Yugoslav Journal of Operations Research
11 (2001), Number 1, 3-14

ON BILEVEL VARIATIONAL INEQUALITIES ARISING IN THE ANALYSIS OF IMPROPER OPTIMIZATION PROBLEMS[1]

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Abstract: A special class of bilevel monotone variational inequalities arising in the parametric analysis of monotone operator equations is investigated. Sufficient conditions for the existence of a solution are given and some numerical methods for these problems are proposed. Improper problems of mathematical programming, complementarity and game theory may be regarded as an area of application of the results of the theory and practice of ill-posed parametric systems of equations and convex inequalities.

Keywords: Monotone operators, lexicographical variational inequalities, stable solvability, improper optimization problems, optimal correction of ill-posed instances.

1. INTRODUCTION

Let X be a real reflexive Banach space having dual X^*, let Q be a convex closed subset of X and let $F(\cdot)$ be a monotone mapping from X into X^*. The variational inequality problem $VI(F,Q)$ is the problem of finding $\tilde{x} \in Q$ such that

$$\langle F(\tilde{x}), x - \tilde{x} \rangle \geq 0 \quad \forall x \in Q, \tag{1}$$

where $\langle x^*, x \rangle$ denotes the value of $x^* \in X^*$ at $x \in X$.

Below we are interested in the case

$$Q = \operatorname{cl} \Omega, \ \ \Omega := \{x \in X : A(x, y) = 0 \ \text{ for some } \ y \in Y\}, \tag{2}$$

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[1] This research was supported by Russian Found of Fundamental Research (project codes 99-01-00136, 00-15-96041).

where $A(x,\cdot)$ denotes a parametric family of monotone nonlinear mappings from another reflexive Banach space Y into its dual Y^*, x being a parameter (see Section 2 for precise definitions).

The instance (1)-(2) arises from [13], [3], [4], [7], [12] and others where singular (unstable, ill-posed, improper) optimization problems and methods of their optimal correction have been analyzed. Along with that, the problem (1)-(2) may be regarded marely as a special two-level monotone lexicographical variational inequality with variables splitting into two groups in such a manner that the variables of the first group enter only in a lower level of the subproblem and the variables of the second group only in an upper level (see [5], [10]). Related results may be found in [9], [6], [1].

Obviously, the main difficulty with the problem (1)-(2) is connected with the non-constructive definition of the set Ω.

The article is organized as follows. In Section 2 we present some definitions and facts from the operator theory used in the sequel. Method description and weak convergence theorems can be found in Section 3. Section 4 contains theorems about strong convergence. Section 5 is devoted to a regularized variant of the method.

2. SOME DEFINITIONS AND CONSTRUCTIONS

We shall use some facts and definitions from the operator theory [2], [8], [11], [14].

A Banach space X is said to be strictly convex if $\|x+y\|<2$ whenever $x, y\in X$, $y\neq x$, $\|y\|\leq 1$, $\|x\|\leq 1$, and uniformly convex if $\|x+y\|\leq 2-g(t)$ whenever $x, y\in X$, $x-y=t$, $\|y\|\leq 1$, $\|x\|\leq 1$ where function $g(t)$ is strictly increasing on $[0,2]$, $g(0)=0$.

A norm of the uniformly convex Banach space satisfies the well-known H-property: a weak convergence $x_n \xrightarrow{w} \tilde{x}$ and a convergence of norms $\|x_n\|\to\|\tilde{x}\|$ both imply a strong convergence $x_n \xrightarrow{s} \tilde{x}$. We shall say that the function $\phi: X\to R^1$ satisfies the H-property too if $x_n \xrightarrow{w} \tilde{x}$ and $\phi(x_n)\to\phi(\tilde{x})$ both imply $x_n \xrightarrow{s} \tilde{x}$.

Let the dual space X^* be strictly convex, $p>1$. The relations

$$\left\langle \Im_p(x), x\right\rangle = \|x\|^p, \quad \|\Im_p(x)\| = \|x\|^{p-1} \ \forall x\in X$$

define the mapping $\Im_p : X\to X^*$, called dual. We can similarly define the dual mapping $\Im_q^* : X^*\to X, q>1$. For uniformly convex spaces X, X^* we have $\Im_p = (\Im_q^*)^{-1}$, $1/p+1/q=1$.

Given an arbitrary mapping $F : X \to 2^{X^*}$, let us define in the usual way its effective domain $D(F) := \{x \in X : F(x) \neq \emptyset\}$ and its graph $Gr(F) := \{(x;u) : x \in D(F), u \in F(x)\}$.

The multi-valued mapping F is said to be

1) bounded over X if $\sup\limits_{x \in G} \sup\limits_{u \in F(x)} \|u\| < \infty$ for all bounded sets $G \subset X$;
2) monotone over X if $\langle u'-u'', x'-x'' \rangle \geq 0 \quad \forall (x';u'), \quad (x'';u'') \in Gr(F)$;
3) strictly monotone over X if $\langle u'-u'', x'-x'' \rangle > 0 \quad \forall (x';u'), \quad (x'';u'') \in Gr(F), \quad x' \neq x''$;
4) strongly monotone over X if there exists a $\mu > 0$ such that $\langle u'-u'', x'-x'' \rangle \geq \mu \|x'-x''\|^2 \quad \forall (x';u'), \quad (x'';u'') \in Gr(F)$;
5) coercive with respect to the unbounded domain $Q \subset X$ if there exists a vector $\breve{x} \in Q \cap D(F)$ such that

$$\lim_{x \in Q \cap D(F), \|x\| \to \infty} \inf_{u \in F(x)} \|x\|^{-1} \langle u, x - \breve{x} \rangle = \infty .$$

The mapping $F : X \to 2^{X^*}$ is said to be maximal monotone if its graph is not properly contained in a graph of any other monotone mapping. It is well-known that for any reflexive Banach space X and any maximal monotone mapping $F : X \to 2^{X^*}$ the set $\operatorname{cl} D(F)$ is convex.

The single-valued mapping $F : X \to X^*$ is said to be demi-continuous if strong convergence $x_n \xrightarrow{s} x$, where $x_n, x \in D(F)$, implies weak convergence $F(x_n) \xrightarrow{w} F(x)$. Demi-continuous monotone mappings defined on the whole space X are simple examples of maximal monotone mappings. In particular, if the space X and its dual are strongly convex then the mappings $\Im_p(\cdot)$ are strictly monotone and demi-continuous.

In the paper the following well-known facts are used:

1. Let a multi-valued mapping $F : X \to 2^{X^*}$ be maximal monotone and let $Q \cap \operatorname{int} D(F) \neq \emptyset$. A vector $\tilde{x}$ is a solution of $VI(F, Q)$ if and only if

   $\langle u, x - \tilde{x} \rangle \geq 0 \quad \forall x \in Q, \forall u \in F(x)$.

2. If in addition to the previous assumptions the mapping $F(\cdot)$ is coercive over an unbounded set Q, or if Q is bounded, then the solution set of $VI(F, Q)$ is nonempty. Moreover, if this mapping $F(\cdot)$ is strictly monotone then $VI(F, Q)$ has the unique solution point.

3. Using the definition of the algebraic sum of two sets, we can define the sum of two multi-valued mappings $F_1, F_2 : X \to 2^{X^*}$ as

$$(F_1 + F_2)(x) = F_1(x) + F_2(x) \quad \text{for} \quad x \in D(F_1 + F_2) = D(F_1) \cap D(F_2) \ .$$

If both mappings F_1, F_2 are maximal monotone and their effective domains have a common interior point then their sum is maximal monotone too.

3. THE METHOD AND ITS WEAK CONVERGENCE

The method we are going to propose for the problem (1)-(2) is based on the following assumptions.

Assumption 1. The parametric mapping $A(\cdot, y)$ satisfies the extended monotonicity condition, i.e. there exists a mapping $B(x,\cdot) : Y \to Y^*$, depending upon parameter $x \in X$, such that

$$\langle A(x'', y'') - A(x', y'), x'' - x' \rangle + \langle B(x'', y'') - B(x', y'), y'' - y' \rangle \geq 0$$

$$\forall x', x'' \in X, y', y'' \in Y \ .$$

In other words it means that the mapping $C : X \times Y \to X^* \times Y^*$ defined by $C(x, y) = A(x, y) \times B(x, y)$ must be monotone over $X \times Y$.

Note that under Assumption 1 all the mappings $B(x,\cdot)$ will be monotone over Y .

Next, let us define multi-valued auxiliary mapping $G_{AB} : X \to 2^{X^*}$ as

$$G_{AB}(x) = \bigcup_{y: A(x, y) = 0} B(x, y) \ . \tag{3}$$

One can verify [9], that under Assumption 1 this mapping will be monotone and $D(G_{AB}) = \{x \in X : G_{AB}(x) \neq \varnothing\} = \Omega$.

Assumption 2. The auxiliary mapping G_{AB} is maximal monotone and its effective domain $D(G_{AB})$ has interior points.

Assumption 2 provides the convexity of the set $\mathrm{cl}\Omega$ and makes it possible to formulate the existence results for the problem (1)-(2) as well as for auxiliary sub-problems involved with the method described below.

The method we present is based on exploiting the following systems of equations with a small parameter

$$A(x, y) = 0, \quad \alpha B(x, y) + F(x) = 0, \quad \alpha > 0 \ \text{ is a small parameter,} \tag{4}$$

and on the following generalized equation (inclusion)

$$\alpha G_{AB}(x) + F(x) \ni 0\,, \tag{5}$$

which is the result of convoluting the system (4) by y.

Directly from the definition of the auxiliary mappings (3) it follows that the problems (4), (5) are solvable or not simultaneously, and the solution set of the equation (5) is just a projection of the solution set of the system (4) onto the subspace of variables x, i.e. any solution $\tilde{x}$ of the inclusion (5) may be completed by some $\tilde{y} \in Y$ to form the solution of system (4) and, vice-versa, the left part of any solution $\tilde{x}; \tilde{y}$ of the system (4) is a solution of the inclusion (5).

Note that the system (4) is defined in a constructive way. To solve it one can apply many different methods. On the other hand, the inclusion (5) involves the auxiliary mapping $G_{AB}(\cdot)$ which has no constructive definition. This inclusion will be used only in the theoretical analysis of (4).

The following proposition describes the relations between the problems (4)-(5) and the original problem (1)-(2).

Lemma 1. Let the Assumptions 1, 2 hold and the mapping $F(\cdot)$ be demi-continuous over X (i.e. maximal monotone). If a sequence $\{x_n\}$ of any solution points of the inclusion (5), associated with $\alpha_n \to +0$, has a weak cluster point $\tilde{x}$ then this point solves the problem (1)-(2).

Proof: According to the assumed properties of $F(\cdot)$ and the convexity of $\mathrm{cl}\Omega$ it is sufficient to prove the relations $\langle F(x), x - \tilde{x} \rangle \geq 0 \quad \forall x \in \Omega$ (see Section 2). Fix an arbitrary $x \in \Omega (\equiv D(G_{AB})), g \in G_{AB}(x)$. From the relation (5) it follows that $g_n := -\alpha_n^{-1} F(x_n) \in G_{AB}(x_n)$. This inclusion and monotonicity of $F(\cdot)$ and $G_{AB}(\cdot)$ imply

$$\langle F(x), x - x_n \rangle \geq \langle F(x_n), x - x_n \rangle = \alpha_n \langle g_n, x_n - x \rangle \geq \alpha_n \langle g, x_n - x \rangle \geq -\alpha_n \|g\| \|x_n - x\|$$

Hence, passing to the limit as $n \to \infty$ one obtains the desired relation.♦

Lemma 1 leads us to the following questions:

1) which assumptions can provide existence of a solution of (1), (5) and guarantee that the solution sets of (5) are bounded for all sufficiently small $\alpha > 0$;

2) which assumptions can provide not only weak but also strong convergence of the method under investigation.

The answer to the first question is given by

Assumption 3. Let monotone mapping $F(\cdot)$ be defined over the whole space and be bounded, demi-continuous and coercive in the following sense: there exists an element $\breve{x}\in\Omega$ such that

$$\lim_{x\in\Omega,\,\|x\|\to\infty}\|x\|^{-1}\langle F(x),x-\breve{x}\rangle=\infty .$$

Indeed, from the previous section it is easy to see that Assumption 3 and Assumptions 1, 2 together guarantee the existence of a solution not only to the variational inequality (1)-(2), but to the generalized equation (inclusion) (5) too; it is sufficient to note that the sum of the mappings on the left-hand side of this inclusion will be maximal monotone and coercive.

Theorem 1. Suppose that Assumptions 1-3 hold. Then the generalized equation (5) is solvable and any sequence $\{x_n\}$ of its solution points, associated with $\alpha_n\to+0$, is bounded and all its weak cluster points solve the problem (1)-(2).

Proof: As it was already noted, under the Assumptions 1-3 the equations (5) are solvable for all $\alpha>0$. According to Lemma 1 it is sufficient to prove only that the sequence $\{x_n\}$ is bounded. Let us assume the contrary, that $\lim_{n\to\infty}\|x_n\|=\infty$. Choose some $\breve{g}\in G_{AB}(\breve{x})$. From (5) it follows that $g_n:=-\alpha_n^{-1}F(x_n)\in G_{AB}(x_n)$. Both this inclusion and the monotonicity of $G_{AB}(\cdot)$ imply

$$\langle F(x_n),x_n-\breve{x}\rangle=\alpha_n\langle g_n,\breve{x}-x_n\rangle\le\alpha_n\langle\breve{g},\breve{x}-x_n\rangle\le\alpha_n\|\breve{g}\|\,\|\breve{x}-x_n\|,$$

i.e.

$$\|x_n\|^{-1}\langle F(x_n),x_n-\breve{x}\rangle\le\alpha_n\|\breve{g}\|(1+\|\breve{x}\|\,\|x_n\|^{-1}).$$

Therefore, passing the limit as $n\to\infty$, one can obtain an obvious contradiction to Assumption 3. ♦

4. HOW TO GET STRONG CONVERGENCE

Let us present two results where strong convergence of the method under consideration is obtained.

Theorem 2. Suppose that Assumptions 1-2 hold and mapping $F(\cdot)$ is bounded, demi-continuous and strongly monotone on X with a constant $\mu>0$. If in the equations (5) one takes $\alpha_n\to+0$, then any corresponding sequence of solution points $\{x_n\}$ strongly converges to the unique solution point $\tilde{x}$ of the problem (1)-(2).

Proof: Since every strongly monotone mapping is coercive, the sequence $\{x_n\}$ weakly converges to the solution point $\tilde{x}$ of the problem (1)-(2) (see Theorem 1). From the strong monotonicity of $F(\cdot)$ it follows that

$$\mu\|x_n - \tilde{x}\|^2 \le \langle F(x_n) - F(\tilde{x}), x_n - \tilde{x}\rangle \le \langle F(x_n), x_n - \tilde{x}\rangle .$$

Let us choose an arbitrary $x^\sigma \in \Omega, \left\|x^\sigma - \tilde{x}\right\| < \sigma, \quad g^\sigma \in G_{AB}(x^\sigma), \quad \sigma > 0$. The relation (5) implies $g_n := -\alpha_n^{-1} F(x_n) \in G_{AB}(x_n)$. Hence

$$\begin{aligned} \mu\|x_n - \tilde{x}\|^2 &\le \langle F(x_n), x_n - \tilde{x}\rangle = \left\langle F(x_n), x_n - x^\sigma\right\rangle + \left\langle F(x_n), x^\sigma - \tilde{x}\right\rangle \le \\ &\le \left\langle F(x_n), x_n - x^\sigma\right\rangle + N\sigma = \alpha_n\left\langle g_n, x^\sigma - x_n\right\rangle + N\sigma \le \\ &\le \alpha_n\left\langle g^\sigma, x^\sigma - x_n\right\rangle + N\sigma \le \alpha_n\left\|g^\sigma\right\|\left\|x^\sigma - x_n\right\| + N\sigma; \end{aligned} \tag{6}$$

where $N > 0$ is some constant not dependent upon σ. Since $\{x_n\}$ is bounded, from the last inequality, passing to the limit as $n \to \infty$ one can conclude

$$\limsup_{(n)} \mu\|x_n - \tilde{x}\|^2 \le N\sigma .$$

Because $\sigma > 0$ is arbitrary, the proof is complete. ♦

Corollary 1: Suppose all the assumptions of Theorem 2 hold and $\tilde{x} \in \Omega$. Then

$$\|x_n - \tilde{x}\| \le \alpha_n \mu^{-1} \min_{g \in G_{AB}(\tilde{x})} \|g\| .$$

Proof: To obtain the desired inequality it is sufficient to replace in (6) vector x^σ by vector $\tilde{x}$ and g^σ by arbitrary $\tilde{g} \in G_{AB}(\tilde{x})$ (so $\sigma = 0$). ♦

The second result will be formulated for the sub-differential[2] mapping $F(\cdot) = \partial\phi(\cdot)$ of some convex function $\phi(\cdot)$, defined on X and satisfying the H-property.

Theorem 3. Let $F(\cdot) = \partial\phi(\cdot)$ be a sub-differential mapping for some convex function $\phi(\cdot)$ defined on X and satisfying the H-property. If this function is strictly convex and coercive (i.e. $\lim_{\|x\|\to\infty} \|x\|^{-1}\phi(x) = \infty$) and if Assumptions 1-2 hold, then the conclusion of Theorem 2 is valid.

Proof: It is clear that under the given assumptions the mapping $F(\cdot)$ is strictly monotone and satisfies Assumption 3. According to Theorem 1 the sequence $\{x_n\}$ weakly converges to the unique solution point $\tilde{x}$ of the problem (1)-(2). Since a convex function is weakly lower semi-continuous, one has

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[2] An element $g \in X$ is said to be a sub-gradient of a convex function $\phi(\cdot)$ at a point $a \in X$ if the inequality $\phi(b) - \phi(a) \ge \langle g, b - a\rangle$ holds for all $b \in X$. A set of all such g is said to be a sub-differential of $\phi(\cdot)$ at point a and is denoted by $\partial\phi(a)$.

$$\liminf_{(n)} \phi(x_n) \ge \phi(\tilde{x}) .$$

To establish the desired strong convergence let us prove the inverse relation (see definition of H-property)

$$\limsup_{(n)} \phi(x_n) \le \phi(\tilde{x}) .$$

Choose an arbitrary $x^{\varepsilon} \in \Omega, \quad \phi(x^{\varepsilon}) < \phi(\tilde{x}) + \varepsilon, \quad g^{\varepsilon} \in G_{AB}(x^{\varepsilon}), \quad \varepsilon > 0$. From (5) it follows that $g_n := -\alpha_n^{-1} F(x_n) \in G_{AB}(x_n)$. This inclusion together with the properties of the sub-differential mapping of a convex function and the monotonicity of $G_{AB}(\cdot)$ imply

$$\phi(x_n) \le \phi(x^{\varepsilon}) + \left\langle F(x_n), x_n - x^{\varepsilon} \right\rangle = \phi(x^{\varepsilon}) + \alpha_n \left\langle g_n, x^{\varepsilon} - x_n \right\rangle \le$$
$$\le \phi(x^{\varepsilon}) + \alpha_n \left\langle g^{\varepsilon}, x^{\varepsilon} - x_n \right\rangle < \phi(\tilde{x}) + \varepsilon + \alpha_n \left\| g^{\varepsilon} \right\| \left\| x^{\varepsilon} - x_n \right\| .$$

Hence, passing to the limit as $n \to \infty$ one gets

$$\limsup_{(n)} \phi(x_n) < \phi(\tilde{x}) + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. ♦

5. TIHONOV'S REGULARIZATON OF THE METHOD

Below the results are presented, where strong convergence is obtained by means of Tihonov's regularization of the system (4), (5).

Let us consider the regularized system (4)

$$A(x, y) = 0, \quad \alpha B(x, y) + \beta \Im_p(x) + F(x) = 0 , \tag{7}$$

and the regularized equation (5) corresponding to it, i.e.

$$\alpha G_{AB}(x) + \beta \Im_P(x) + F(x) \ni 0 \tag{8}$$

where $\alpha, \beta > 0$ are small parameters and $\Im_p(\cdot) : X \to X^*$ is a dual mapping mentioned above. Let us recall that

$$\left\langle \Im_p(x), x \right\rangle = \|x\|^p, \ \left\| \Im_p(x) \right\| = \|x\|^{p-1} \ \ \forall x \in X \quad (p > 1) .$$

As before, the system (7) and inclusion (8) are solvable or not simultaneously, and the solution set of (8) is just the projection of the solution set of (7) onto the subspace of variables x. The properties of the dual mapping $F(\cdot)$ (see Section 2) make it possible to assert the existence of a solution of the systems (7), (8) for all $\alpha, \beta > 0$ without any coercive assumptions for the mapping $F(\cdot)$. Nevertheless, Assumption 3 will be used below to provide solvability of the original problem (1)-(2) and guarantee that the solution sets of inclusions (8) are bounded.

Lemma 2. Suppose that all the assumptions of Lemma 1 hold. If a sequence $\{x_n\}$ of solution points of the inclusion (8) with $\alpha_n \to +0$, $\beta_n \to +0$ has a weak cluster point $\tilde{x}$, this point solves the problem (1)-(2).

Proof: Obviously, it is sufficient to prove that

$$\langle F(x), x-\tilde{x}\rangle \geq 0 \quad \forall x \in \Omega .$$

As before, choose an arbitrary $x \in \Omega (\equiv D(G_{AB}))$, $g \in G_{AB}(x)$. From the relation (8) it follows that $g_n := -\alpha_n^{-1}(F(x_n)+\beta_n \Im_p(x_n)) \in G_{AB}(x_n)$. These inclusions and the monotonicity of $F(\cdot)$, $\Im_p(\cdot)$ and $G_{AB}(\cdot)$ imply

$$\langle F(x), x-x_n\rangle \geq \langle F(x_n), x-x_n\rangle = \alpha_n \langle g_n, x_n - x\rangle + \beta_n \langle \Im_p(x_n), x_n - x\rangle \geq$$
$$\geq \alpha_n \langle g, x_n - x\rangle + \beta_n \langle \Im_p(x_n), x_n - x\rangle \geq$$
$$\geq -\alpha_n \|g\| \, \|x_n - x\| - \beta_n \|x\|^{\,p-1} \|x_n - x\| .$$

Hence passing to the limit as $n \to \infty$, one obtains the desired relation.♦

Lemma 3. Suppose that Assumptions 1-3 hold. Then the generalized equation (8) is solvable and, if $\alpha_n \to +0$, $\beta_n \to +0$, then any corresponding sequence of its solutions $\{x_n\}$ is bounded.

Proof: The proof is similar to the scheme applied in Theorem 1. Let us assume the contrary, that $\lim\limits_{n\to\infty} \|x_n\| = \infty$. Choose some $\breve{g} \in G_{AB}(\breve{x})$. From the relation (8) it follows that $g_n := -\alpha_n^{-1}(F(x_n)+\beta_n \Im_n(x_n)) \in G_{AB}(x_n)$. This fact and monotonicity of $\Im_p(\cdot)$, $G_{AB}(\cdot)$ imply

$$\langle F(x_n), x_n - \breve{x}\rangle = \alpha_n \langle g_n, \breve{x} - x_n\rangle + \beta_n \langle \Im_p(x_n), \breve{x} - x_n\rangle \leq$$
$$\leq \alpha_n \langle \breve{g}, \breve{x} - x_n\rangle + \beta_n \langle \Im_p(\breve{x}), \breve{x} - x_n\rangle \leq (\alpha_n \|\breve{g}\| + \beta_n \|\breve{x}\|^{-1}) \|\breve{x} - x_n\| ,$$

i.e.

$$\|x_n\|^{-1} \langle F(x_n), x_n - \breve{x}\rangle \leq (\alpha_n \|\breve{g}\| + \beta \|\breve{x}\|^{p-1}) (1+\|\breve{x}\| \, \|x_n\|^{-1}) .$$

Passing to the limit as $n \to \infty$ here, we get an obvious contradiction with Assumption 3. ♦

In what follows we need

Assumption 4. Suppose that a solution point of the problem (1)-(2) with the minimal norm belongs to Ω.

The proof of the following proposition is based on the H-property of the dual mapping.

Theorem 4. Suppose that Assumptions 1-4 hold, space X is uniformly convex and $\{x_n\}$ is an arbitrary sequence of solution points of the equations (8), associated with $\alpha_n \to +0$, $\beta_n \to +0$, where $r_n := \alpha_n \beta_n^{-1} \to +0$. Then $x_n \xrightarrow{s} \tilde{x}$, where $\tilde{x}$ is a solution of the problem (1)-(2) with the minimal norm.

Proof: The existence of weak cluster points of the sequence $\{x_n\}$ is established by Lemma 3. Let $\tilde{x}$ be one of them. According to Lemma 2 this point solves (1)-(2). Since the norm (as the convex function) is lower semi-continuous, we have

$$\liminf_{(n)} \|x_n\| \ge \|\overline{x}\| \ge \|\tilde{x}\| .$$

Let us prove the inverse inequality

$$\limsup_{(n)} \|x_n\| \le \|\tilde{x}\| \le \|\overline{x}\| .$$

As before, choose an arbitrary $\tilde{g} \in G_{AB}(\tilde{x})$. From (8) it follows that $g_n := -\alpha_n^{-1}(F(x_n) + \beta_n \Im_p(x_n)) \in G_{AB}(x_n)$. Taking into account the properties of the inequality (1)-(2), the properties of the dual mapping and the properties of mapping $G_{AB}(\cdot)$, we can write

$$\begin{aligned}
&\|x_n\|^p = \langle \Im_p(x_n), \tilde{x} \rangle + \langle \Im_p(x_n), x_n - \tilde{x} \rangle = \\
&= \langle \Im_p(x_n), \tilde{x} \rangle + \beta_n^{-1} \langle F(x_n), \tilde{x} - x_n \rangle + r_n \langle g_n, \tilde{x} - x_n \rangle \le \\
&\le \|\Im_p(x_n)\| \|\tilde{x}\| + r_n \langle \tilde{g}, \tilde{x} - x_n \rangle \le \\
&\le \|x_n\|^{p-1} \|\tilde{x}\| + r_n \|\tilde{g}\| \|\tilde{x} - x_n\| \le \|x_n\|^{p-1} \|\tilde{x}\| + N r_n ,
\end{aligned}$$

where $N > 0$ is some constant, not dependent upon n. Therefore any separated from zero subsequence $\{x_{n_k}\}$ satisfies

$$\|x_{n_k}\| \le \|\tilde{x}\| + N r_n \|x_{n_k}\|^{1-p}$$

(if all sub-sequences converge to zero then the proof is immediate). Passing to the limit as $n_k \to \infty$, we can conclude that

$$\limsup_{(k)} \|x_{n_k}\| \le \|\tilde{x}\| .$$

Hence, every weak cluster point appears to be a strong cluster point and its norm is equal to the norm of $\tilde{x}$, being the projection of zero onto the convex closed solution set of (1)-(2). Since under our assumptions this projection is unique, all weak cluster points are the same and are equal to $\tilde{x}$. ♦

Assumption 4 looks a bit difficult. But, as simple examples show, when dropping it we will need to correlate the speed of decrease of $r_n = \alpha_n \beta_n^{-1}$ with the speed of increase of $\|g_n\|$ for $\{x_n\}$ converging to boundary points of Ω which are not properly in this set.

Assumption 5. Suppose that the a priori estimation is known

$$\forall\sigma > 0 \quad \exists x^{\sigma} \in \Omega \quad \exists g^{\sigma} \in G_{AB}(x^{\sigma}) : \left\|x^{\sigma} - \tilde{x}\right\| < \sigma, \quad \left\|g^{\sigma}\right\| < \omega(\sigma);$$

where $\omega(\cdot)$ is a scalar function defined for a positive argument, $\omega(\sigma) \to +\infty$ whenever $\sigma \to +0$, $\tilde{x}$ is the minimal (with respect to norm) solution point (1)-(2).

The following proposition is valid (a new parameter $\gamma > 0$ plays an auxiliary role).

Theorem 5. Suppose that Assumptions 1-3, 5 hold, space X is uniformly convex and $\{x_n\}$ is some sequence of solution points to (8). If $\alpha_n \to +0$, $\beta_n \to +0$, $\gamma_n \to +0$ and $r_n^{'} := \gamma_n \beta_n^{-1} \to 0$, $r_n^{''} := \alpha_n \beta_n^{-1} \omega(\gamma_n) \to 0$, then $x_n \xrightarrow{s} \tilde{x}$.

Proof: The proof corresponds to the previous schemes. The main difference from the proof of Theorem 4 is as follows. When proving the inequality $\limsup_{(n)} \|x_n\| \le \|\tilde{x}\|$ we now have to apply auxiliary sequence $\{\tilde{x}_n\} \subset \Omega$ and $\{\tilde{g}_n\} \subset G_{AB}(\tilde{x}_n)$ such that $\|\tilde{x} - \tilde{x}_n\| < \gamma_n$, $\|\tilde{g}_n\| < \omega(\gamma_n)$. They exist due to Assumption 5.

As before, from (8) it follows that $g_n := -\alpha_n^{-1}(F(x_n) + \beta_n \Im_p(x_n)) \in G_{AB}(x_n)$. Taking into account the properties of the variational inequality (1)-(2), the properties of the dual mapping and mapping $G_{AB}(\cdot)$, as well as the fact that the sequence $\alpha_n g_n$ is bounded, we have

$$\begin{aligned}
&\|x_n\|^p = \left\langle \Im_p(x_n), \tilde{x} \right\rangle + \left\langle \Im_p(x_n), x_n - \tilde{x} \right\rangle = \\
&= \left\langle \Im_p(x_n), \tilde{x} \right\rangle + \beta_n^{-1} \left\langle F(x_n), \tilde{x} - x_n \right\rangle + \alpha_n \beta_n^{-1} \left\langle g_n, \tilde{x} - x_n \right\rangle \le \\
&\le \left\|\Im_p(x_n)\right\| \|\tilde{x}\| + \alpha_n \beta_n^{-1} \left\langle g_n, \tilde{x} - \tilde{x}_n \right\rangle + \alpha_n \beta_n^{-1} \left\langle g_n, \tilde{x}_n - x_n \right\rangle \le \\
&\le \left\|\Im_p(x_n)\right\| \|\tilde{x}\| + \alpha_n \beta_n^{-1} \left\langle g_n, \tilde{x} - \tilde{x}_n \right\rangle + \alpha_n \beta_n^{-1} \left\langle \tilde{g}_n, \tilde{x}_n - x_n \right\rangle \le \\
&\le \left\|\Im_p(x_n)\right\| \|\tilde{x}\| + \alpha_n \beta_n^{-1} \|g_n\| \|\tilde{x} - \tilde{x}_n\| + \alpha_n \beta_n^{-1} \|\tilde{g}_n\| \|\tilde{x}_n - x_n\| \le \\
&\le \left\|\Im_p(x_n)\right\| \|\tilde{x}\| + \alpha_n \beta_n^{-1} \gamma_n \|g_n\| + \alpha_n \beta_n^{-1} \omega(\gamma_n) \|\tilde{x}_n - x_n\| \le \\
&\le \|x_n\|^{p-1} \|\tilde{x}\| + N r_n^{'} + M r_n^{''},
\end{aligned}$$

where $N > 0$, $M > 0$ are some constants, not depending upon n. The last inequality shows that every separated from zero subsequence $\{x_{n_k}\}$ (if any exists) satisfies

$$\left\|x_{n_k}\right\| \le \|\tilde{x}\| + (N r_n' + M r_n'')\left\|x_{n_k}\right\|^{1-p}.$$

Hence, passing to the limit as $n_k \to \infty$, we get

$$\limsup_{(k)} \left\|x_{n_k}\right\| \le \|\tilde{x}\|.$$

The rest of the proof is the same as the proof of Theorem 4 and is omitted here ♦

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