\mathrm{Fix}\varphi_2(\cdot)},$$

where $\mathrm{Fix}\varphi_i(\cdot)$ is a symbol for notation of the immovability sets of the maps $\varphi_i(\cdot)$. The sets of solutions of $(6.6)_1^0$ and $(6.6)_2^0$ are these immovability sets. According to the scheme for construction of the map $\varphi(z)$ from Theorem 6.1 it is possible to use $\varphi_1(\mathbf{x}, t)$ and $\varphi_2(\mathbf{y}, t)$ to establish a mapping $\varphi(\mathbf{x}, \mathbf{y}, t)$, which has the following form: $\varphi(\mathbf{x}, \mathbf{y}, t) = \{[\bar{\mathbf{x}}, \bar{\mathbf{y}}, \frac{t' + t''}{2}]\}$, where $\bar{\mathbf{x}}$ is the trace of the vector $z_1 \in \varphi_1(\mathbf{x}, t)$ in $\mathbf{R}^n$ as the subspace of the space $\mathbf{R}^n \times \mathbf{R}$, $\bar{\mathbf{y}}$ is the trace of the vector $z_2 \in \varphi_2(\mathbf{y}, t)$ in $\mathbf{R}^m$ as the subspace of the space $\mathbf{R}^m \times \mathbf{R}$, t' and t'' are the traces of z_1 and z_2 in one-dimensional subspace $\mathbf{R}$ of the spaces $\mathbf{R}^n \times \mathbf{R}$ and $\mathbf{R}^m \times \mathbf{R}$.

Thus, the mapping $\varphi(\mathbf{x}, \mathbf{y}, t)$ will be closed and fejer for the set of solutions of the system

$$F_{\mathbf{v}}^{(1)}(\mathbf{x}, t) \le 0,\ \forall \mathbf{v} \in N; \quad F_{\mathbf{w}}^{(2)}(\mathbf{y}, t) \le 0,\ \forall \mathbf{w} \in M. \tag{6.14}$$

The system (6.14) is the union of the systems $(6.6)_1^0$ and $(6.6)_2^0$.

The requirements of $\mathbf{x} \in M$ and $\mathbf{y} \in N$ are taken into account with the help of using the projecting operators in M and N as shown in (6.13).

Vector $[\mathbf{Pr}_M(\bar{\mathbf{x}}), \mathbf{Pr}_N(\bar{\mathbf{y}}), \frac{t' + t''}{2}]$ from (6.13) realizes the projection of the vector $[\bar{\mathbf{x}}, \bar{\mathbf{y}}, \frac{t' + t''}{2}] \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$ in $M \times N \times \mathbf{R}$, and the mapping $\tilde{\varphi}(\mathbf{x}, \mathbf{y}, t)$ realizes the unity of all vector projections from $\varphi(\mathbf{x}, \mathbf{y}, t)$ in the mentioned set. Mapping $\varphi(\mathbf{x}, \mathbf{y}, t)$, as shown above, is a fejer mapping for the set of solutions of the system (6.14), and the projection operator is continuous and fejer mapping for the set on which the projecting is done (Lemma 2.4). That is why their superposition $\tilde{\varphi}(\mathbf{x}, \mathbf{y}, t)$ is a closed fejer map for the set Fix$\tilde{\varphi}(\cdot)$ (Theorem 2.1, Property 2). Using the statement of Lemma 2.5, we get the validity of Theorem 6.2.

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