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ON THE PALLET LOADING PROBLEM

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Abstract: In this paper we found an upper bound on the number of items of the rectangular form $a\times b$ that can be loaded onto a rectangular pallet $A\times B$, such that the sides of the loaded items are parallel to the sides of the pallet and the interiors of the loaded items do not overlap.

Key words: Combinatorial optimization, packing, pallet problem.

1. INTRODUCTION

A rectangular form $A\times B$ $(A, B\in R)$ is called a pallet (or box). The pallet loading problem is to load as many as possible rectangular items $a\times b$ $(a, b\in R)$ (called also bricks) onto a pallet, such that the interiors of the bricks do not overlap and their sides are parallel to the sides of the pallet. The problem is NP complete and there exist some heuristic and exact methods to solve the problem ([4], [5]).

We call the problem integer if the dimensions of the pallet and bricks are integer numbers. The optimal solution for the integer pallet problem is known in some special cases such as harmonic bricks (if one dimension of the brick is a multiple of the other) (Brualdi and Foregger [3]) and for sufficiently large pallets (Barnes [1]). Some generalizations for three and higher dimensions are also known ([2],[3]).

Little is known about problems with real dimensions, but some results (Barnes [2]) suggest an essential difference in their behavior. A slight generalization of the result in [2] shows that if the sides of the bricks have a common divisor, i.e. $\frac{a}{b}$ is a rational number, then the real pallet problem could be reduced to the integer case.

The aim of this paper is to extend some results to the real dimensional case. First, we show that we can reduce the dimensions of the box to the integer combinations of the dimensions of the bricks without changing the number of loaded bricks. Second, we prove that an expression giving the optimal solution in the integer case gives an upper bound of the optimal solution in the real case and could be used to prove optimality.

2. MAIN RESULT

To fix the layout, we choose a coordinate system Oxy such that the pallet corresponds to the rectangular $P=\{(x,y)\,|\,0\le x\le A,\ 0\le y\le B\}$. Every loaded brick is then a rectangular with down-left corner (u,v), horizontal dimension d and vertical dimension h, $\{d,h\}=\{a,b\}$.

Let us first show two lemmas:

Lemma 1. Two bricks with different down-left corners (u_i,v_i), lengths d_i and heights h_i, $i=1,2$ have no common interior point if and only if:

if $u_1\le u_2, v_1\le v_2$, then $u_2-u_1\ge d_1$ or $v_2-v_1\ge h_1$;

if $u_1\le u_2, v_2\le v_1$, then $u_2-u_1\ge d_1$ or $v_1-v_2\ge h_2$.

Proof: Obvious.

Lemma 2. The function $f(c)=\max\{xa+yb\,|\,xa+yb\le c;\ x,y\ge 0,\ x,y\in Z\}$ is well defined for $c\ge 0$, nondecreasing and $f(u+a)\ge f(u)+a$, $f(u+b)\ge f(u)+b$ for every $u\ge 0$.

Proof: The feasible set is nonempty and finite and increases with c. Hence, for every $c\ge 0$, there exists nonnegative integers m and n such that $f(c)=ma+nb$ and $c_1\le c_2\Rightarrow f(c_1)\le f(c_2)$. From $f(u)+a=(ma+nb)+a=(m+1)a+b$ we conclude $f(u+a)\ge f(u)+a$. The rest follows in the same manner.

Let us now show two theorems:

Theorem 1. A pallet $A\times B$ loaded with bricks $a\times b$ can be reduced to a pallet $\hat{A}\times\hat{B}$, where $\hat{A}$ and $\hat{B}$ are integer combinations of a and b, having the same number of loaded bricks.

Proof: Let the pallet $A\times B$ be loaded with some bricks $a\times b$. Let $\hat{A}=f(A)$, $\hat{B}=f(B)$ where f is the function defined in Lemma 2. We translate every loaded brick with down-left corner (u,v) having horizontal dimension d and vertical dimension h, $(\{d,h\}=\{a,b\})$, such that the translate has down-left corner $(f(u),f(v))$. Since $u+d\le A$, $v+h\le B$, it follows that $f(u)+d\le f(u+d)\le f(A)=\hat{A}$ and $f(v)+h\le f(v+h)\le f(B)=\hat{B}$, using the properties of f. Hence, the translates are on the pallet $\hat{A}\times\hat{B}$. It remains to show that any two translates have no common interior point. Consider any two originally loaded bricks with down-left corners (u_i, v_i), lengths d_i and heights h_i, $i = 1,2$. Since they have no common interior point, using Lemma 1, we conclude:

$$\text{if } u_1 \le u_2, v_1 \le v_2, \text{ then } u_2 - u_1 \ge d_1 \text{ or } v_2 - v_1 \ge h_1$$

and

$$\text{if } u_1 \le u_2, v_2 \le v_1, \text{ then } u_2 - u_1 \ge d_1 \text{ or } v_1 - v_2 \ge h_2 .$$

Using the properties of f, we have in the first case,

$$f(u_1) \le f(u_2),\ f(v_1) \le f(v_2) \text{ and } f(u_2) - f(u_1) \ge d_1 \text{ or } f(v_2) - f(v_1) \ge h_1$$

and, in the second case

$$f(u_1) \le f(u_2),\ f(v_2) \le f(v_1) \text{ and } f(u_2) - f(u_1) \ge d_1 \text{ or } f(v_1) - f(v_2) \ge h_2$$

Using Lemma 1, we conclude that corresponding translates have no common interior point and the theorem is proved.

Now, we find an upper bound for the number of loaded bricks on the pallet.

Let p, q, r and s be the remainders of the division of A and B respectively by a and b. In other words, let

$$A = \alpha a + p = \beta b + r,\ \alpha, \beta \in N \cup \{0\},\ 0 \le p < a,\ 0 \le r < b$$

and

$$B = \gamma a + q = \delta b + s,\ \gamma, \delta \in N \cup \{0\},\ 0 \le q < a,\ 0 \le s < b .$$

Then, we can state

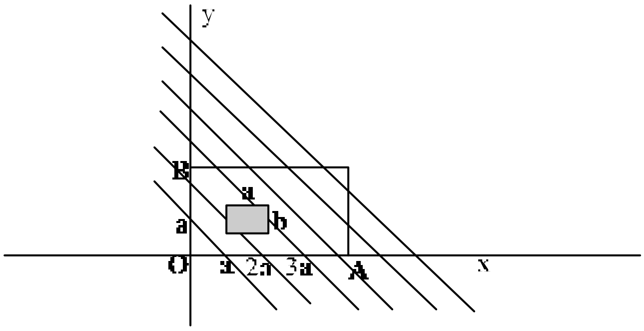
Theorem 2. The number of bricks on the pallet does not exceed

$$\left\lfloor \frac{AB - \max\{\min\{pq, (a-p)(a-q)\}, \min\{rs, (b-r)(b-s)\}\}}{ab} \right\rfloor .$$

That means that in every feasible packing, the wasted area is at least

$$\max\{\min\{pq, (a-p)(a-q)\}, \min\{rs, (b-r)(b-s)\}\} .$$

Proof: Consider a coordinate system Oxy such that the palette equals the rectangle $P = \{(x, y) \mid 0 \le x \le A, 0 \le y \le B\}$. Let $S = \{(x, y) \mid x + y = ka, k \in Z\}$.



The set S has the property that the intersection with any correctly loaded item consists of one or more line segments having the total length $b\sqrt{2}$. The intersection $S \cap P$ consists of line segments having the total length

$$L = \sqrt{2}\{[a + 2a + \cdots + \left\lfloor \frac{A+B}{a} \right\rfloor a] - [(a-p) + (2a-p) + \cdots + (\left\lfloor \frac{p+B}{a} \right\rfloor a - p)] -$$
$$-[(a-q) + (2a-q) + \cdots + (\left\lfloor \frac{q+A}{a} \right\rfloor a - q)]\}$$

If $p+q < a$, then $\left\lfloor \frac{A+B}{a} \right\rfloor = \alpha + \gamma$, $\left\lfloor \frac{p+B}{a} \right\rfloor = \gamma$, $\left\lfloor \frac{q+A}{a} \right\rfloor = \alpha$, so that

$$L = \sqrt{2}\{\frac{(\alpha+\gamma)(\alpha+\gamma+1)}{2} a - \frac{\gamma(\gamma+1)}{2} a - \frac{\alpha(\alpha+1)}{2} a + p\gamma + q\alpha\} = \sqrt{2}\{\frac{AB - pq}{a}\}.$$

If $p+q \geq a$, then $\left\lfloor \frac{A+B}{a} \right\rfloor = \alpha + \beta + 1$, $\left\lfloor \frac{p+B}{a} \right\rfloor = \gamma + 1$, $\left\lfloor \frac{q+A}{a} \right\rfloor = \alpha + 1$, so that

$$L = \sqrt{2}\{A\gamma - a + q\alpha + p + q\} = \sqrt{2}\{\frac{AB - (a-p)(a-q)}{a}\}.$$

In general, we have

$$L = \frac{AB - \min\{pq, (a-p)(a-q)\}}{a} \sqrt{2}.$$

Dividing by $b\sqrt{2}$, we obtain that the number of loaded items does not exceed

$$\left\lfloor \frac{AB - \min\{pq, (a-p)(a-q)\}}{ab} \right\rfloor.$$

Interchanging the roles of a and b, we obtain that the number of loaded items does not exceed $\left\lfloor \frac{AB - \min\{rs,(b-r)(b-s)\}}{ab} \right\rfloor$. Using the best from the proved upper bounds, we obtain the theorem.

The theorem asserts that in every feasible loading the uncovered area is greater or equal than

$$\max\{\min\{pq,(a-p)(a-q)\},\min\{rs,(b-r)(b-s)\}\}.$$

If the uncovered area is zero, each side of the brick has to divide one side of the pallet. Notice that, using Theorem 1, each side of the box is an integer combination of the sides of the brick. This gives new proof of Klarner's result ([6]) for real dimensions.

In some cases the proved upper bound gives the optimal value for the pallet problem. This is always true if we find a packing with this number of loaded bricks. In the harmonic case (if b divides a, using Theorem 1 we could suppose that b also divides A and B) there exists a packing with the uncovered area $\min\{pq,(a-p)(a-q)\}$.

If $p+q<a$, this is shown in Fig. 1 and if $p+q\geq a$, in Fig. 2.

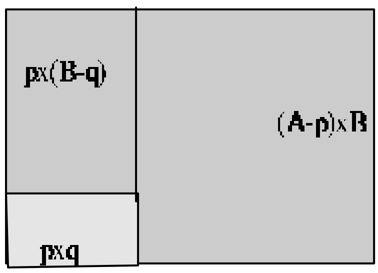


Figure 1.

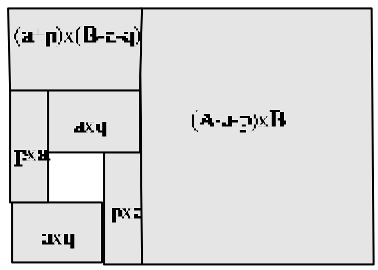


Figure 2.

From $b|a$ and $b|A$ follows $b|p$. Hence, $p\times(B-q)$ and $(A-p)\times B$ could be trivially loaded with bricks. Similarly, $b|a$ and $b|B$ implies $b|q$. Hence, $a\times q$, $p\times a$, $(A-a-p)\times B$ and $(a+p)\times(B-a-q)$ could be trivially loaded with bricks.

The types of packing shown in Figs. 1 (linear) and 2 (turbulent) give the optimal solution in some nonharmonic cases. The following example confirms this assertion.

Example 1. Let $A=\sqrt{5}+1.1$, $B=\sqrt{5}+2.1$, $a=\sqrt{5}$, $b=1$. Using Theorem 1, we can reduce the box to $\hat{A}=\sqrt{5}+1$, $\hat{B}=\sqrt{5}+2$ such that $p=1$, $q=2$, $r=s=\sqrt{5}-2$ and

$\max\{\min\{pq,(a-p)(a-q)\},\min\{rs,(b-r)(b-s)\}\}=(a-p)(a-q)=7-3\sqrt{5}$. Using Theorem 2, we can load maximum $\left\lfloor\frac{(\sqrt{5}+1)(\sqrt{5}+2)-(7-3\sqrt{5})}{\sqrt{5}}\right\rfloor=6$ bricks. Figure 3 shows that this is possible.

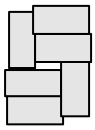


Figure 3.

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