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A NOTE ON THE CONTINUOUS p-DEFENSE-SUM PROBLEM*

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Abstract: The continuous p-defense-sum problem consists of locating p facilities in a convex polyhedron, such that the sum of the distances among all their pairs is maximized. We indicate that it is sufficient to search for optimal sites at the polyhedron's vertices only, and show that the optimal solution can be degenerate, i.e., more than one facility being located at the same point. An integer programming formulation is also given, taking the possible degeneracy into account.

Keywords: Optimization, location, p-defense-sum, degeneracy.

The discrete p-defense-sum problem is one of the known noxious facility location problems. Among a given set F of n potential undesirable facility sites (nuclear waste, garbage dumpsites, missile silos, chemical plants etc.), find p sites such that the sum of the distances among all pairs of facilities is maximized. It is usually formulated as

$$\max_{X \subseteq F} \sum_{i \in X} \sum_{j \in X} d_{ij} \qquad (1)$$

subject to $|X| = p$,

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where $|F|=n$ and d_{ij} are distances (shortest paths or some other appropriate metric). This strongly NP-hard problem [5] is formulated as an 0-1 integer program in [7], with $O(n^2)$ constraints using $O(n^2)$ 0-1 variables and in [3] involving $O(n)$ variables and $O(n)$ constraints. An exact solution method based on branch and bound is suggested in [1], while heuristic methods are proposed for examples in [2], [6] and [4].

The continuous version we shall consider here consists of finding p facilities in a polyhedron $S \subset R^q$ using any kind of norm or gauge distance evaluation (see e.g. [8]), e.g. Euclidean distances. It can be formulated as follows:

$$\max \sum_{i=1}^{p} \sum_{j=1}^{p} d(x_i, x_j) \tag{2}$$

subject to

$$x_j \in S = \{x \in R^q \mid Ax \le b\}, \quad j = 1,\dots,p.$$

where $b \in R^m$, $A \in R^{mq}$ and each d_{ij} is some gauge on R^q.

Proposition 1. There exists some optimal solution $x_j^*, j = 1,\dots,p$ of the continuous p-defense-sum problem (2), with each point x_j^* located at the extreme point of polyhedron S.

Proof: Let $X = (x_1, x_2, \dots, x_p) \in R^{q \cdot p}$ and let $S^p = S \times S \times \dots \times S$. Then (2) can be expressed as

$$\max_{X \in S^p} g(X) = \sum_{i=1}^{p} \sum_{j=1}^{p} d(x_i, x_j) \tag{3}$$

Since $d(x_i, x_j)$ is a convex function of X, (see for example [8]), $g(X)$ is convex as well (as a sum of convex functions). Now (3) is a convex maximization problem and thus its optimal value is reached at some extreme point X^* of S^p. Extreme points of product convex sets are products of extreme points, i.e. $ext(S^p) = (ext(S))^p$, hence $(x_1^*, x_2^*, \dots, x_p^*) = (e_1, e_2, \dots, e_p)$, where $e_j \in ext(S)$. ❑

From proposition 1 it is clear that continuous p-defense-sum problems (2) can be reformulated as discrete maximization problems. The next proposition shows that the thus obtained problem is not equivalent to the usual discrete p-defense-sum problem (1). We first define a *degenerate* solution of (2) as a solution where not all points x_j are different.

Proposition 2. The optimal solution of the continuous p-defense-sum problem (2) can be degenerate, if $p > 3$.

Proof: It is easy to see that the solution is never degenerate if $p \leq 3$ and $|ext(S)| > 2$. We construct a counter-example with $n = 4$ and $p = 4$. Let S be defined by the following extreme points: e_1 is the center of a ball with radius 10; three other extreme points are located on the boundary of this ball with distances among them equal to 1, 1 and 2 respectively. If $(x_1, x_2, x_3, x_4) = (e_1, e_2, e_3, e_4)$, then the corresponding objective function value is equal to $10+10+10+2+1+1 = 34$. But if $(x_1, x_2, x_3, x_4) = (e_1, e_2, e_3, e_1)$, then we have $f = 10+10+0+1+10+10 = 41$. Thus a larger (better) value is obtained in the degenerate solution since e_1 is repeated twice. ❑

The difference between the two formulations is of course the fact that in a discrete problem it is easy to impose the (implicit) additional constraint that all chosen facility sites should be different, while in a continuous problem this is much less evident. Indeed, constraints such as $x_i \neq x_j$ define open feasible sets, and this leads to the possibility of poor formulation without optimal solutions, as clearly shown by the previous counterexample.

In order to obtain a discrete formulation of the continuous p-defense-sum problem (2), the set of extreme points of S is copied $p-3$ times to allow duplications (Proposition 2). Therefore the cardinality of the set of possible facility locations F is $n = |F| = n' + (p-3)\cdot n' = (p-2)\cdot n'$, where $n' = |ext(S)|$. The new $p-3$ levels of the extreme points set $ext(S)$ are constructed such that the following holds

$$d(x_{i+\ell n'}, x_{j+kn'}) = d(x_i, x_j), \forall i, j \in ext(S), \forall \ell, k = 1, \ldots, p-3. \tag{4}$$

Now combinatorial formulation (1) can be applied.

The integer programming formulation can be given as the 0-1 *quadratic knapsack problem*:

$$\max \sum_{\ell=1}^{p-2} \sum_{k=1}^{p-2} \sum_{i=1}^{n'} \sum_{j=1}^{n'} d(x_i, x_j) y_{i+(\ell-1)n'} \cdot y_{j+(k-1)n'} \tag{5}$$

subject to

$$\sum_{k=1}^{n} y_k = p$$

$$y_k \in \{0,1\}, \quad 1 \leq k \leq n,$$

where y_k is set to 1 if the candidate point k is selected and 0 otherwise.

There is, however, a combinatorial problem with this formulation, in the sense that the same solution may be encoded in different ways by selecting the same site at other levels, leading to many symmetrical equivalent solutions. One way to avoid this is to break this symmetry by imposing that some site may be used at a given level only when all previous levels have been used also, which may be expressed by the additional symmetry breaking constraint

$$y_{i+\ell n'} \leq y_{i+kn'} \text{ for all } i \text{ and all } \ell < k .$$

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