Yugoslav Journal of Operations Research
11 (2001), Number 1 15-30

A NEW AND CONSTRUCTIVE PROOF OF TWO BASIC RESULTS OF LINEAR PROGRAMMING*

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Abstract: In this paper a new, elementary and constructive proof of Farkas' lemma is given. The basic idea of the proof is extended to derive the strong duality theorem of linear programming. Zhang's algorithms used, in the proofs of Farkas' lemma and the strong duality theorem, are criss-cross type algorithms, but the pivot rules give more flexibility than the original criss-cross rule of T. Terlaky. The proof of the finiteness of the second algorithm is technically more complicated than that for the original criss-cross algorithm.

Both of the algorithms defined in this paper have all the nice theoretical characteristics of the criss-cross method, i.e. they solve the linear programming problem in one phase; they can be initialized with any, not necessarily primal feasible basis, bases generated during the solution of the problem, are not necessarily primal or dual feasible.

Keywords: Farkas lemma, strong duality theorem, criss-cross type pivot rules.

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* 1991 Mathematics Subject Classification. 90C05.

The first author was supported in part by the Hungarian Ministry of Education under the grant FKFP No. 0152/1999 and by the Hungarian National Research Fund under the grant OTKA TO29775.

The first version of this paper was prepared during the visit of the first author to W•rzburg University, Institute of Applied Mathematics and Statistics, sponsored by a scholarship from the Deutscher Akademischer Austaushdienst, DAAD A/98/07349.

1. INTRODUCTION

On the 150th anniversary of the birth of Gyula Farkas in 1997, S. Zhang [14] published two new and finite pivot algorithms for solving linear programming problems. Zhang's algorithms are generalizations of Terlaky's Criss-cross method [11, 12, 13].[1] Klafszky and Terlaky [7,8] gave a constructive proof to the well-known lemma of Gy. Farkas [2, 3].

Using the first algorithm (FILO/LOFI rule) of Zhang [14, 15] and the so-called orthogonality theorem (see for instance [7, 8, 5, 6]) we give herein a constructive proof to Farkas' lemma in a similar way as Klafszky and Terlaky did in their papers [7, 8]. This kind of constructive proof can be extended to verify the well-known strong duality theorem. We use Zhang's second algorithm with the most-often selected rule [14, 15]. Our proofs of the finiteness of Zhang's algorithms are simpler than the original one.

Let $A \in \mathbf{R}^{m\times n}$, $\mathbf{c}, \mathbf{x} \in \mathbf{R}^n$, $\mathbf{y}, \mathbf{b} \in \mathbf{R}^m$ and $I = \{1,2,\dots,n\}$. Without loss of generality we may assume that the rank of A is m, thus A has full row rank. Let $\mathbf{a}^{(i)} \in \mathbf{R}^n$ denote the i^{th} row vector of the matrix A, while $\mathbf{a}_j \in \mathbf{R}^m$ denotes the j^{th} column vector of it. In our paper the following form of the Farkas lemma is proved in Section 2.

Theorem 1.1. (Farkas' lemma) From the following two systems of linear inequalities exactly one is solvable:

$$\left.\begin{array}{l} A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{array}\right\} \quad (A_1) \qquad\qquad \left.\begin{array}{l} \mathbf{y}^T A \leq 0 \\ \mathbf{y}^T \mathbf{b} = 1 \end{array}\right\} \quad (A_2)$$

Our second goal in this note is to give constructive proof for the strong duality theorem of the linear programming problem (Section 3).

Now, let us consider the primal and dual linear programming problems in the following form

$$\left.\begin{array}{l} \min \mathbf{c}^T \mathbf{x} \\ A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{array}\right\} \quad (P) \qquad\qquad \left.\begin{array}{l} \max \mathbf{y}^T \mathbf{b} \\ \mathbf{y}^T A \leq \mathbf{c} \end{array}\right\} \quad (D)$$

Furthermore, let P be the set of primal feasible solutions[2], namely

$$P := \{\mathbf{x} \in \mathbf{R}^n_{\oplus} \mid A\mathbf{x} = \mathbf{b})$$

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[1] Zhang [14, 15] proved the finiteness of one of his algorithm, following the steps of Terlaky's original proof [11, 12].

[2] The $\mathbf{R}^n_{\oplus}$ is the positive orthant, thus $\mathbf{R}^n_{\oplus} = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \geq 0\}$.

and let the set of dual feasible solutions, D, be given as

$$D = \{\mathbf{y} \in \mathbf{R}^m \mid \mathbf{y}^T A \le \mathbf{c}\}.$$

Theorem 1.2. (Strong duality theorem) From the following two statements exactly one holds:

(1) There exists $\hat{\mathbf{x}} \in P$ and $\hat{\mathbf{y}} \in D$ such that $\mathbf{c}^T \hat{\mathbf{x}} = \hat{\mathbf{y}}^T \mathbf{b}$.
(2) $P = \emptyset$ or $D = \emptyset$.

Let us introduce the *(*primal*)* pivot tableau for the (primal) linear programming problem, as follows

A	$\mathbf{b}$
$\mathbf{c}^T$	*

where all the data related to the problem are arranged. Under the assumption that matrix A has full row rank, there exists an $m \times m$ nonsingular submatrix A_B of A. Let us interchange the columns of A to obtain the following partition $A = (A_B, A_N)$, where the submatrix A_N contains those columns of A which do not belong to A_B. Now the linear system $A\mathbf{x} = \mathbf{b}$ can be written as $A_B \mathbf{x}_B + A_N \mathbf{x}_N = \mathbf{b}$, where we group the unknowns in the same way as the columns of matrix A, namely $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$. Similarly, we can reorder the components of the vector $\mathbf{c}$ as $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$.

Now we are ready to restate some well-known concepts of linear algebra and linear programming such as basis, basic solution, feasible basic solution, optimal solution and orthogonality.

Definition 1.3.

1. Any $m \times m$ nonsingular submatrix A_B of A is called a basis.
2. The $\mathbf{x}_B = A_B^{-1}\mathbf{b}$, $\mathbf{x}_N = \mathbf{0}$ is a basic solution of $A\mathbf{x} = \mathbf{b}$ for a given A_B.
3. Variables grouped in $\mathbf{x}_B$ are called basic variables, while those corresponding to $\mathbf{x}_N$ are called nonbasic variables.
4. If $A_B^{-1}\mathbf{b} \ge \mathbf{0}$ then we say that $(\mathbf{x}_B, \mathbf{x}_N)$ is a (primal) feasible solution and A_B is a (primal) feasible basis.
5. The vector $\mathbf{y} = (\mathbf{c}_B^T A_B^{-1})^T \in \mathbf{R}^m$ is called a dual basic solution.
6. If $\mathbf{c}_B^T A_B^{-1} A \le \mathbf{c}$ holds then A_B is said to be a dual feasible basis.
7. The primal feasible solution $\overline{\mathbf{x}} \in P$ is said to be an optimal solution of the primal problem, if $\mathbf{c}^T \overline{\mathbf{x}} \le \mathbf{c}^T \mathbf{x}$ holds for all $\mathbf{x} \in P$.

8. The dual feasible solution $\overline{\mathbf{y}} \in D$ is said to be an optimal solution of the dual problem, if $\mathbf{b}^T \overline{\mathbf{y}} \geq \mathbf{b}^T \mathbf{y}$ holds for all $\mathbf{y} \in D$.

The (primal) pivot tableau corresponding to the basis A_B for the LP problem[3] is the following

$A_B^{-1} A$	$A_B^{-1} \mathbf{b}$
$\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A$	$-\mathbf{c}_B^T A_B^{-1} \mathbf{b}$

and let us introduce the following notations

$$T = A_B^{-1} A, \quad \overline{\mathbf{b}} = A_B^{-1} \mathbf{b}, \quad \overline{\mathbf{c}}^T = \mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A, \text{ and } z = -\mathbf{c}_B^T A_B^{-1} \mathbf{b}.$$

The set of basic indices corresponding to the basis A_B is denoted by B, while the set of nonbasic indices is denoted by N. Trivially, $I = B \cup N$.

We need the concept of orthogonality among vectors.

Definition 1.4. Let $\mathbf{a}, \mathbf{b} \in \mathbf{R}^k$ then vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal, if $\mathbf{a}^T \mathbf{b} = 0$.

Using the pivot tableau we can introduce the following n-dimensional (column) vectors:

$$\mathbf{t}^{(i)} = (\mathbf{t}_k^{(i)})_{k=1}^n = \begin{cases} t_{ik}, & \text{if } k \in N \\ 1, & \text{if } k = i \\ 0, & \text{if } k \in B, k \neq i \end{cases}$$

and

$$\mathbf{t}_j = (\mathbf{t}_{(j)k})_{k=1}^n = \begin{cases} t_{kj}, & \text{if } k \in B \\ -1, & \text{if } k = j \\ 0, & \text{if } k \in N, k \neq j \end{cases}$$

where $\mathbf{t}^{(i)}$, $i \in B$ is equal to the i^{th} row of T, while $\mathbf{t}_j$, $j \in N$ is formed from the j^{th} column of T extended by an $(n-m)$ - dimensional negative unity vector.[4]

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[3] For the system (A_1) the pivot tableau corresponding to the basis of A_B is simpler as you may see:

$A_B^{-1} A$	$A_B^{-1} \mathbf{b}$

The following useful observation is called the orthogonality theorem (see for instance [7, 8, 6]).

Proposition 1.5. Let a linear system $A\mathbf{x} = \mathbf{b}$ be given. Furthermore, $A_{B'}$ and $A_{B''}$ are bases of the linear system, then

$$(\mathbf{t}'^{(i)})^T \mathbf{t}''_j = 0 \text{ for all } i \in B' \text{ and for all } j \in N'',$$

holds, where B' and B'' are the index sets corresponding to the bases $A_{B'}$ and $A_{B''}$, respectively.

Theorem 1.1 is proved in Section 2. First we define an algorithm to solve the system (A_1) and prove its finiteness. The algorithm either solves (A_1) or gives a certificate for the nonexistence of a solution. In this second case, using elementary computations, we can compute the solution of system (A_2). The solvability of the LP problem (P) is discussed in Section 3. A pivot algorithm is defined using the (most-often-selected variable) pivot rule of Zhang, [14, 15]. The finiteness of this second algorithm is proved. The strong duality theorem, Theorem 1.2., is obtained as an easy consequence of the finiteness of the algorithm.

Both of the presented algorithms have the general property of the criss-cross method [4], namely that the system (A_1) is solved without introducing artificial variables and using the so-called first phase objective function (or other techniques like the big-M method, [9]). Consequently, we do not need two phases to solve problem (P), because the algorithm can be initiated by any (not necessarily primal feasible) basis.

Our proofs are purely combinatorial, therefore the only information that is used is the sign of the entries of the pivot tableau. Thus, we use the Balinski-Tucker [1] notation which is very convenient for our purposes. Positive, nonnegative, negative and nonpositive numbers are denoted by $+$, $\oplus$, $-$, $\ominus$ signs, respectively. If an entry in the tableau is denoted by $*$ then there is no information about the sign of that entry.

2. PROOF OF THE FARKAS LEMMA

First, let us deal with the solution of the system (A_1). We introduce the following mapping $\mathbf{u}_r : I \to \mathbf{N}_0$, and let $\mathbf{u}_0 = (0,0,\ldots,0)$ and

$$u_r(i) = \begin{cases} r, & \text{if the } i^{th} \text{ variable moves in the } r^{th} \text{ iteration} \\ u_{r-1}(i), & \text{otherwise} \end{cases}$$

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[4] The vector $\mathbf{t}_j$ is a column of the dual simplex tableau as it is defined in [10].

for $r=1,2,\dots,k$. It is easy to show that $\mathbf{u}_r \geq \mathbf{u}_{r-1}$ and $\mathbf{u}_r \neq \mathbf{u}_{r-1}$.

The basic idea of the pivot rule is the following: from the infeasible variables choose the one to leave the current basis which entered most recently and from those which are candidates to enter the basis choose the one which has left the basis most recently.

Algorithm 2.1.

Let a basis A_B for the system (A_1) be given with the corresponding pivot tableau. Let $r=1$.

Step 1. Let $J := \{i \in B : x_i < 0\}$.

If $J = \emptyset$ **then** the system (A_1) is solved, STOP

else let $J_{\max} := \{ j \in J : u_{r-1}(j) \geq u_{r-1}(i),\ \text{for all } i \in J\}$

choose an arbitrary index $k \in J_{\max}$ and go to Step 2.

Step 2. Let $K := \{ j \in N : t_{kj} < 0\}$.

If $K = \emptyset$ **then** the system (A_1) is solved, STOP

else let $K_{\max} := \{ j \in K : u_{r-1}(j) \geq u_{r-1}(i),\ \text{for all } i \in K\}$,

choose an arbitrary index $l \in K_{\max}$ and go to Step 3.

Step 3. Now, x_k leaves and x_l enters the current basis.

Let us update the vector $\mathbf{u}$ as follows

$$u_r(i) = \begin{cases} r, & \text{if } i=k \text{ or } i=l \\ u_{r-1}(i), & \text{otherwise} \end{cases}$$

Increase the value of r by 1, namely $r := r+1$ and go to Step 1.

Let us extend the definition of the vectors $\mathbf{t}^{(i)}$ and $\mathbf{t}_j$ to the column of $\mathbf{b}$, as well. From now on, we assume that the index b (which belongs to the column vector $\mathbf{b}$) is always in the set of N. Now we can apply the orthogonality theorem for the matrix $[A, \mathbf{b}]$ and then the vectors $\mathbf{t}^{(i)}$ and $\mathbf{t}_j$ become $n+1$ - tuples.

Lemma 2.2. Algorithm 2.1 is finite.

Proof: By contradiction. Let us assume that the algorithm is not finite. This means that there exists (at least) one example in which the algorithm is not finite, thus it generates an infinite sequence of pivot tableaus, i.e. infinite sequence of bases. But the number of all possible bases for a given problem (with an $m \times n$ matrix A) is finite (at most $\binom{n}{m}$, therefore some of the bases should occur infinitely many times.[5] From those examples for which cycling occurs choose one with the smallest possible size. For such problems all the variables have to change their basis status during a cycle.

Let us consider the sequence of pivot tableaus generated by the algorithm and let us denote by $T_{B'}$ that which satisfies the following criteria:

- there is a variable x_q which changes its basic status for the first time;
- after this pivot tableau all the variables have changed their basic status at least once.

We have two choices: the variable x_q either enters or leaves the bases at the pivot tableau $T_{B'}$. It follows from our counter assumption that there should be another basic tableau, $T_{B''}$ such that the variable x_q will change its basic status for the first time since $T_{B'}$. Let us analyze the (sign) structure of the pivot tableaus $T_{B'}$ and $T_{B''}$.

Case 1. Let us deal with the case when the variable x_q leaves the pivot tableau $T_{B'}$ and enters $T_{B''}$. Then the sign structure of the pivot tableaus $T_{B'}$ and $T_{B''}$ are as follows

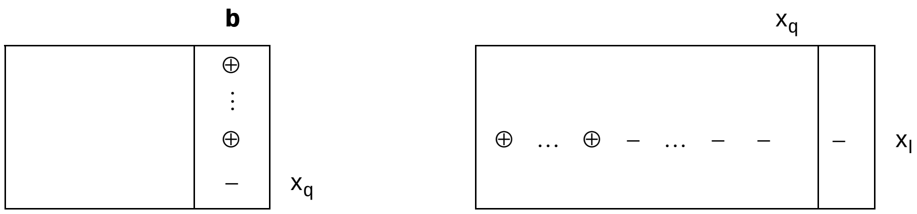

Figure 1.

Using the orthogonality theorem (Proposition 1.5) the vectors $\mathbf{t}'_b$ and $\mathbf{t}''^{(l)}$ are orthogonal. On the other hand, based on the pivot rule of Algorithm 2.1 if $t''_{li} < 0$, for some $i \in N'' \setminus \{q, b\}$ then $u_{r-1}(i) < u_{r-1}(q)$, means that the variable x_i did not change its basic status since the pivot tableau $T_{B'}$, therefore $i \in N'$, so $t'_{ib} = 0$. From this observation we may get that

$$0 = (\mathbf{t}''^{(l)})^T \mathbf{t}'_b \geq t''_{lb} t'_{bb} + t''_{lq} t'_{qb} > 0,$$

because $t'_{bb} = -1$, $t'_{qb} < 0$, $t''_{lb} < 0$ and $t''_{lq} < 0$. A contradiction is obtained.

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[5] This phenomena is known in the literature as cycling, see for instance [9, 10, 5, 6].

Case 2. Let us assume that the variable x_q enters the basis at pivot tableau $T_{B'}$ and leaves at $T_{B''}$.

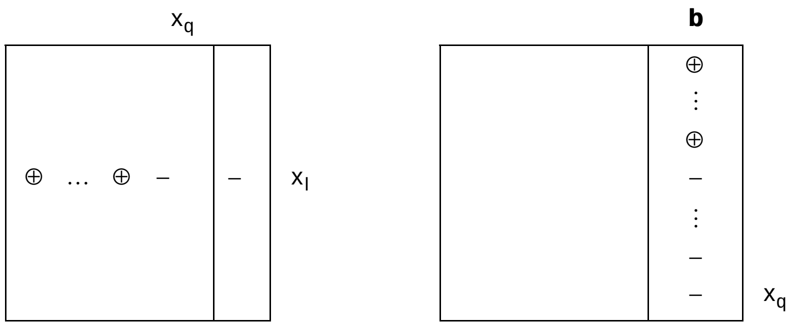

Figure 2.

Taking into consideration the sign structure of these tableaus, the pivot rule of Algorithm 2.1. and using the orthogonality theorem (Proposition 1.5), as in the previous case, we can show that both $T_{B'}$ and $T_{B''}$ cannot occur in the sequence of pivot tableaus generated by Algorithm 2.1. Therefore Algorithm 2.1. is not cycling.

Now, we are ready to prove the Farkas' Lemma.

Proof of Theorem 1.1: Let us assume that both (A_1) and (A_2) have a solution, then from $A\mathbf{x} = \mathbf{b}$ it follows that $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b}$. Taking into consideration that

$$\mathbf{y}^T A \leq \mathbf{0} \text{ and } \mathbf{x} \geq \mathbf{0}$$

it follows that

$$0 \geq \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b} = 1 ,$$

because $\mathbf{y}^T \mathbf{b} = 1$ holds. This is a contradiction, thus both systems cannot have a solution.

We need to show that one of the systems is solvable. Let us apply Algorithm 2.1. to system (A_1). According to Lemma 2.2. Algorithm 2.1. is finite, therefore it either stops in Step 1 with $J = \emptyset$, which means that a (basic) feasible solution of the system (A_1) is found, or it reports that $K = \emptyset$ (Step 2), thus we obtain a pivot tableau such that $\mathbf{t}^{(k)} \geq \mathbf{0}$ and $\overline{b}_k < 0$.[6] Now it is obvious that the following system of inequalities

[6] This is known from the literature as primal infeasibility criteria, see for instance [9, 10, 5, 6].

$$(\mathbf{t}^{(k)})^T \mathbf{x} = \overline{\mathbf{b}}_k, \quad \mathbf{x} \geq \mathbf{0} \tag{1}$$

has no solution. Therefore the system (A_1) cannot have a solution. Using the corresponding basis we can compute a solution[7] of the system (A_2) as

$$\mathbf{y} = \frac{1}{\overline{b}_k}((\mathbf{e}_k)^T A_B^{-1})^T ,$$

where $\mathbf{e}_k \in \mathbf{R}^m$ is the k^{th} unity vector.

3. PROOF OF THE STRONG DUALITY THEOREM

Let us consider the primal linear programming problem, (P). Let us introduce the following mapping $\mathbf{v}_r : I \to \mathbf{N}_0$, and let $\mathbf{v}_0 = (0,0,\ldots,0)$, furthermore

$$v_r(i) = \begin{cases} v_{r-1}(i) + 1, & \text{if the } i^{th} \text{ variable moves in the } r^{th} \text{ iteration} \\ v_{r-1}(i), & \text{otherwise.} \end{cases}$$

Vector $\mathbf{v}_r$ counts how many times the variables have changed their basic status until the end of the r^{th} iteration.

Algorithm 3.1.

Let a basis A_B of the primal linear programming problem (P) be given with the corresponding pivot tableau and let $r = 1$.

Step 1. Let $J := \{i \in I : x_i < 0 \text{ or } \overline{c}_i < 0\}$.

If $J = \emptyset$ **then** the current basic solution is optimal, STOP,

else let $J_{max} := \{ j \in J : v_{r-1}(j) \geq v_{r-1}(i), \quad \forall i \in J\}$ and choose an arbitrary index $k \in J_{max}$.

Step 2. (a) Primal iteration: $k \in N$.

Define the set of indices $K_P = \{i \in B : t_{ik} > 0\}$.

If $K_p = \emptyset$ **then** $D = \emptyset$, there is no dual feasible solution, STOP,

else let $K_{P,max} := \{ j \in K_P : v_{r-1}(j) \geq v_{r-1}(i), \quad \forall i \in K_P\}$

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[7] Using the orthogonality theorem (Proposition 1.5) it is easy to check that if the current pivot tableau is denoted by T and the corresponding basis by A_B then

$$t_{kj} = (\mathbf{z}_k)^T \mathbf{a}_j \text{ and } \overline{b}_k = (\mathbf{z}_k)^T \mathbf{b},$$

where $\mathbf{z}_k = ((\mathbf{e}_k)^T A_B^{-1})^T$ and $\mathbf{e}_k \in \mathbf{R}^m$ is the k^{th} unity vector. See for instance [7, 8, 5, 6].

and choose an arbitrary index $l \in K_{P,\max}$.

Now, x_l leaves the basis, while x_k enters it, and

$$v_r(i) = \begin{cases} v_{r-1}(i)+1, & \text{if } i=k \text{ or } i=l \\ v_{r-1}(i), & \text{otherwise.} \end{cases}$$

Increase the value of r by 1 and go to Step 1.

(b) Dual iteration: $k \in B$.

Define the set of indices $K_D := \{i \in N : t_{ki} < 0\}$.

If $K_D = \emptyset$ **then** $P = \emptyset$, there is no primal feasible solution, STOP,

else let $K_{D,\max} := \{ j \in K_D : v_{r-1}(j) \geq v_{r-1}(i), \quad \forall i \in K_D \}$

and choose an arbitrary index $l \in K_{D,\max}$.

Now, x_k leaves the basis, while x_l enters it, and

$$v_r(i) = \begin{cases} v_{r-1}(i)+1, & \text{if } i=k \text{ or } i=l \\ v_{r-1}(i), & \text{otherwise.} \end{cases}$$

Increase the value of r by 1 and go to Step 1.

The most often selected infeasible variable is chosen by the pivot rule of the algorithm in Step 1. Using exactly the same rule in Step 2 from the candidate variables, the most-often selected is chosen again. If we have more than one candidate either in Step 1 (elements of $J_{\max}$) or in Step 2 (in case (a) the elements of $K_{P,\max}$ and in case (b) the elements of $K_{D,\max}$) then we may choose from them arbitrarily.

The finiteness of the algorithm will be proved using the orthogonality theorem (Proposition 1.5). In the case of linear programming, the vectors $\mathbf{t}^{(i)} \in \mathbf{R}^{n+1}$ and $\mathbf{t}_j \in \mathbf{R}^{n+1}$, furthermore $\mathbf{t}^{(i)}$ belongs to the row space, while $\mathbf{t}_j$ belongs to the null space of the following matrix

$$\begin{pmatrix} A & \mathbf{b} \\ \mathbf{c}^T & 0 \end{pmatrix}.$$

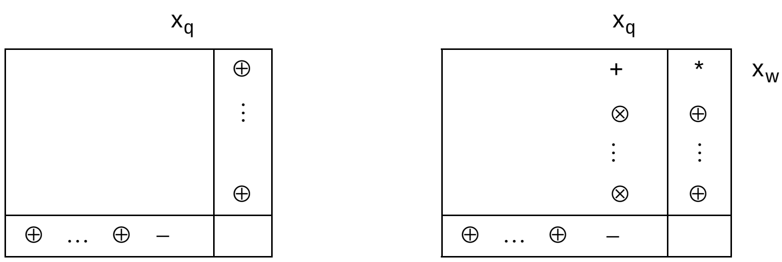
From now on we assume that the index c (which belongs to the row vector $\mathbf{c}$) is always in the set of B.

Lemma 3.2. The Algorithm 3.1. is finite.

Proof: The proof of this lemma is very similar to the proof of Lemma 2.2.

Let us assume to the contrary, that the Algorithm 3.1. is not finite. But the number of possible bases is finite, therefore at least one basis should be repeated infinitely many times. Thus cycling must occur. From those examples where cycling occurs choose one with the smallest size, which means all the variables enter and leave the basis during a cycle.

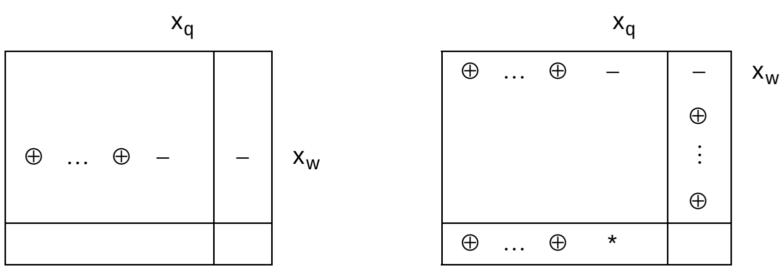
Let x_q be the variable which moves last and $A_{B'}$ the first basis when x_q changes its basic status. (Without loss of generality we may assume that x_q enters at basis $A_{B'}$.) Let us denote by $A_{B''}$ that basis when x_q moves for the second time. We assume that after the basis $A_{B'}$, all the variables have changed their basic status at least once. It may happen that another variable x_w, together with x_q, changes its basic status at $A_{B'}$ for the first time. We now have the following cases:



(3a) (3b)

Figure 3: At primal iteration x_q enters the basis and (a) x_q is the only candidate to change its basic status in Step 1.; (b) both x_q and x_w change their basic status for the first time

If $\bar{b}_w \geq 0$ then (3a) and (3b) are equivalent tableaus.



(4a) (4b)

Figure 4: At dual iteration x_q enters the basis: (a) x_q has been selected uniquely; (b) both x_q and x_w change their basic status for the first time

In (4a) and (4b) the sign structure of the row of x_w is the same. $A_{B''}$ is the basis when the variable x_q leaves the basis for the first time. We have two cases: x_q leaves the basis either in Step 2 (a), primal iteration, or in Step 2 (b), dual iteration.

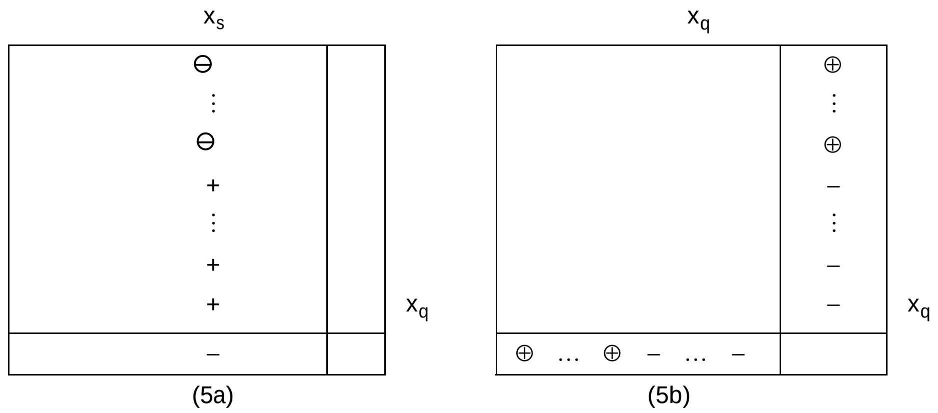


Figure 5: The variable x_q leaves the basis $A_{B''}$ at primal (5a) or at dual (5b) iteration

If $t''_{is} > 0$ and $i \in B'' \setminus \{q\}$ (see Fig. 5, part (5a)) then, according to the pivot rule of Algorithm 3.1, $v_{r-1}(i) = v_{r-1}(q) = 1$, thus $i \in B'$. Similarly, if $t''_{ci} < 0\,(t''_{jb} < 0)$, where $i \in N''\,(j \in B'' \setminus \{q\})$ then $i \in N'\,(j \in B')$ holds because $v_{r-1}(i) = v_{r-1}(q) = 1\,(v_{r-1}(j) = = v_{r-1}(q) = 1)$ using the case (5b).

Now we have the following four possible cases, namely: the variable x_q enters the basis $A_{B'}$ at

a) primal iteration and leaves the basis $A_{B''}$ at primal iteration;
b) primal iteration and leaves the basis $A_{B''}$ at dual iteration;
c) dual iteration and leaves the basis $A_{B''}$ at primal iteration;
d) dual iteration and leaves the basis $A_{B''}$ at dual iteration.

Let us deal with the case (a), where we have the sign structures given at (3a) (or (3b)) and (5a). Because $\mathbf{t}'^{(c)}$ is the same vector for both (3a) and (3b) we do not need to separate these two subcases. From the pivot tableau shown on (5a) we use the vector $\mathbf{t}''_s$. We know that $(\mathbf{t}'^{(c)})^T \mathbf{t}''_s = 0$, according to the orthogonality theorem (Proposition 1.5). Taking into consideration the signs of the entries in $\mathbf{t}'^{(c)}$ and $\mathbf{t}''_s$, especially if $t''_{is} > 0, i \in B'' \setminus \{q\}$ then $i \in B'$, thus $t'_{ci} = 0$, we have

$$0 = (\mathbf{t}'^{(c)})^T \mathbf{t}''_s \le t'_{cq} t''_{qs} + t'_{cc} t''_{cs} < 0.$$

The last (strict) inequality holds because $t''_{cq} < 0, \quad t''_{cs} < 0, \quad t'_{cc} = 1$ and $t''_{qs} > 0$. Thus we have obtained a contradiction, namely (3a) (or (3b)) cannot occur in the sequence of pivot tableaus produced by the algorithm together with the tableau shown in (5a).

Case (b). Let us consider the vectors $\mathbf{t}'^{(c)}$ and $\mathbf{t}'_b$ from the pivot tableau (3a), while the vectors $\mathbf{t}''^{(c)}$ and $\mathbf{t}''_b$ are from the pivot tableau (5b). Applying the orthogonality theorem (Proposition 1.5) twice and summing the terms we get $(\mathbf{t}'^{(c)})^T \mathbf{t}''_b + (\mathbf{t}'_b)^T \mathbf{t}''^{(c)} = 0$. Using the remark given after Figure 5, we can compute the previous expression in more detail, thus

$$\begin{aligned} 0 &= (\mathbf{t}'^{(c)})^T \mathbf{t}''_b + (\mathbf{t}''^{(c)})^T \mathbf{t}'_b \\ &\geq t'_{cc}t''_{cb} + t'_{cb}t''_{bb} + t'_{cq}t''_{qb} + t''_{cc}t'_{cb} + t''_{cb}t'_{bb} + t''_{cq}t'_{qb} \\ &= z'' - z' + t'_{cq}t''_{qb} + z' - z'' > 0, \end{aligned} \tag{2}$$

because $t'_{bb} = t''_{bb} = -1, t'_{cc} = t''_{cc} = 1, t'_{qb} = t''_{cq} = 0, t'_{cb} = z', t''_{cq} = z'', t'_{cq} < 0$ and $t''_{qb} < 0$. Therefore both pivot tableaus (3a) and (5b) cannot occur.

Now we need to pay more attention to the case when pivot tableaus (3b) and (5b) are considered.

In expression (2) the term $t''_{cw}\, t'_{wb}$ appears. Unfortunately, we have no information about the sign of the element t'_{wb} (see Fig. (3b)). The element t'_{wb} can be both negative and nonnegative. Furthermore, it may happen that the element t''_{cw} is negative or nonnegative. Therefore we have four subcases depending on the sign of t'_{wb} and t''_{cw}.

If $t'_{wb} \geq 0$ and $t''_{cw} \geq 0$ then the proof goes along the same lines as for the tableaus (3a) and (5b).

If $t''_{cw} < 0$ then let us consider the vectors $\mathbf{t}'_q$ and $\mathbf{t}''^{(c)}$. Using the orthogonality theorem (Proposition 1.5) we have $(\mathbf{t}'_q)^T \mathbf{t}''^{(c)} = 0$. Then

$$0 = (\mathbf{t}''^{(c)})^T \mathbf{t}'_q \leq t''_{cw}t'_{wq} + t''_{cq}t'_{qq} + t''_{cb}t'_{bq} = t''_{cw}t'_{wq} < 0$$

where $t''_{cq} = 0$ and $t'_{bq} = 0$, because $q \in B''$ and $b \in N'$, furthermore $t''_{cw} < 0$ and $t'_{wq} > 0$. Thus we have a contradiction in this subcase.

Let us now consider the subcase when $t'_{wb} < 0$ and $t''_{cw} \geq 0$. In this situation cycling may occur in two different ways: (i) if the variables x_q and x_w change their basic status at the basis $A_{B''}$, or (ii) if the variable x_q leaves the basis $A_{B''}$ and x_w enters a basis coming after $A_{B''}$.

Let us analyze first the case (i). Now, the vectors $\mathbf{t}''^{(q)}$ and $\mathbf{t}'_b$ are considered. The sign structure of $\mathbf{t}''^{(q)}$ has the following properties:

$$t''_{qb} < 0, t''_{qw} < 0 \text{ and if } t''_{qi} < 0 \text{ and } i \neq w \text{ then } i \in N', \text{ namely } t'_{ib} = 0 .$$

From the orthogonality theorem (Proposition 1.5) we know that $(\mathbf{t}'_b)^T \mathbf{t}''^{(q)} = 0$ and using the previous information we may compute in more details as follows

$$0 = (t''^{(q)})^T \mathbf{t}'_b \geq t''_{qw} t'_{wb} + t''_{qb} t'_{bb} + t''_{qq} t'_{qb} = t''_{qw} t'_{wb} - t''_{qb} > 0,$$

because $t''_{qw} < 0, t'_{wb} < 0, t''_{qb} < 0, t'_{bb} = -1, t''_{qq} = 1$ and $t'_{qb} = 0$. Thus we have obtained contradiction once more.

Now, we need to analyze the case (ii), thus we take into consideration the first basis $A_{B'''}$ after $A_{B''}$ such that x_w enters the basis and x_q is a nonbasic variable at $A_{B'''}$[8]. The sign structure of $A_{B'''}$ is

$$\begin{array}{|ccccccc|c|}\hline &&&&&&& \oplus \\ &&&&&&& \vdots \\ &&&&&&& \oplus \\ &&&&&&& - \\ &&&&&&& \vdots \\ &&&&&&& - \\ \hline \oplus & \dots & \oplus & - & - & \dots & - & \\ \hline \multicolumn{7}{c}{x_w} & \end{array}$$

and because $q \in N'''$ then according to the pivot rule we have $t'''_{cq} \geq 0, t'''_{cw} < 0$. Furthermore, if $t'''_{ci} < 0, i \neq w$ then $i \in N'$, therefore

$$0 = (\mathbf{t}'''^{(c)})^T \mathbf{t}'_q \leq t'''_{cw} t'_{wq} + t'''_{cc} t'_{cq} < 0,$$

since $t'''_{cc} = 1$ and $t'_{cq} < 0$. Thus a contradiction is obtained.

After this complicated case let us analyze (c) and (d), which are similar to case (a).

In case (c), we consider the vectors $\mathbf{t}'^{(w)}$ and $\mathbf{t}''_s$. Using the orthogonality theorem (Proposition 1.5) and the sign structure of the vectors we have

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[8] The existence of such basis $A_{B'''}$ is necessary to get a cycle, because at $A_{B'}$ the variable x_w was in the basis, while x_q was out of the basis.

$$0 = (\mathbf{t}'^{(w)})^T \mathbf{t}''_s \le t'_{wq}t''_{qs} + t'_{wb}t''_{bs} + t'_{wc}t''_{cs} = t'_{wq}t''_{qs} < 0,$$

where $t''_{bs} = t'_{wc} = 0, t'_{wq} < 0$ and $t''_{qs} > 0$. Thus a contradiction is obtained.

In the last case, (d), we consider the vectors $\mathbf{t}'^{(w)}$ and $\mathbf{t}''_b$ and instead of the orthogonality of the vectors (proved in Proposition 1.5) we get

$$0 = (\mathbf{t}'^{(w)})^T \mathbf{t}''_b \ge t'_{wq}t''_{qb} + t'_{wb}t''_{bb} = t'_{wq}t''_{qb} - t'_{wb} > 0,$$

since $t'_{wq} < 0, t''_{qb} < 0$ and $t'_{wb} < 0.$ This contradiction shows that case (d) cannot occur, as well.

This completes the proof, because none of the possible cases can occur.

Now we are ready to prove the strong duality theorem.

Proof of the strong duality theorem (Theorem 1.2): The two statements of the theorem exclude each other. Let us apply Algorithm 3.1. for the linear programming problem (P). The algorithm terminates with one of the following cases:

1. Variable x_k leaves the current basis and we cannot choose any variable to enter the basis (Algorithm 3.1, Step 2 (b)). Then we have $x_k = \bar{b}_k < 0$ and $t_{ki} \ge 0$, for all $i \in I$ thus $P = \emptyset$.

2. Variable x_k enters the current basis and we cannot choose any variable to leave the basis (Algorithm 3.1, Step 2 (a)). Then we have $\bar{c}_k < 0$ and $t_{ik} \le 0$, for all $i \in B$ thus $D = \emptyset$.

3. According to the pivot rule of Algorithm 3.1. we cannot choose a variable either to leave or to enter the current basis, thus $x_i \ge 0$ and $\bar{c}_i \ge 0,$ for all $i \in I$ (Step 1., $J = \emptyset$). Therefore the current basis is optimal, so an optimal solution is found to the primal problem.

It is obvious that if 1 or 2 occurs then the statement (2) of the strong duality theorem is obtained. For this, we only need to show that statement 1 and 2 are true.

Statement 1 is proved during the verification of the Farkas Lemma (see (1)). Statement 2[9] can be verified as follows. Let us assume to the contrary, that there exists a dual feasible basis A_B for which $\mathbf{t}^{(c)} \ge 0$ has to be orthogonal to the vector $\mathbf{t}'_k \le 0$, but

$$0 = (\mathbf{t}^{(c)})^T \mathbf{t}'_k \le t_{cc}t'_{ck} + t_{cb}t'_{bk} = t'_{ck} < 0,$$

[9] Known in the literature as dual infeasibility criteria, see for instance [9, 10, 5, 6].

because $t'_{bk} = 0, t_{cc} = 1$ and $t'_{ck} < 0,$ gives a contradiction.

Statement 3 is true because if we denote the basic feasible solution, produced by the algorithm, by $\hat{\mathbf{x}} = (A_B^{-1}\mathbf{b}, \mathbf{0})$ where $A_B^{-1}\mathbf{b} \geq 0,$ and $\hat{\mathbf{y}} = (\mathbf{c}_B^T A_B^{-1})^T$ then

$$\mathbf{c}^T \hat{\mathbf{x}} = \mathbf{c}_B^T A_B^{-1} \mathbf{b} = \hat{\mathbf{y}}^T \mathbf{b}.$$

Now applying the weak duality theorem of linear programming [9, 10, 5, 6], we may show the optimality of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, thus statement (1) of the strong duality theorem is obtained if Algorithm 3.1. stops in Step 1.

This completes the proof of the strong duality theorem.

Acknowledgement. The authors are grateful to Filiz Erbilen for reading an earlier version of this paper and making useful comments that improved the presentation of this paper.

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