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[0,1]-VALUED LOGIC: A NATURAL GENERALIZATION OF BOOLEAN LOGIC

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Abstract: [0,1]-valued logic can be seen as a basis of decision making and reasoning complementary to "black & white" reasoning. A new [0,1]-valued logic is presented in this paper. Classical (Boolean) {0,1}-valued logic can be treated as a special case of new [0,1]-valued logic. [0,1]-valued logical functions are a generalization of Boolean and/or pseudo-Boolean functions. All properties of classical logic are preserved in [0,1]-valued logic.

Keywords: [0,1]-valued logical function, structural function of a logical formula, structural vector of a logical formula, basic logical function, vector of a basic logical function - weighted vector of a logical function, t-norms

1. INTRODUCTION

The problems of aggregating (fusing) many relevant aspects (attributes, symptoms, characteristics,...) into one value (degree of importance, satisfaction, similarity, suitability, compatibility, operability, etc.) are very often encountered in many domains of OR. These problems are by nature logical problems and mathematical logic should be the principal tool for their solution. Classical {0,1}-valued Boolean logic is adequate for problems which are "black & white" or two-valued problems. Many real problems from the logical point of view are many-valued and even infinite-valued and/or [0,1]-valued problems. There are a great number of generalizations of Boolean two-valued logic to multi-valued logic and/or to [0,1]-valued logic [13]. There are many surveys [6], [2], [7] of these generalizations. Until now, all known approaches have been only partially complementary to Boolean logic.

This paper presents [0,1]-valued logic complementary to Boolean {0,1}-valued logic. The new [0,1]-valued logic is a generalization of Boolean logic so all the main characteristics are preserved and, even more, new light is thrown on classical logic.

An n-ary [0,1]-valued logical function is composed of two main constituents: (a) a structural set function of logical formula and (b) basic logical functions.

The structural function of an n-ary logical formula is a set function of the elements of the power set of atomic symbols. The vector of values of the structural function of an n-ary logical formula is a structural vector. A Boolean function and any of its generalizations (multi-valued and/or [0,1]-valued) based on the same logical formula have the same structural function and/or structural vector. Since structural vectors of logical formulae are {0,1} vectors, all properties of classical logic are preserved (Boolean logic algebra is defined on a set of structural vectors of all n-ary logical formulae).

The structural function of an n-ary pseudo-Boolean formula (a linear convex combination of n-ary Boolean formulae) is a pseudo-logical structural set function and the corresponding vector is a pseudo-structural vector. The pseudo-structural vector of any n-ary pseudo-logical formula can be represented as a linear convex combination of structural vectors of n-ary logical formulae.

Geometrically, the set of all logical (0-1) structural and pseudo-logical structural vectors of n-ary formulae is a structural hyper-cube. In the n-ary case, the structural hyper-cube has dimension 2^n or it is a space defined by $[0,1]^{2^n}$ set, with 2^{2^n} vertex - logical structural vectors.

The basic vectors of a structural hyper-cube have only one component different from zero and equal to 1. A logical function whose structural vector is a basic structural vector is a basic logical function. Since any structural and/or pseudo-structural vector can be represented as a linear combination of corresponding basic vectors - any n-ary [0,1]-valued logical function can be represented as a linear combination of corresponding n-ary basis logical functions.

A basic logical function depends on the values of [0,1]-valued atomic symbols as variables and the chosen continual logical operators as parameters. For different continual logical operators there are different [0,1]-valued logic. The two most important cases of the continual logical operators for [0,1]-valued logical functions are analyzed in this paper: (a) the AND operator defined as an algebraic product and (b) the AND operator defined as a standard intersection.

An n-ary [0,1]-valued logical function is a linear convex combination of the components of its structural (0-1 or pseudo-logical) vector and a basic logical function is a weighted coefficient of the corresponding component of a structural vector.

This introduction includes a list of basic notions. Section 2 presents the structural function and structural vector of a logical formula. The logical structural function and the structural vector of a logical formula are analyzed first. Then the pseudo-logical structural function is analyzed and the pseudo-logical structural vector.

Section 3 defines basic logical functions and the weighted vector of a logical formula and analyzes their general properties and their properties as a function of accepted continual logical operators.

The definition and main properties of an n-ary [0,1]-valued logical function are given in Section 4. First we analyze the properties of the [0,1]-valued logical function for 0-1 structural vectors and then the case of pseudo-logical structural vectors. An n-ary [0,1]-valued logical function is presented for two different continual logical AND operators: (a) an algebraic product and (b) a standard intersection.

1.1. Basic notions

- a $[0,1]$ - valued logical variable (atomic symbol, atomic logical formula)
- $\Omega = \{a_1, ..., a_n\}$ set of $[0,1]$-valued logical variables (atomic symbols)
- $\mathbf{P}(\Omega)$ power set of atomic symbols
- $A, B \subseteq \Omega$, or $A, B \in \mathbf{P}(\Omega)$
- $|A|$ cardinal number of set $A \in \mathbf{P}(\Omega)$
- $A^c = \Omega \setminus A$ complement of a set $A \in \mathbf{P}(\Omega)$
- j logical formula
- sj pseudo-logical formula
- $L_j(a_1, ..., a_n)$ n-ary Boolean (0-1) logical function ($L_j : \{0,1\}^n \to \{0,1\}$),
- $L_{sj}(a_1, ..., a_n)$ pseudo-Boolean function ($L_{sj} : \{0,1\}^n \to [0,1]$),
- e_A characteristic vector
- $\boldsymbol{j}(n)$ set of all n-ary Boolean (0-1) logical formulae $|\boldsymbol{j}(n)| = 2^{2^n}$
- $m_{a_i}(A)$, $a_i \in \Omega$, $A \in \mathbf{P}(\Omega)$, structural function of an atomic formula
- $\vec{m}_{a_i}(A)$, $a_i \in \Omega$, atomic structural vector
- $m_j(A)$, $A \in \mathbf{P}(\Omega)$ logical (0-1) structural function of logical formula j ($m_j : \mathbf{P}(\Omega) \to \{0,1\}$)
- $\vec{m}_j$ logical (0-1) structural vector of a logical formula
- $\vec{\blacksquare}(n)$ set of all logical (0-1) structural vectors of n-ary logical formulae $|\vec{\blacksquare}(n)| = 2^{2^n}$
- $\boldsymbol{B}_{\blacksquare} = \langle \vec{\blacksquare}(n), \vee, \wedge \rangle$; $n \in N_0$ Boolean algebra on a set of all 0-1 structural vectors of n-ary logical formulae
- $m_{sj}(A)$, $A \in \mathbf{P}(\Omega)$ pseudo-logical structural function of pseudo-logical formula ($m_{sj} : \mathbf{P}(\Omega) \to [0,1]$)
- $\vec{m}_{sj}$ pseudo-logical structural vector of a pseudo-logical formula
- $\blacksquare(n)$ structural-space - space of all structural and pseudo-structural vectors of n-ary formulae - $[0,1]^{2^n}$ hyper-cube

- $m_1(\mathrm{A})=1$; $\mathrm{A}\in \mathbf{P}(\Omega)$ structural function of logical constant 1
- $m_0(\mathrm{A})=0$; $\mathrm{A}\in \mathbf{P}(\Omega)$ structural function of logical constant 0
- $\vec{m}_{\mathrm{B(A)}}$ basic vector of structural - space, basic structural vector
- $\mathrm{L}_{\mathrm{B(A)}}(a_1,\ldots,a_n)$ basic function, $\mathrm{A}\in \mathbf{P}(\Omega)$
- $\vec{\mathrm{L}}_{\mathrm{B}}(a_1,\ldots,a_n)$ vector of a basic function - weighted coefficient vector of an n-ary logical function
- $\mathrm{L}m_j(a_1,\ldots,a_n)$ $[0,1]$-valued logical function
- $\mathrm{L}m_{sj}(a_1,\ldots,a_n)$ $[0,1]$-valued pseudo-logical function
- $\mathrm{C}_m(a_1,\ldots,a_n)$ discrete Choquet integral
- $\mathrm{LC}_m(a_1,\ldots,a_n)$ logical Choquet integral
- $\mathrm{OWA}_m(a_1,\ldots,a_n)$ Ordered weighted averaging operator

2. THE STRUCTURAL FUNCTION AND STRUCTURAL VECTOR OF LOGICAL FORMULAE

In this section we analyze the structural logical (pseudo-logical) function and structural logical (pseudo-logical) vector of a logical (pseudo-logical) formula. The logical (pseudo-logical) structural function as a set function and the logical (pseudo-logical) structural vector of an n-ary logic (pseudo-logic) formula are defined in [10] as a general measure and general measure vector.

A logical structural function is a characteristic of logical formulae. A logical formula is defined as in classical logic.

A logical structural function is a set function defined on a power set of atomic symbols and it is characteristic of logical formulae. The logical structural vector of a logical formula is defined on the basis of a structural function. Boolean algebra is defined on the set of all logical structural vectors of n-ary logical formulae.

A pseudo logical structural function and a pseudo-logical structural vector are characteristics of a pseudo-Boolean formula. Every pseudo-logical formula can be represented as a linear convex combination of logical formulae. As a consequence, every pseudo-logical structural vector can be represented as a linear convex combination of 0-1 structural vectors.

2.1. The logical (0-1) structural function and structural vector of a logical formulae

In this subsection we present the logical structural function and vector of logical formula j .

An atomic formula is characterized by a 0-1 structural function which is a set function given by the following definition:

Definition 1. The 0-1 structural function of atomic formula $a_i \in \Omega$ is the set function $m_{a_i} : \mathbf{P}(\Omega) \to \{0,1\}$ defined by:

$$m_{a_i}(A) = \begin{cases} 1 & a_i \in A \\ 0 & a_i \notin A \end{cases}; \quad A \in \mathbf{P}(\Omega).$$

For the definition of a [0,1]-valued logical function we need to define the structural vector of an atomic formula - an atomic structural vector.

Definition 2. Atomic structural vector $\vec{m}_{a_i}$ has components equal to the values of the structural function of atomic formula a_i in a given order (lexicographic or modified lexicographic, for example).

Definition 3. The modified lexicographic order of the elements of a power set means that elements $A, B \in \mathbf{P}(\Omega)$ are ordered in the following way: $A \prec B$ if and only if $|A| < |B|$ or $|A| = |B| = s$ and A is before B in lexicographic order between s-ary sets ordered by increasing index.

Example 4. Elements of power set $\mathbf{P}(\Omega)$, for $\Omega = \{a_1, a_2, a_3\}$ and components of the structural vector in partial lexicographic order are given in the following table:

		$\vec{m}$
1.	$\emptyset$	$m(\emptyset) = m_0$
2.	$\{a_1\}$	$m(\{a_1\}) = m_1$
3.	$\{a_2\}$	$m(\{a_2\}) = m_2$
4.	$\{a_3\}$	$m(\{a_3\}) = m_3$
5.	$\{a_1, a_2\}$	$m(\{a_1, a_2\}) = m_{1,2}$
6.	$\{a_1, a_3\}$	$m(\{a_1, a_3\}) = m_{1,3}$
7.	$\{a_2, a_3\}$	$m(\{a_2, a_3\}) = m_{2,3}$
8.	$\{a_1, a_2, a_3\}$	$m(\{a_1, a_2, a_3\}) = m_{1,2,3}$

A Boolean (two-valued) n-ary logical function $L_j : \{0,1\}^n \to \{0,1\}$ is defined by a truth-table and/or its generalization by [0,1]-valued logic function $Lm_j : [0,1]^n \to [0,1]$, which is equivalent to the corresponding Boolean formula for $\Omega \subset \{0,1\}^n$.

Example 5. Atomic structural vectors for $\Omega = \{a_1, a_2, a_3\}$ are given in the following table:

	$\vec{m}_{a_1}$	$\vec{m}_{a_2}$	$\vec{m}_{a_3}$
m_0	0	0	0
m_1	1	0	0
m_2	0	1	0
m_3	0	0	1
$m_{1,2}$	1	1	0
$m_{1,3}$	1	0	1
$m_{2,3}$	0	1	1
$m_{1,2,3}$	1	1	1

The logical (0-1) structural function of a logical formula is given by the following definition:

Definition 6. The 0-1 structural function of logical formula j, is the set function $m_{L_j} : \mathbf{P}(\Omega) \to \{0,1\}$, such that for every $A \in \mathbf{P}(\Omega)$

$$m_j(A) = L_j(e_A),$$

where: $e_A = ((e_A)_i) \subset \{0,1\}^{|\Omega|}$ is the characteristic vector of $A \in \mathbf{P}(\Omega)$ and components $(e_A)_i = 1$ if and only if $a_i \in A$ and $L_j(e_A)$ 0-1 is the truth-value of logical function L_j for e_A.

For the definition of a [0,1]-valued logical function we need to define the structural vector of a logical formula.

Definition 7. Structural vector $\vec{m}_j$ of logical formula j, has components equal to the values of the structural function of that logical formula in a given order (lexicographic or modified lexicographic, for example).

The structural vector of a logical formula corresponds to the column of that formula in a classical - Boolean truth-table. A very important property of structural vectors, which follows from the definition of the structural function of a logical formula, is given in the following corollary:

Corollary 8. All n-ary logical functions L_{m_j} ([0,1]-valued, multi-valued and/or $\{0,1\}$-valued) which map

$$L_{m_j}: \{0,1\}^n \to \{0,1\}$$

in the same way, have the same structural function and as a consequence the same structural vector.

Example 9. The structural vector of binary logic function $L_{a_1 \otimes a_2}(a_1, a_2)$ can be represented in the following way:

$$\begin{aligned}\vec{m}_{a_1 \otimes a_2} &= [L_{a_1 \otimes a_2}(0,0), L_{a_1 \otimes a_2}(1,0), L_{a_1 \otimes a_2}(0,1), L_{a_1 \otimes a_2}(1,1)]^T \\ &= [m_{a_1 \otimes a_2}(\emptyset), m_{a_1 \otimes a_2}(\{a_1\}), m_{a_1 \otimes a_2}(\{a_2\}), m_{a_1 \otimes a_2}(\{a_1, a_2\})]^T\end{aligned}$$

Example 10. Logical structural "vectors" (scalars) for nullary logical constants are:

	$\vec{m}_{\mathbf{0}}$	$\vec{m}_{\mathbf{1}}$
m_0	0	1

Example 11. Logical structural vectors for unary logical formulae $j \in \{\mathbf{0}, a, \bar{a}, \mathbf{1}\}$, are

	$\vec{m}_{\mathbf{0}}$	$\vec{m}_a$	$\vec{m}_{\bar{a}}$	$\vec{m}_{\mathbf{1}}$
m_0	0	0	1	1
m_a	0	1	0	1

Example 12. Logical structural vectors for binary logical formulae:

$$\begin{aligned} j \in \{&\mathbf{0}, a_1 \Leftarrow\!| \, a_2, a_1 \,|\!\Rightarrow a_2, a_1 \underline{\vee} a_2, a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\} \cup \\ \{&\mathbf{1}, a_1 \Rightarrow a_2, a \Leftarrow a_2, a_1 \Leftrightarrow a_2, a_1 \barwedge a_2, \bar{a}_1, \bar{a}_2, a_1 \,\bar{\vee}\, a_2\}\end{aligned}$$

are given in the following table:

	$\vec{m}_{\mathbf{0}}$	$\vec{m}_{a_1 \bar{\vee} a_2}$	$\vec{m}_{a_1 \Leftarrow\mid a_2}$	$\vec{m}_{a_1 \mid\Rightarrow a_2}$	$\vec{m}_{a_1 \wedge a_2}$	$\vec{m}_{\bar{a}_2}$	$\vec{m}_{\bar{a}_1}$	$\vec{m}_{a_1 \Leftrightarrow a_2}$
m_0	0	1	0	0	0	1	1	1
m_1	0	0	1	0	0	1	0	0
m_2	0	0	0	1	0	0	1	0
$m_{1,2}$	0	0	0	0	1	0	0	1

	$\vec{m}_{a_1 \underline{\vee} a_2}$	$\vec{m}_{a_1}$	$\vec{m}_{a_2}$	$\vec{m}_{a_1 \bar{\wedge} a_2}$	$\vec{m}_{a_1 \Leftarrow a_2}$	$\vec{m}_{a_1 \Rightarrow a_2}$	$\vec{m}_{a_1 \vee a_2}$	$\vec{m}_{\mathbf{1}}$
m_0	0	0	0	1	1	1	0	1
m_1	1	1	0	1	1	0	1	1
m_2	1	0	1	1	0	1	1	1
$m_{1,2}$	0	1	1	0	1	1	1	1

The value of Boolean $\{0,1\}$-valued n-ary logical function $L_j(a_1,\dots,a_n)$ of formula j

$$L_j : \{0,1\}^n \to \{0,1\}$$

and its generalization multi-valued and/or $[0,1]$-valued logical function $Lm_j(a_1,\dots,a_n)$

$$Lm_j : [0,1]^n \to [0,1]$$

are equal for $\{0,1\}$-valued atomic symbols

$$L_j(a_1,\dots,a_n) = Lm_j(a_1,\dots,a_n).$$

From the definition of the structural vector of a logical formula it follows that every extension of Boolean function (multi-valued and/or [0,1]-valued) has the same structural function and/or structural vector as the original Boolean formula.

Nullary logical formulae - logical constants have trivial structural vectors:

$$m_{\mathbf{1}}(A) = 1; \quad A \in \mathbf{P}(\Omega),$$

$$m_{\mathbf{0}}(A) = 0; \quad A \in \mathbf{P}(\Omega),$$

or

$$\vec{m}_{\mathbf{1}} = \mathbf{1},$$

$$\vec{m}_{\mathbf{0}} = \mathbf{0}.$$

Unary logical operations on a structural vector:

A structural vector of the negation of logical formula j :

$$\vec{m}_{\bar{j}} = \bar{\vec{m}}_j = \vec{m}(\mathbf{1}) - \vec{m}_j ,$$

or the components of the structural vector of the negation of the logical formula:

$$\bar{m}_j(A) = 1 - m_j(A); \quad A \in \mathbf{P}(\Omega).$$

Corollary 13. As a consequence of the definition of the structural vector of a logical function's negation:

$$\overline{\overline{\vec{m}}}_j = \vec{m}_j$$

$$\overline{\overline{\vec{m}}}_{\mathbf{1}} = \vec{m}_{\mathbf{1}}$$

$$\overline{\overline{\vec{m}}}_{\mathbf{0}} = \vec{m}_{\mathbf{0}}$$

$$\overline{\vec{m}}_{\mathbf{1}} = \vec{m}_{\mathbf{0}}$$

$$\overline{\vec{m}}_{\mathbf{0}} = \vec{m}_{\mathbf{1}}$$

Remark 1. The following symbols have the same meaning:

$$\vec{m}_{\overline{j}} = \overline{\vec{m}}_j = \overline{\vec{m}_j} \text{ .}$$

The structural vector of the negation of a logical function is equal to the negation of the structural vector of that logical function.

Structural vector $\vec{m}_{j_1 \otimes j_2}$ of a binary operation on two logical formulae j_1 and j_2 is equal to a binary logical operation on these two structural vectors $\vec{m}_{j_1}$ and $\vec{m}_{j_2}$:

$$\vec{m}_{j_1 \otimes j_2} = \vec{m}_{j_1} \otimes \vec{m}_{j_2} ,$$

or in scalar form:

$$m_{j_1 \otimes j_2}(A) = m_{j_1}(A) \otimes m_{j_2}(A) , \quad A \in \mathbf{P}(\Omega)$$

where $\otimes \in \{\Leftarrow\!|, |\!\Rightarrow, \underline{\vee}, \wedge, \vee, \Rightarrow, \Leftarrow, \Leftrightarrow, \overline{\wedge}, \overline{\vee}\}$.

Corollary 14. Binary operations of a structural vector with a logical constant vector are:

$$\vec{m}_j \vee \vec{m}_{\mathbf{0}} = \vec{m}_j \text{ ; } \quad \vec{m}_j \vee \vec{m}_{\mathbf{1}} = \vec{m}_{\mathbf{1}},$$
$$\vec{m}_j \wedge \vec{m}_{\mathbf{1}} = \vec{m}_j \text{ ; } \quad \vec{m}_j \wedge \vec{m}_{\mathbf{0}} = \vec{m}_{\mathbf{0}},$$
$$\vec{m}_j \vee \vec{m}_{\overline{j}} = \vec{m}_{\mathbf{1}}\text{; or } \vec{m}_j \vee \overline{\vec{m}}_j = \vec{m}_{\mathbf{1}},$$
$$\vec{m}_j \wedge \vec{m}_{\overline{j}} = \vec{m}_{\mathbf{0}}\text{; or } \vec{m}_j \wedge \overline{\vec{m}}_j = \vec{m}_{\mathbf{0}}.$$

Corollary 15. Binary operations of logical constant vectors are:

$$\vec{m}_{\mathbf{1}} \wedge \vec{m}_{\mathbf{1}} = \vec{m}_{\mathbf{1}}$$
$$\vec{m}_{\mathbf{1}} \vee \vec{m}_{\mathbf{1}} = \vec{m}_{\mathbf{1}}$$
$$\vec{m}_{\mathbf{0}} \wedge \vec{m}_{\mathbf{0}} = \vec{m}_{\mathbf{0}}$$
$$\vec{m}_{\mathbf{0}} \vee \vec{m}_{\mathbf{0}} = \vec{m}_{\mathbf{0}}$$
$$\vec{m}_{\mathbf{1}} \wedge \vec{m}_{\mathbf{0}} = \vec{m}_{\mathbf{0}}$$
$$\vec{m}_{\mathbf{1}} \vee \vec{m}_{\mathbf{0}} = \vec{m}_{\mathbf{1}}$$

Structural vector $\vec{m}_{\Diamond(j_1,\ldots,j_n)}$ of an n-ary operation $\Diamond$ on n logical formulae $j_1,\ldots,j_n$ is equal to an n-ary logical operation on these n structural vectors $m_1,\ldots,m_n$:

$$\vec{m}_{\Diamond(j_1,\ldots,j_n)} = \Diamond(\vec{m}_{j_1},\ldots,\vec{m}_{j_n}),$$

or in scalar form:

$$m_{\Diamond(j_1,\ldots,j_n)}(\mathrm{A}) = \Diamond(m_{j_1}(\mathrm{A}),\ldots,m_{j_n}(\mathrm{A})),\quad \mathrm{A} \in \mathbf{P}(\Omega).$$

Definition 16. $\vec{\blacksquare}(\mathrm{n})$ is the set of structural vectors of all n-ary logical formulae $\boldsymbol{j}(\mathrm{n})$:

$$\vec{\blacksquare}(\mathrm{n}) = \{\vec{m}_{j_i} \mid j_i \in \boldsymbol{j}(\mathrm{n})\};\quad \mathrm{n} \in \mathrm{N}_0.$$

The dimension of a structural vector of n-ary logical function is 2^{n} and the number of n-ary 0-1 structural vectors is $2^{2^{\mathrm{n}}}$.

Boolean algebra $\mathbf{B}_{\blacksquare}$ on a set $\vec{\blacksquare}(\mathrm{n})$ of all 0-1 structural vectors of n-ary logical functions is defined as the quadruple:

$$\mathbf{B}_{\blacksquare} = \langle \vec{\blacksquare}(\mathrm{n}), \vee, \wedge, - \rangle$$

where $\vee, \wedge$ are binary operations and $-$ is a unary operation on $\vec{\blacksquare}(\mathrm{n})$ for which the following properties are satisfied.

(B1)	Idempotence	$\vec{m}_j \vee \vec{m}_j = \vec{m}_j$
		$\vec{m}_j \wedge \vec{m}_j = \vec{m}_j$
(B2)	Commutativity	$\vec{m}_{j_1} \vee \vec{m}_{j_2} = \vec{m}_{j_2} \vee \vec{m}_{j_1}$
		$\vec{m}_{j_1} \wedge \vec{m}_{j_2} = \vec{m}_{j_2} \wedge \vec{m}_{j_1}$
(B3)	Associativity	$(\vec{m}_{j_1} \vee \vec{m}_{j_2}) \vee \vec{m}_{j_3} = \vec{m}_{j_1} \vee (\vec{m}_{j_2} \vee \vec{m}_{j_3})$
		$(\vec{m}_{j_1} \wedge \vec{m}_{j_2}) \wedge \vec{m}_{j_3} = \vec{m}_{j_1} \wedge (\vec{m}_{j_2} \wedge \vec{m}_{j_3})$
(B4)	Absorption	$\vec{m}_{j_1} \vee (\vec{m}_{j_1} \wedge \vec{m}_{j_2}) = \vec{m}_{j_1}$
		$\vec{m}_{j_1} \wedge (\vec{m}_{j_1} \vee \vec{m}_{j_2}) = \vec{m}_{j_1}$
(B5)	Distributivity	$\vec{m}_{j_1} \wedge (\vec{m}_{j_2} \vee \vec{m}_{j_3}) = (\vec{m}_{j_1} \wedge \vec{m}_{j_2}) \vee (\vec{m}_{j_1} \wedge \vec{m}_{j_3})$
		$\vec{m}_{j_1} \vee (\vec{m}_{j_2} \wedge \vec{m}_{j_3}) = (\vec{m}_{j_1} \vee \vec{m}_{j_2}) \wedge (\vec{m}_{j_1} \vee \vec{m}_{j_3})$
(B6)	Universal bounds	$\vec{m}_j \vee \vec{m}_{\mathbf{0}} = \vec{m}_j$; $\vec{m}_j \vee \vec{m}_{\mathbf{1}} = \vec{m}_{\mathbf{1}}$
		$\vec{m}_j \wedge \vec{m}_{\mathbf{1}} = \vec{m}_j$; $\vec{m}_j \wedge \vec{m}_{\mathbf{0}} = \vec{m}_{\mathbf{0}}$

(B7)	Complementarity	$\vec{m}_j \vee \vec{m}_{\bar{j}} = \vec{m}_{\mathbf{1}}$; or $\vec{m}_j \vee \bar{\vec{m}}_j = \vec{m}_{\mathbf{1}}$
		$\vec{m}_j \wedge \vec{m}_{\bar{j}} = \vec{m}_{\mathbf{0}}$; or $\vec{m}_j \wedge \bar{\vec{m}}_j = \vec{m}_{\mathbf{0}}$
		$\vec{m}_{\bar{\mathbf{1}}} = \vec{m}_{\mathbf{0}}$; or $\bar{\vec{m}}_{\mathbf{1}} = \vec{m}_{\mathbf{0}}$
(B8)	Involution	$\vec{m}_{\bar{\bar{j}}} = \vec{m}_j$; or $\bar{\bar{\vec{m}}}_j = \vec{m}_j$
(B9)	Dualization	$\overline{\vec{m}_{j_1} \vee \vec{m}_{j_2}} = \vec{m}_{\bar{j}_1} \wedge \vec{m}_{\bar{j}_2}$; $\overline{\vec{m}_{j_1} \vee \vec{m}_{j_2}} = \bar{\vec{m}}_{j_1} \wedge \bar{\vec{m}}_{j_2}$
		$\overline{\vec{m}_{j_1} \wedge \vec{m}_{j_2}} = \vec{m}_{\bar{j}_1} \vee \vec{m}_{\bar{j}_2}$; or $\overline{\vec{m}_{j_1} \wedge \vec{m}_{j_2}} = \bar{\vec{m}}_{j_1} \vee \bar{\vec{m}}_{j_2}$

Corollary 17. All properties (B1)-(B9) are valid for 0-1 structural functions as components of structural vectors.

Properties (B1)-(B4) are common to all lattices. Boolean algebras are therefore lattices that are distributive (B5), bounded (B6), and complemented (B7)-(B9).

From classical Boolean logic it follows that a structural vector $\vec{m}_j$ of any n-ary logical formula j can be represented by structural vectors $\vec{m}_{a_i}$ of atomic symbols $i = 1,\ldots,n$; or their negation in normal forms:

(a) Disjunctive normal form of a Boolean vector:

$$\vec{m}_j = \bigvee_{A \in \mathbf{P}(\Omega)} m_j(A) \bigwedge_{a_i \in A} \vec{m}_{a_i} \bigwedge_{a_j \in A^c} \vec{m}_{\bar{a}_j}$$

(b) Conjunctive normal form of a Boolean vector:

$$\vec{m}_j = \overline{\bigwedge_{A \in \mathbf{P}(\Omega)} m_j(A) \bigvee_{a_i \in A} \vec{m}_{\bar{a}_i} \bigvee_{a_j \in A^c} \vec{m}_{a_j}} .$$

Every conjunctive normal form is dual to a corresponding disjunctive normal form:

$$\bigvee_{A \in \mathbf{P}(\Omega)} m_j(A) \bigwedge_{a_i \in A} \vec{m}_{a_i} \bigwedge_{a_j \in A^c} \vec{m}_{\bar{a}_j} = \overline{\bigwedge_{A \in \mathbf{P}(\Omega)} m_j(A) \bigvee_{a_i \in A} \vec{m}_{\bar{a}_i} \bigvee_{a_j \in A^c} \vec{m}_{a_j}}$$

Every structural vector is equivalent to a corresponding conjunctive or disjunctive normal form, and can be reduced to it.

Example 18. A vector of the values of a structural function of logical implication in normal form:

$$\begin{aligned}\vec{m}_{a_1\Rightarrow a_2} &= \vec{m}_{a_1} \Rightarrow \vec{m}_{a_2}\\ &= \bar{\vec{m}}_{a_1} \vee \vec{m}_{a_2}.\end{aligned}$$

$$\vec{m}_{a_1\Rightarrow a_2} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \vee \begin{bmatrix}0\\0\\1\\1\end{bmatrix}.$$

Example 19. A structural vector of logical binary XOR in normal forms:

$$\begin{aligned}\vec{m}_{a_1\underline{\vee} a_2} &= \vec{m}_{a_1} \underline{\vee} \vec{m}_{a_2}\\ &= (\vec{m}_{a_1} \wedge \bar{\vec{m}}_{a_2}) \vee (\bar{\vec{m}}_{a_1} \wedge \vec{m}_{a_2})\\ &= \overline{(\bar{\vec{m}}_{a_1} \vee \vec{m}_{a_2}) \wedge (\vec{m}_{a_1} \vee \bar{\vec{m}}_{a_2})}.\end{aligned}$$

or

$$\begin{aligned}\vec{m}_{a_1\underline{\vee} a_2} &= \begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \wedge \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right) \vee \left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \wedge \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right).\\ &= \overline{\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \vee \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right) \wedge \left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \vee \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right)}.\end{aligned}$$

Example 20. A structural vector of a logical equivalence in normal forms:

$$\begin{aligned}\vec{m}_{a_1\Leftrightarrow a_2} &= \vec{m}_{a_1} \Leftrightarrow \vec{m}_{a_2}\\ &= (\vec{m}_{a_1} \wedge \vec{m}_{a_2}) \vee (\bar{\vec{m}}_{a_1} \wedge \bar{\vec{m}}_{a_2})\\ &= \overline{(\vec{m}_{a_1} \vee \vec{m}_{a_2}) \wedge (\bar{\vec{m}}_{a_1} \vee \bar{\vec{m}}_{a_2})}.\end{aligned}$$

or

$$\begin{aligned}\vec{m}_{a_1\Leftrightarrow a_2} &= \begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \vee \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right) \wedge \left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \vee \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right).\\ &= \overline{\left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \wedge \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right) \vee \left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \wedge \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)}.\end{aligned}$$

2.2. The structural function of pseudo-logical formula and pseudo-structural vector

Here we present the structural function of n-ary pseudo-Boolean formula sj. Pseudo-Boolean n-ary function $L_{sj} : \{0,1\}^n \to [0,1]$, of pseudo-Boolean formula sj, is defined as:

$$L_{sj}(a_1,...,a_n) = \sum_{A \in P(\Omega)} L_{sj}(e_A) \prod_{a_i \in A} a_i \prod_{a_j \in A^c} \bar{a}_j$$

$$\Omega = \{a_1,...,a_n\} \subset \{0,1\}^n$$

$$L_{sj}(e_A) \in [0,1]\,; \quad A \in \mathbf{P}(\Omega)\,.$$

Every pseudo-Boolean n-ary logical function can be represented as a linear convex combination of n-ary Boolean functions.

The structural function of a pseudo-Boolean formula and its generalization is determined by the truth-values of the pseudo-Boolean function for characteristic vectors.

Definition 21. The structural function of pseudo-logical formula sj (pseudo-logical structural function) is the set function $m_{sj} : \mathbf{P}(\Omega) \to [0,1]$; such that for every $A \in \mathbf{P}(\Omega)$

$$m_{sj}(A) = L_{sj}(e_A)\,,$$

where: $e_A = ((e_A)_i) \subset \{0,1\}^{|\Omega|}$ is the characteristic vector of $A \in \mathbf{P}(\Omega)$ and components $(e_A)_i = 1$ if and only if $a_i \in A$ and $L_{sj}(e_A)$ is the truth-value of that pseudo-logic formula for e_A.

For definition of the [0,1]-valued logical function we need to define the pseudo-logical structural vector of a pseudo-Boolean formula.

Definition 22. The pseudo-logical structural vector $\vec{m}_{sj}$ of a pseudo-Boolean formula sj, has components equal to the values of the pseudo-structural function of that pseudo-logical formula in a given order (lexicographic or modified lexicographic, for example).

Every pseudo-structural vector can be represented as a linear convex combination of structural vectors:

$$\vec{m}_{sj} = \sum_i l_i \vec{m}_{j_i}$$

$$\sum_i l_i = 1, \quad l_i \ge 0.$$

In the general case a component in a linear convex combination can be a null-vector too.

Geometrically the structural vectors of logical formulae are the $2^{2^{|\Omega|}}$ vertices of structural hyper-cube $[0,1]^{2^{|\Omega|}}$ which is a structural vector space. All other vectors of structural hyper-cubes are pseudo-logical structural vectors.

Unary logic operations on a pseudo-structural vector:

The pseudo-structural vector of the negation of logical function:

$$\vec{m}_{\overline{sj}} = \bar{\vec{m}}_{sj} = \vec{m}(\mathbf{1}) - \vec{m}_{sj} \ ,$$

or components of the pseudo-logical structural vector of the negation of logical function:

$$\bar{m}_{sj}\,(\mathrm{A}) = (1 - m_{sj}\,(\mathrm{A})) \in [0,1]\,;\ \ \mathrm{A} \in \mathbf{P}(\Omega)\ .$$

Binary logical operations are $\otimes \in \{\Leftarrow\!|, |\!\Rightarrow, \underline{\vee}, \wedge, \vee, \Rightarrow, \Leftarrow, \Leftrightarrow, \overline{\wedge}, \overline{\vee}\}$ on a pseudo-structural vector:

(A) The pseudo-structural vector of a binary logical operation on pseudo-logical structural vector $\vec{m}_{sj}$ of pseudo-Boolean formula sj and 0-1 structural vector $\vec{m}_j$ of logic formula j is:

$$\begin{aligned}\vec{m}_{sj\,\otimes j} &= \vec{m}_{sj} \otimes \vec{m}_j \\ &= \sum_{\mathrm{i}} l_{\mathrm{i}}\, \vec{m}_{j_{\mathrm{i}}} \otimes \vec{m}_j \\ &= \sum_{\mathrm{i}} l_{\mathrm{i}}\, \vec{m}_{j_{\mathrm{i}} \otimes j}\end{aligned}$$

$$\sum_{\mathrm{i}} l_{\mathrm{i}} = 1, \quad l_{\mathrm{i}} \geq 0\ .$$

(B) The pseudo-structural vector of a binary logical operation on two pseudo-logical formulae is:

$$\begin{aligned}\vec{m}_{sj_1 \otimes sj_2} &= \vec{m}_{sj_1} \otimes \vec{m}_{sj_2} \\ &= \sum_{\mathrm{i}} l_{\mathrm{i}} \vec{m}_{j_{1,\mathrm{i}}} \otimes \sum_{\mathrm{j}} l_{\mathrm{j}} \vec{m}_{j_{2,\mathrm{j}}} \\ &= \sum_{\mathrm{i}} \sum_{\mathrm{j}} l_{\mathrm{i}} l_{\mathrm{j}} \vec{m}_{j_{1,\mathrm{i}}} \otimes \vec{m}_{j_{2,\mathrm{j}}} \\ &= \sum_{\mathrm{i}} \sum_{\mathrm{j}} l_{\mathrm{i}} l_{\mathrm{j}} \vec{m}_{j_{1,\mathrm{i}} \otimes j_{2,\mathrm{j}}}\,; \\ \sum_{\mathrm{i}} l_{\mathrm{i}} = 1, \quad &\sum_{\mathrm{j}} l_{\mathrm{j}} = 1, \quad l_{\mathrm{i}}, l_{\mathrm{j}} \geq 0\ .\end{aligned}$$

Logical functions whose formulae have structural functional vectors equal to basic vectors - basic logical functions are very important for the definition of [0,1]-valued logical functions.

3. BASIC LOGICAL FUNCTIONS AND THE WEIGHTED VECTOR OF A LOGICAL FUNCTION

In this section we present basic logical functions of n-ary logical functions. A structural vector of an n-ary logic (and/or pseudo-logic) formula (structural and/or pseudo-structural vector) can be represented as a linear combination of n-ary logical structural basic vectors. Logical functions of basic structural vectors are basic logical functions. The value of a basic logical function for given values of [0,1]-valued logical atomic symbols is the weight coefficient for the corresponding basic vector - a component of the structural vector. A basic logical function depends, as parameters, on the chosen continual logical operators (AND and/or OR; t-norms and/or t-conorms). The vector of all n-ary basic logical functions (in the same order as the components in the structural function vectors) is the weighted vector of a logical and/or pseudo-logical formula.

Definition 23. Basic n-ary structural vectors are the vectors from the set of n-ary logical formula structural vectors $\vec{\blacksquare}(n)$ with only one component equal to 1 (and every other equal to 0).

Example 24. Basic Boolean vectors for binary logical formulae $(\Omega = \{a_1, a_2\})$

	$\vec{m}_{B(\emptyset)}$	$\vec{m}_{B(\{a_1\})}$	$\vec{m}_{B(\{a_2\})}$	$\vec{m}_{B(\{a_1,a_2\})}$
$m_{B(\cdot)}(\emptyset)$	1	0	0	0
$m_{B(\cdot)}(\{a_1\})$	0	1	0	0
$m_{B(\cdot)}(\{a_2\})$	0	0	1	0
$m_{B(\cdot)}(\{a_1, a_2\})$	0	0	0	1

Definition 25. Basis logical function $L_{B(A)}(a_1,\ldots,a_n)$ has a basic vector $\vec{m}_{B(A)}$ as a structural vector.

The set of structural vectors $\vec{\blacksquare}(n)$ has 2^n basic structural vectors of logical formulae and/or basic logical functions.

Example 26. Basic logical functions and corresponding basic structural vectors for the binary logical functions $(\Omega = \{a_1, a_2\})$:

Basic logical function	$m(\emptyset)$	$m(\{a_1\})$	$m(\{a_2\})$	$m(\{a_1, a_2\})$
$a_1 \overline{\vee} a_2$	1	0	0	0
$a_1 \Leftarrow\!| a_2$	0	1	0	0
$a_2 \Leftarrow\!| a_1$	0	0	1	0
$a_1 \wedge a_2$	0	0	0	1

where:

$L_{B(\emptyset)}(a_1, a_2) = a_1 \,\overline{\vee}\, a_2 = 1 - a_1 - a_2 + a_1 \wedge a_2$
$L_{B(\{a_1\})}(a_1, a_2) = a_1 \Leftarrow\mid a_2 = a_1 - a_1 \wedge a_2$
$L_{B(\{a_2\})}(a_1, a_2) = a_2 \Leftarrow\mid a_1 = a_2 - a_1 \wedge a_2$
$L_{B(\Omega)}(a_1, a_2) = a_1 \wedge a_2$

For a concise representation of [0,1]-valued logical functions it is useful to introduce a weighted vector of logical functions.

Definition 27. A weighted vector of a logical function is

$$\vec{\mathbf{L}}_B(a_1,\dots,a_n) = (L_{B(A)}(a_1,\dots,a_n) \mid A \in \mathbf{P}(\Omega);\ a_1,\dots,a_n \in [0,1])^T ,$$

where elements are basic logical functions in a defined order (lexicographic or modified lexicographic, for example).

Proposition 28. Components of weighted vector - weighted logical functions are the following basic logical functions of $a_i \in [0,1],\ i = 1,\dots,n$

$$L_{B(\emptyset)}(a_1,\dots,a_n) = \overline{\bigvee_{a_i \in \Omega} a_i}$$

$$L_{B(A)}(a_1,\dots,a_n) = \left(\bigwedge_{a_i \in A} a_i\right) \Leftarrow\mid \left(\bigvee_{a_j \in A^c} a_j\right); \quad A \subset \Omega,\ A^c = \Omega \setminus A ,$$

$$L_{B(\Omega)}(a_1,\dots,a_n) = \bigwedge_{a_i \in \Omega} a_i .$$

Proof: Corresponding 0-1 structural vectors of weighted logical functions are basic structural vectors. ❑

Any structural vector can be represented by relevant basic logical vectors. Here is an orthonormal basis and its logical interpretation:

$$\left[1 \quad \overline{\mathbf{0}}_{1\times(2^n-1)}\right]^T = \vec{m}_{\left[\left(\overline{\bigvee_{a_i \in \Omega} a_i}\right)\right]},$$

and

$$\left[\overline{\mathbf{0}}_{1\times m} \quad 1 \quad \overline{\mathbf{0}}_{1\times(2^n-1-m)}\right]^T = \vec{m}_{\left[\left(\bigwedge_{a_i \in A} a_i\right) \Leftarrow\mid \left(\bigvee_{a_j \in \Omega \setminus A} a_j\right)\right]},$$

where, m is the order of subset $A \in \mathbf{P}(\Omega)$ and

$$\begin{bmatrix} \mathbf{0}_{1\times(2^n-1)} & 1 \end{bmatrix}^T = \vec{m}\left[\left(\bigwedge_{a_i \in \Omega} a_i\right)\right].$$

Structural vector $\vec{m}_j = [m_j(A) \mid A \in \mathbf{P}(\Omega)]^T$ can be represented in this (the most natural) orthonormal basis as:

$$\vec{m} = m(\emptyset)\vec{m}\left[\overline{\bigvee_{a_i \in \Omega} a_i}\right] + \sum_{A \subset \Omega} m(A)\vec{m}\left[\left(\bigwedge_{a_i \in A} a_i\right) \Leftarrow \left(\bigvee_{a_j \in \Omega \setminus A} a_j\right)\right] + m(\Omega)\vec{m}\left[\bigwedge_{a_i \in \Omega} a_i\right].$$

In the case $\Omega \in \{0,1\}^n$ only one basic function is different from 0 (and equal to 1).

$$L_{B(A)}(e_C) = \begin{cases} 1 & C = A \\ 0 & C \neq A \end{cases}; \quad A, C \in \mathbf{P}(\Omega).$$

Basic logical functions - weighted coefficients $L_{B(A)}(a_1, \ldots, a_n)$; $A \in \mathbf{P}(\Omega)$, depending on the AND operator are:

$$L_{B(A)}(a_1, \ldots, a_n) = \sum_{n=0}^{|A^c|} (-1)^n \sum_{\substack{C \in \mathbf{P}(A^c) \\ |C| = n}} \bigwedge_{a_i \in A \cup C} a_i \; ; \; A \in \mathbf{P}(\Omega).$$

Every basic logical function is nonnegative :

$$L_{B(A)}(a_1, \ldots, a_n) \geq 0, \; A \in \mathbf{P}(\Omega), \; \Omega \subset [0,1]^n.$$

This is a constraint in choosing the AND operator:

$$\sum_{n=0}^{|A^c|} (-1)^n \sum_{\substack{C \in \mathbf{P}(A^c) \\ |C| = n}} \bigwedge_{a_i \in A \cup C} a_i \geq 0 \; ; \; A \in \mathbf{P}(\Omega).$$

3.1. Weighted logical functions for different continual logical AND operators

Here we analyze AND operators defined as (a) standard intersection $(a_1 \wedge a_2 := \min(a_1, a_2))$ and (b) algebraic product $(a_1 \wedge a_2 := a_1 \cdot a_2)$. Both AND operators satisfy the above constraint.

(A) For AND operator (t-norms) = algebraic product $(a \wedge b := a \cdot b)$ the basic logical functions are:

$$L_{B(\emptyset)}(a_1,\ldots,a_n) = \prod_{a_i \in \Omega} \bar{a}_i \ ,$$

$$L_{B(A)}(a_1,\ldots,a_n) = \prod_{a_i \in A} a_i \prod_{a_j \in A^c} \bar{a}_j \ ; \ A \subset \Omega, \ \ A^c = \Omega \setminus A \ ,$$

$$L_{B(\Omega)}(a_1,\ldots,a_n) = \prod_{a_i \in \Omega} a_i \ .$$

(B) For AND operator (t-norms) = standard intersection $(a \wedge b := \min(a,b))$ the basic logical functions are:

$$L_{B(A)}(a_1,\ldots,a_n) = \sum_{n=0}^{|A^c|} (-1)^n \sum_{\substack{C \in \mathbf{P}(A^c) \\ |C| = n}} \min_i \{a_i \mid a_i \in A \cup C\} \ ; \ A \in \mathbf{P}(\Omega) \ .$$

Remark 2. The value of the components of the weighted vector of an n-ary logical function depends only on the chosen AND operator (t-norm) and values of [0,1]-valued atomic symbols a_i, $i = 1,\ldots,n$.

4. THE [0,1]-VALUED LOGICAL FUNCTION

In this section we define the [0,1]-valued logical function as a natural generalization of the Boolean and pseudo-Boolean function. The two main parts of [0,1]-valued functions are a logical (pseudo-logical) structural vector and the weighted vector of a logical function - the vector of a logical basic function.

From the property that any logical (and/or pseudo-logical) structural vector $\vec{m}_j$ ($\vec{m}_{sj}$) can be represented by basic structural vectors $\vec{m}_{B(A)}, A \in \mathbf{P}(\Omega)$;

$$\vec{m}_j = \sum_{A \in \mathbf{P}(\Omega)} m_j(A) \vec{m}_{B(A)}$$

and that to every basic logical vector $\vec{m}_{B(A)}$ there corresponds a basic logical function $L_{B(A)}(a_1,\ldots,a_n)$, the definition of the [0,1]-valued logical function follows:

Definition 29. An n-ary [0,1]-valued logical function is:

$$\boxed{Lm_j(a_1,\ldots,a_n) = \sum_{A \in \mathbf{P}(\Omega)} m_j(A) L_{B(A)}(a_1,\ldots,a_n)}$$

or in vector form:

$$\boxed{Lm_j(a_1,\ldots,a_n) = \vec{m}_j^T \cdot \vec{\mathbf{L}}_B(a_1,\ldots,a_n)},$$

where: $a_1,\ldots,a_n$ are [0,1]-valued atomic symbols;

$\vec{m}_j$ is a structural (and/or pseudo-structural) vector of logical formula j (pseudo logical formula sj), whose components are the values of structural logical function $m_j(A) = L_j(e_A)$ (pseudo-logical function $m_{sj}(A) = L_{sj}(e_A)$) of j, (sj) for $A \in \mathbf{P}(\Omega)$;

$e_A = ((e_A)_i) \in \{0,1\}^{|\Omega|}$ is the characteristic vector of $A \in \mathbf{P}(\Omega)$ and components $(e_A)_i = 1$ if and only if $a_i \in A$;

$\vec{\mathbf{L}}_B(a_1,\ldots,a_n)$ is the weighted vector of the logical expression, whose components are basic logical functions $L_{B(A)}(a_1,\ldots,a_n)$, $A \in \mathbf{P}(\Omega)$ defined as:

$$L_{B(A)}(a_1,\ldots,a_n) = \left(\bigwedge_{a_i \in A} a_i \right) \Leftarrow \left| \left(\bigvee_{a_j \in A^c} a_j \right) \right., \; A \in \mathbf{P}(\Omega).$$

4.1. Properties of [0,1]-valued logical functions for logical structural vectors

Here we give some properties of [0,1]-valued logical functions for logical (0-1) structured vectors, which are direct consequences of the definition.

(P1) - The [0,1]-valued logical function for a basic structural vector is equal to the corresponding basic (weighted) logical function:

$$Lm_{B(A)}(a_1,\ldots,a_n) = L_{B(A)}(a_1,\ldots,a_n), \; A \in \mathbf{P}(\Omega).$$

(P2) - The [0,1]-valued logical function for a logical formula vector composed of basic vectors is equal to the sum of corresponding weighted logic functions

$$L\left(\sum_{A \in P} m_{B(A)} \right)(a_1,\ldots,a_n) = \sum_{A \in P} L_{B(A)}(a_1,\ldots,a_n), \; P \subset \mathbf{P}(\Omega).$$

Corollary 30. The sum of weighted logical functions (basic logical functions) for all $a_i \in [0,1]$, $i = 1,\ldots,n$; is identical to 1:

$$\sum_{A \in \mathbf{P}(\Omega)} L_{B(A)}(a_1,\ldots,a_n) \equiv 1.$$

Proof: Since

$$Lm_{\mathbf{1}}(a_1,\ldots,a_n) \equiv 1,$$

and

$$\vec{m}_{\mathbf{1}} = \sum_{A \in \mathbf{P}(\Omega)} \vec{m}_{B(A)},$$

and

$$\begin{aligned} \mathrm{L}m_{\mathbf{1}}(a_1,\ldots,a_n) &= \mathrm{L}\left(\sum_{A\in\mathbf{P}(\Omega)} m_{B(A)}\right)(a_1,\ldots,a_n) \\ &= \sum_{A\in\mathbf{P}(\Omega)} \mathrm{L}m_{B(A)}(a_1,\ldots,a_n) \\ &= \sum_{A\in\mathbf{P}(\Omega)} \mathrm{L}_{B(A)}(a_1,\ldots,a_n), \end{aligned}$$

it follows that:

$$\sum_{A\in\mathbf{P}(\Omega)} \mathrm{L}_{B(A)}(a_1,\ldots,a_n) \equiv 1\,. \qquad \square$$

A [0,1]-valued logical function is a linear convex combination of the components of the structural vector of a logical formula. In classical logic, the value of the logical function is equivalent to the value of the corresponding component of the structural vector.

A [0,1]-valued logical function in scalar form is:

$$\mathrm{L}m_j(a_1,\ldots,a_n) = \sum_{A\in\mathbf{P}(\Omega)} m_j(A)\left[\left(\bigwedge_{a_i\in A} a_i\right) \Leftarrow \left(\bigvee_{a_j\in A^c} a_j\right)\right].$$

Logical operations on [0,1]-valued logical functions are logical operations on their structural vectors. Here are some logical operations on [0,1]-valued logical functions.

(A) Nullary logical operations (logical constants):

$$\mathrm{L}m_1(a_1,\ldots,a_n) \equiv 1,$$
$$\mathrm{L}m_0(a_1,\ldots,a_n) \equiv 0.$$

(B) Unary logical operations (atomic formula and its negations):

$$\mathrm{L}m_{a_i}(a_1,\ldots,a_n) = a_i,\ \ a_i \in [0,1],\ \ i = 1,\ldots,n,$$
$$\mathrm{L}m_{\bar{a}_i}(a_1,\ldots,a_n) = \bar{a}_i,\ \ a_i \in [0,1],\ \ i = 1,\ldots,n,$$

$$\begin{aligned} \overline{\mathrm{L}m_j(a_1,\ldots,a_n)} &= \mathrm{L}m_{\bar{j}}(a_1,\ldots,a_n) \\ &= 1 - \mathrm{L}m_j(a_1,\ldots,a_n). \end{aligned}$$

(C) Binary logical operations:

$$\begin{aligned} \mathrm{L}m_{j_1}(a_1,\ldots,a_n) \otimes \mathrm{L}m_{j_2}(a_1,\ldots,a_n) &= \mathrm{L}(m_{j_1} \otimes m_{j_2})(a_1,\ldots,a_n) \\ &= \mathrm{L}m_{(j_1\otimes j_2)}(a_1,\ldots,a_n), \end{aligned}$$

where: $\otimes \in \{\Leftarrow|, |\Rightarrow, \underline{\vee}, \wedge, \vee, \Rightarrow, \Leftarrow, \Leftrightarrow, \overline{\wedge}, \overline{\vee}\}$.

(D) N-ary logical operations:

$$\Diamond\,(Lm_{j_1}(a_1,\dots,a_n),\dots,Lm_{j_n}(a_1,\dots,a_n)) = L(\Diamond(m_{j_1},\dots,m_{j_n}))(a_1,\dots,a_n) = Lm_{\Diamond(j_1,\dots,j_n)}(a_1,\dots,a_n).$$

Example 31. (A) The classical law of the excluded middle

$$a \vee \bar{a} = 1\,,$$

is valid in [0,1]-valued logic too, since $a \vee \bar{a} = 1$ is a unary logical formula and the corresponding structural vector is a logical constant $\mathbf{1}$:

$$\vec{m}_a \vee \vec{m}_{\bar{a}} = \begin{bmatrix}0\\1\end{bmatrix} \vee \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} = \mathbf{1}$$

it follows that:

$$\begin{aligned} Lm_a(a) \vee L\overline{m}_a(a) &= L(m_a \vee \overline{m}_a)(a)\\ &= Lm_{a\vee\bar{a}}(a)\\ &= Lm_{\mathbf{1}}(a) \equiv 1. \end{aligned}$$

(B) The contradiction law (dual to the law of the excluded middle)

$$a \wedge \bar{a} = 0\,,$$

is valid in [0,1]-valued logic too; the corresponding structural vector is logical constant $\mathbf{0}$:

$$\vec{m}_a \wedge \vec{m}_{\bar{a}} = \begin{bmatrix}0\\1\end{bmatrix} \wedge \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \mathbf{0}$$

it follows that:

$$\begin{aligned} Lm_a(a) \wedge L\overline{m}_a(a) &= L(m_a \wedge \overline{m}_a)(a)\\ &= Lm_{a\wedge\bar{a}}(a)\\ &= Lm_{\mathbf{0}}(a) \equiv 0. \end{aligned}$$

Example 32. The following four logical formulae are equivalent in classical logic:

$$a \Rightarrow b = \bar{a} \vee b = \bar{a} \vee (a \wedge b) = (\bar{a} \wedge \bar{b}) \vee b\,;$$

in [0,1]-valued logic they are equivalent too, since the structural vectors are equal:

$$\vec{m}_a \Rightarrow \vec{m}_b = \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix},$$

$$\vec{m}_{\bar{a}} \vee \vec{m}_b = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \vee \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix},$$

$$\vec{m}_{\bar{a}} \vee (\vec{m}_a \wedge \vec{m}_b) = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \vee \left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \wedge \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\\1\end{bmatrix},$$

$$\vec{m}_b \vee (\vec{m}_{\bar{a}} \wedge \vec{m}_{\bar{b}}) = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \vee \left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \wedge \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\\1\end{bmatrix},$$

and

$$\mathrm{L}m_a(a,b) \Rightarrow \mathrm{L}m_b(a,b) = \mathrm{L}(m_a \Rightarrow m_b)(a,b)\ ,$$
$$\mathrm{L}\overline{m}_a(a,b) \vee \mathrm{L}m_b(a,b) = \mathrm{L}(\overline{m}_a \vee m_b)(a,b)\ ,$$
$$\mathrm{L}\overline{m}_a(a,b) \vee (\mathrm{L}m_a(a,b) \wedge \mathrm{L}m_b(a,b)) = \mathrm{L}(\overline{m}_a \vee (m_a \wedge m_b))(a,b)\ ,$$
$$\mathrm{L}m_b(a,b) \vee (\mathrm{L}\overline{m}_a(a,b) \wedge \mathrm{L}\overline{m}_b(a,b)) = \mathrm{L}(m_a \vee (\overline{m}_a \wedge \overline{m}_b))(a,b)\ ,\quad a,b \in [0,1]\,.$$

4.2. Properties of [0,1]-valued logical functions for pseudo-logical structural vectors

Here we give some properties of [0,1]-valued logical formulae for pseudo-structural vectors, which are direct consequences of the definition.

A [0,1]-valued logical formula for a pseudo-structural vector:

$$\mathrm{L}m_{sj}(a_1,\ldots,a_n) = \vec{m}_{sj}^{\mathrm{T}} \cdot \vec{\mathbf{L}}_B(a_1,\ldots,a_n)$$
$$= \sum_{A \in \mathbf{P}(\Omega)} m_{sj}(A) \left(\bigwedge_{a_p \in A} a_p \Leftarrow\!\!| \bigvee_{a_q \in A^c} a_q \right)$$

Since, any pseudo-structural vector can be represented as a linear convex combination of structural vectors $\vec{m}_{sj} = \sum_i l_i \vec{m}_{j_i}$ it follows that:

$$\begin{aligned} \mathrm{L}m_{sj}(a_1,...,a_n) &= \mathrm{L}\left(\sum_i l_i \vec{m}_{j_i}\right)(a_1,...,a_n) \\ &= \sum_i l_i \vec{m}_{j_i}^{\mathrm{T}} \cdot \vec{\mathbf{L}}_{\mathrm{B}}(a_1,...,a_n) \\ &= \sum_i l_i \mathrm{L}m_{j_i}(a_1,...,a_n). \end{aligned}$$

The negation of a pseudo-Boolean vector in vector form is:

$$\vec{m}_{sj} + \vec{\overline{m}}_{sj} = \vec{m}_{\mathbf{1}}$$

or in scalar form:

$$\overline{m}_{sj}(\mathrm{A}) = 1 - \overline{m}_{sj}(\mathrm{A}); \quad \mathrm{A} \in \mathbf{P}(\Omega).$$

From the definition of the [0,1]-valued logical function, it follows for negation in the case of a pseudo-structural vector that:

$$\begin{aligned} \overline{\mathrm{L}m_{sj}(a_1,...,a_n)} &= \mathrm{L}\overline{m}_{sj}(a_1,...,a_n) \\ &= 1 - \mathrm{L}m_{sj}(a_1,...,a_n). \end{aligned}$$

In the case of a pseudo-structural vector $\vec{m}_{sj} = a\vec{m}_j$, where: $a \in [0,1]$ and $\vec{m}_j$ is a 0-1 structural vector:

$$\begin{aligned} \mathrm{L}m_{sj}(a_1,...,a_n) &= \mathrm{L}(am_j)(a_1,...,a_n) \\ &= a\,\mathrm{L}m_j(a_1,...,a_n); \quad a \in [0,1]. \end{aligned}$$

Binary logical operations $\otimes \in \{\Leftarrow\!|, |\!\Rightarrow, \underline{\vee}, \wedge, \vee, \Rightarrow, \Leftarrow, \Leftrightarrow, \overline{\wedge}, \overline{\vee}\}$:

(A) On function $\mathrm{L}m_{sj}(a_1,...,a_n)$ with pseudo-logical structural vector m_{sj} and function $\mathrm{L}m_j(a_1,...,a_n)$ with logical structural vector m_j:

$$\begin{aligned} \mathrm{L}m_{sj}(a_1,...,a_n) \otimes \mathrm{L}m_j(a_1,...,a_n) &= \sum_i l_i \mathrm{L}m_{j_i}(a_1,...,a_n) \otimes \mathrm{L}m_j(a_1,...,a_n) \\ &= \sum_i l_i \mathrm{L}(m_{j_i} \otimes m_j)(a_1,...,a_n) \\ &= \sum_i l_i \mathrm{L}m_{j_i \otimes j}(a_1,...,a_n) \end{aligned}$$

(B) on two functions with pseudo-logic structural vectors:

$$\begin{aligned}
Lm_{sj_1}(a_1,\dots,a_n)\otimes Lm_{sj_2}(a_1,\dots,a_n) &= Lm_{\sum_i l_i j_{1,i}}(a_1,\dots,a_n)\otimes Lm_{\sum_j l_j j_{2,j}}(a_1,\dots,a_n)\\
&=\sum_i l_i\, Lm_{j_{1,i}}(a_1,\dots,a_n)\otimes\sum_j l_j\, Lm_{j_{2,j}}(a_1,\dots,a_n)\\
&=\sum_i\sum_j l_i l_j\, Lm_{j_{1,i}}(a_1,\dots,a_n)\otimes Lm_{j_{2,j}}(a_1,\dots,a_n)\\
&=\sum_i\sum_j l_i l_j\, L(m_{j_{1,i}}\otimes m_{j_{2,j}})(a_1,\dots,a_n)\\
&=\sum_i\sum_j l_i l_j\, Lm_{j_{1,i}\otimes j_{2,j}}(a_1,\dots,a_n)
\end{aligned}$$

where:

$$\vec{m}_{sj_1}=\sum_i l_i\vec{m}_{j_{1,i}},\quad \sum_i l_i=1,\quad l_i\ge 0,$$
$$\vec{m}_{sj_2}=\sum_j l_j\vec{m}_{j_{2,j}},\quad \sum_j l_j=1,\quad l_j\ge 0.$$

A [0,1]-valued logical function depends on the chosen continual logical operators as parameters. Different continual AND operators generate different [0,1]-valued logics. Here we analyze two very important cases: the AND operator is defined as (a) an algebraic product and (b) a standard intersection. The analysis of other AND continual operators (t-norms) in [0,1]-valued logic(s) will be the subject of a forth-coming paper.

4.3. [0,1]-valued logical functions for continual logical AND - algebraic product

[0,1]-valued logical functions for a continual AND operator (t-norms) defined as an algebraic product $(a_1\wedge a_2:=a_1\cdot a_2)$ have the following form:

$$Lm_j(a_1,\dots,a_n)=\sum_{A\in\mathbf{P}(\Omega)} m_j(A)\prod_{a_i\in A}a_i\prod_{a_j\in A^c}\bar{a}_j\ .$$

Since $m_j(A)=L_j(e_A),\ A\in\mathbf{P}(\Omega)$; by definition, it follows that:

$$Lm_j(a_1,\dots,a_n)=\sum_{A\in\mathbf{P}(\Omega)} L_j(e_A)\prod_{a_i\in A}a_i\prod_{a_j\in A^c}\bar{a}_j\ .$$

The disjunctive canonical form of the $\{0,1\}$-valued logical function:

$$L_j(a_1,\dots,a_n)=\bigvee_{A\in\mathbf{P}(\Omega)} L_j(e_A)\prod_{a_i\in A}a_i\prod_{a_j\in A^c}\bar{a}_j\ ;$$

is very similar to the [0,1]-valued logical function for a t-norm defined as an algebraic product $(a\wedge b:=ab)$:

$$Lm_j(a_1,\ldots,a_n) = \sum_{A\in\mathbf{P}(\Omega)} L_j(e_A) \prod_{a_i\in A} a_i \prod_{a_j\in A^c} \bar{a}_j \ ;$$

$$\Omega \subset [0,1]^n \ .$$

Example 33. A [0,1]-valued logical function for a binary case $(\Omega = \{a_1, a_2\})$:

$$\begin{aligned} Lm_j(a_1,a_2) = {} & m_j(\emptyset)\,(1 - a_1 - a_2 + a_1 \wedge a_2) + \\ & m_j(\{a_1\})\,(a_1 - a_1 \wedge a_2) + \\ & m_j(\{a_2\})\,(a_2 - a_1 \wedge a_2) + \\ & m_j(\{a_1,a_2\})\; a_1 \wedge a_2. \end{aligned}$$

when the AND operator is an algebraic product, $a_1 \wedge a_2 := a_1 \cdot a_2$, is

$$\begin{aligned} Lm_j(a_1,a_2) = {} & m_j(\emptyset)\,(1 - a_1)(1 - a_2) + \\ & m_j(\{a_1\})\; a_1(1 - a_2) + \\ & m_j(\{a_2\})\,(1 - a_1)a_2 + \\ & m_j(\{a_1,a_2\})\; a_1 a_2. \end{aligned}$$

Example 34. The connection between a probability function and a [0,1]-valued logical function.
Set of elementary events (universe) $\Omega = \{\Pi, \Gamma\}$, power set of universe (elementary events) $\mathbf{P}(\Omega) = \{\emptyset, \{\Pi\}, \{\Gamma\}, \{\Pi,\Gamma\}\}$. The structural vectors of elementary events (atomic symbols) - values of measure of elementary events:

	$\vec{m}_\Pi$	$\vec{m}_\Gamma$
$m(\emptyset)$	0	0
$m(\{\Pi\})$	1	0
$m(\{\Gamma\})$	0	1
$m(\{\Pi,\Gamma\})$	1	1

The structural function of logical event $\Pi \otimes \Gamma$ defined by the structural function of elementary events:

$$\begin{aligned} m_{\Pi\otimes\Gamma}(\emptyset) &= m_\Pi(\emptyset) \otimes m_\Gamma(\emptyset) \\ m_{\Pi\otimes\Gamma}(\{\Pi\}) &= m_\Pi(\{\Pi\}) \otimes m_\Gamma(\{\Pi\}) \\ m_{\Pi\otimes\Gamma}(\{\Gamma\}) &= m_\Pi(\{\Gamma\}) \otimes m_\Gamma(\{\Gamma\}) \\ m_{\Pi\otimes\Gamma}(\{\Pi,\Gamma\}) &= m_\Pi(\{\Pi,\Gamma\}) \otimes m_\Gamma(\{\Pi,\Gamma\}) \end{aligned}$$

or in vector form

$$\vec{m}_{\Pi\otimes\Gamma} = \vec{m}_{\Pi} \otimes \vec{m}_{\Gamma}$$

where $\otimes \in \{\vee, \overline{\vee}, \wedge, \overline{\wedge}, \underline{\vee}, \Leftrightarrow, \Leftarrow|, \Rightarrow, |\Rightarrow, \Leftarrow\}$.

Probability of elementary events:

$$\mathrm{p}(\Pi) = \mathrm{p}_{\Pi}$$
$$\mathrm{p}(\Gamma) = \mathrm{p}_{\Gamma}$$
$$\mathrm{p}_{\Pi} + \mathrm{p}_{\Gamma} = 1$$

The probability of logical event $\Pi \otimes \Gamma$, as a function of the probability of elementary event p_{Π} (or p_{Γ}) since $\mathrm{p}_{\Pi} + \mathrm{p}_{\Gamma} = 1$, is:

$$\begin{aligned} \mathrm{P}m_{(\Pi\otimes\Gamma)}(\mathrm{p}_{\Pi}) &= m_{\Pi}(\emptyset) \otimes m_{\Gamma}(\emptyset)\,\overline{\mathrm{p}}_{\Pi}\,\overline{\mathrm{p}}_{\Gamma} + m_{\Pi}(\{\Pi\}) \otimes m_{\Gamma}(\{\Pi\})\,\mathrm{p}_{\Pi}\,\overline{\mathrm{p}}_{\Gamma} + \\ &\quad + m_{\Pi}(\{\Gamma\}) \otimes m_{\Gamma}(\{\Gamma\})\,\overline{\mathrm{p}}_{\Pi}\,\mathrm{p}_{\Gamma} + m_{\Pi}(\{\Pi,\Gamma\}) \otimes m_{\Gamma}(\{\Pi,\Gamma\})\,\mathrm{p}_{\Pi}\,\mathrm{p}_{\Gamma} \\ &= m_{\Pi\otimes\Gamma}(\emptyset)\,\overline{\mathrm{p}}_{\Pi}\,\overline{\mathrm{p}}_{\Gamma} + m_{\Pi\otimes\Gamma}(\{\Pi\})\,\mathrm{p}_{\Pi}\,\overline{\mathrm{p}}_{\Gamma} + m_{\Pi\otimes\Gamma}(\{\Gamma\})\,\overline{\mathrm{p}}_{\Pi}\,\mathrm{p}_{\Gamma} + m_{\Pi\otimes\Gamma}(\{\Pi,\Gamma\})\,\mathrm{p}_{\Pi}\,\mathrm{p}_{\Gamma}, \end{aligned}$$

or in vector form:

$$\mathrm{P}m_{(\Pi\otimes\Gamma)}(\mathrm{p}_{\Pi}) = \vec{m}^{\mathrm{T}}_{\Pi\otimes\Gamma}\ \vec{\mathrm{p}}_{\Pi,\Gamma},$$

where: $\vec{m}_{\Pi\otimes\Gamma}$ is the structural vector of logical formula $\Pi \otimes \Gamma$

$$\vec{m}_{\Pi\otimes\Gamma} = [m_{\Pi\otimes\Gamma}(\emptyset) \quad m_{\Pi\otimes\Gamma}(\{\Pi\}) \quad m_{\Pi\otimes\Gamma}(\{\Gamma\}) \quad m_{\Pi\otimes\Gamma}(\{\Pi,\Gamma\})]^{\mathrm{T}},$$

$\vec{\mathrm{p}}_{\Pi,\Gamma}$ is a vector of the basic logical function - a vector of the weighted coefficients of the logical formula

$$\vec{\mathrm{p}}_{\Pi,\Gamma} = [\overline{\mathrm{p}}_{\Pi}\,\overline{\mathrm{p}}_{\Gamma} \quad \mathrm{p}_{\Pi}\,\overline{\mathrm{p}}_{\Gamma} \quad \overline{\mathrm{p}}_{\Pi}\,\mathrm{p}_{\Gamma} \quad \mathrm{p}_{\Pi}\,\mathrm{p}_{\Gamma}]^{\mathrm{T}}.$$

Structural vectors of binary events are:

	$\vec{m}_{(\mathbf{0})}$	$\vec{m}_{(\Pi\overline{\vee}\Gamma)}$	$\vec{m}_{(\Pi\Leftarrow\mid\Gamma)}$	$\vec{m}_{(\Pi\mid\Rightarrow\Gamma)}$	$\vec{m}_{(\Pi\wedge\Gamma)}$	$\vec{m}_{(\overline{\Gamma})}$	$\vec{m}_{(\overline{\Pi})}$	$\vec{m}_{(\Pi\Leftrightarrow\Gamma)}$
$m(\emptyset)$	0	1	0	0	0	1	1	1
$m(\{\Pi\})$	0	0	1	0	0	1	0	0
$m(\{\Gamma\})$	0	0	0	1	0	0	1	0
$m(\{\Pi,\Gamma\})$	0	0	0	0	1	0	0	1

	$\vec{m}_{(\Pi \veebar \Gamma)}$	$\vec{m}_{(\Pi)}$	$\vec{m}_{(\Gamma)}$	$\vec{m}_{(\Pi \barwedge \Gamma)}$	$\vec{m}_{(\Pi \Leftarrow \Gamma)}$	$\vec{m}_{(\Pi \Rightarrow \Gamma)}$	$\vec{m}_{(\Pi \vee \Gamma)}$	$\vec{m}_{(\mathbf{1})}$
$m(\emptyset)$	0	0	0	1	1	1	0	1
$m(\{\Pi\})$	1	1	0	1	1	0	1	1
$m(\{\Gamma\})$	1	0	1	1	0	1	1	1
$m(\{\Pi,\Gamma\})$	0	1	1	0	1	1	1	1

4.4. [0,1]-valued logical functions for continual logical AND - standard intersection

[0,1]-valued logical functions for a continual AND operator (t-norms) defined as a standard intersection: $(a_1 \wedge a_2 := \min(a_1, a_2))$ have the following form:

$$Lm_j(a_1,\ldots,a_n) = \sum_{A \in \mathbf{P}(\Omega)} m_j(A) \left(\sum_{n=0}^{|A^c|} (-1)^n \sum_{\substack{C \in \mathbf{P}(A^c) \\ |C|=n}} \min_i \{a_i \mid a_i \in A \cup C\} \right);$$

Example 35. In the case of $a_1 \wedge a_2 = \min(a_1, a_2)$ the values of the following binary logical functions $(a_1 \wedge a_2, a_1 \vee a_2, a_1 \Rightarrow a_2$ and $a_1 \Leftrightarrow a_2)$ for $a_i \in \{0, \frac{1}{2}, 1\}; i = 1,2$ are given in the next table

a_1	a_2	$Lm_{(a_1 \wedge a_2)}(a_1, a_2)$	$Lm_{(a_1 \vee a_2)}(a_1, a_2)$	$Lm_{(a_1 \Rightarrow a_2)}(a_1, a_2)$	$Lm_{(a_1 \Leftrightarrow a_2)}(a_1, a_2)$
0	0	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1

The values are the same as in the Lukasiewicz three-valued logic [6]

a_1	a_2	$a_1 \wedge a_2$	$a_1 \vee a_2$	$a_1 \Rightarrow a_2$	$a_1 \Leftrightarrow a_2$
0	0	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1

but in the new [0,1]-valued logic:

$$\mathrm{L}m_{(a_i \wedge \bar{a}_i)}(a_1, a_2) \equiv 0; \quad i = 1,2$$
$$\mathrm{L}m_{(a_i \vee \bar{a}_i)}(a_1, a_2) \equiv 1; \quad i = 1,2$$

contrary to Lukasiewicz:

$$a_1 \wedge \bar{a}_1 = \tfrac{1}{2}; \quad a_1 = \tfrac{1}{2},$$
$$a_1 \vee \bar{a}_1 = \tfrac{1}{2}; \quad a_1 = \tfrac{1}{2}.$$

Very interesting representations of [0,1]-valued logical functions for the AND operator defined as a standard intersection are the discrete Choquet integral [1] for monotone structural function (fuzzy measures) and the discrete logical-Choquet integral [10] for any structural function (structural and/or pseudo-structural vectors).

For a monotone (pseudo-logical in the general case) structural function m_{m} - fuzzy measure [11]:

(a) $m_{\mathrm{m}}(\emptyset) = 0,\ m_{\mathrm{m}}(\Omega) = 1$;

(b) if $A, B \in \mathbf{P}(\Omega)$ and $A \subseteq B$ then $m_{\mathrm{m}}(A) \le m_{\mathrm{m}}(B)$;

and the AND logical operator defined as standard intersection $(a_1 \wedge a_2 := \min(a_1, a_2))$, then the [0,1]-valued n-ary logic function is equivalent to the discrete fuzzy Choquet integral

$$\mathrm{L}m_{\mathrm{m}}(\mathrm{a}_1,...,\mathrm{a}_\mathrm{n}) = \mathrm{C}_{m_\mathrm{m}}(\mathrm{a}_1,...,\mathrm{a}_\mathrm{n}).$$

Corollary 36. OWA - An ordered weighted averaging operator is an n-ary [0,1]-valued logical function for a symmetric monotone structural function m_{sm} and with the AND operator defined as a standard intersection

$$\mathrm{OWA_w}(\mathrm{a}_1,...,\mathrm{a}_\mathrm{n}) = \mathrm{L}m_{\mathrm{sm}}(\mathrm{a}_1,...,\mathrm{a}_\mathrm{n}).$$

Any [0-1]-valued n-ary logical function for a given structural function and the AND logical operator defined as a standard intersection is equivalent to the generalized logical-Choquet integral for that structural function [10]. A logical-Choquet integral is given by the following definition:

Definition 37. A structural (pseudo-logical in the general case) function m is a set function $m:\mathbf{P}(\Omega) \to [0,1]$, where $\Omega \in [0,1]^\mathrm{n}$, and $\mathrm{n} = |\Omega|$. The generalized discrete logical-Choquet integral with respect to m is defined by

$$\mathrm{LC}_m(\mathrm{a}_1,...,\mathrm{a}_\mathrm{n}) \coloneqq \sum_{\mathrm{i}=1}^{\mathrm{n}+1}(\mathrm{a}_{(\mathrm{i})} - \mathrm{a}_{(\mathrm{i}-1)})m(\mathrm{A}_{(\mathrm{i})})$$

where: $\cdot_{(\mathrm{i})}$ indicates that the indices have been permuted so that $0 \le \mathrm{a}_{(1)} \le ... \le \mathrm{a}_{(\mathrm{n})} \le 1$, and

$\mathrm{A}_{(\mathrm{i})} \coloneqq \{\mathrm{a}_{(\mathrm{i})},...,\mathrm{a}_{(\mathrm{n})}\}$,

$\mathrm{A}_{(\mathrm{n}+1)} \coloneqq \emptyset; \quad \Rightarrow m(\mathrm{A}_{(\mathrm{n}+1)}) = m(\emptyset) \ge 0$, and

$\mathrm{a}_{(0)} = 0$ and $\mathrm{a}_{(\mathrm{n}+1)} = 1$.

Example 38. For attribute values $\mathrm{a}_1 = 0.2$, $\mathrm{a}_2 = 0.6$ and for the following structural function of logical formulae, the values of the logical-Choquet integral are given in the next table:

m_j	m_0	m_1	m_2	$m_{1,2}$	$\mathrm{L}m_j(0.2,0.6) = \mathrm{LC}_{m_j}(0.2,0.6)$
$\vec{m}_{\mathrm{a}_1}$	0	1	0	1	0.2
$\vec{m}_{\overline{\mathrm{a}}_1}$	1	0	1	0	0.8
$\vec{m}_{\underline{\vee}}$	0	1	1	0	0.4
$\vec{m}_{\Leftrightarrow}$	1	0	0	1	0.6

5. CONCLUSION

The new [0,1]-valued logic includes the classical Boolean {0,1}-valued logic as a special case. All properties of Boolean logic are fulfilled (for example: logic laws of the excluded middle and contradiction; the number of logical formulae (2^{2^n}) depends only on n-numbers of atomic symbols; equivalent Boolean formulae have equivalent [0,1]-valued logical functions, etc.)

Works on fuzzy measures and fuzzy integrals ingeniously initialized by professor Sugeno in the early seventies [11] contain the main directions to the new [0,1]-valued logic.

An n-ary [0,1]-valued logical function is composed of (a) the structural function of its formula and/or a structural vector and (b) basic logical functions and/or a vector of weighting coefficients.

The structural function of logical formulae is a set function defined on a power set of atomic symbols - and doesn't depend on the values of the logical variables. The values of the structural function of a logical formula are arranged in a structural vector. Boolean algebra is defined on a set of structural vectors of all n-ary formulae.

Basis logical functions depend on the values of [0,1]-valued atoms (elementary logical variables). The values of basic logical functions are arranged in a weighted vector. The sum of the values of the components of a weighted vector is 1 and a [0,1]-valued logical function is a linear convex combination of the components of the structural vector. In the case when atomic symbols are {0,1}-valued, a weighted vector has only one component different from zero and equal to 1. This is the reason why only the truth-values of a Boolean, {0,1}-valued, function are completely determined by the truth-values of components of its structural vector (truth-table column) and Boolean logic can be applied directly to atomic values.

Logic operations (nullary, unary, binary or n-ary) on n-ary [0,1]-valued logical functions are logic operations on their structural vectors.

Basis logical functions and, as a consequence, a [0,1]-valued logical function, depend on (chosen) continual logical operators as parameters. For different logical operators we have different logics. In this paper the two most important cases are analyzed: (a) an AND continual logical operator defined as an algebraic product and (b) AND as a standard intersection. [0,1]-logical functions for other AND operators will be the subject of a forthcoming paper.

It is shown by an example that new [0,1]-valued logical binary functions for an AND operator defined as a standard intersection can be {0,1/2,1}-valued logic and have the same truth values as the Lukasiewicz binary conjunction, disjunction, equivalence and implication but have a different interpretation.

It is also shown by an example that new [0,1]-valued logical binary functions for an AND operator defined as an algebraic multiplication can be used for determination of the probability of an experiment (or game) on the basis of the probability of an elementary event. Since the probability of an elementary event in probability theory corresponds to the truth-value of atomic symbol, problems from probability theory can be treated as a special class of problems from [0,1]-valued logic. Probability as a class of [0,1]-valued logic will be the subject of a forthcoming paper.

The generalization of an n-ary pseudo-Boolean function is presented too. To an n-ary pseudo-Boolean function there corresponds a pseudo-logical structural function and/or a pseudo-logical structural vector. Any pseudo-logical structural vector can be represented as a linear convex combination of 0-1 structural vectors.

An n-ary [0,1]-valued logical function for a monotone structural set function and/or monotone pseudo-logical structural function and an AND continual logic operator defined as a standard intersection is equivalent to the corresponding discrete fuzzy Choquet integral [1], [11]. In a special case, when a monotone structural set function is symmetric too, a [0,1]-valued logical function is equivalent to the corresponding OWA operator [12].

A logical-Choquet integral [10] is equivalent to the corresponding [0,1]-valued logic function for an AND continual logic operator defined as a standard intersection, without constraints on the type of structural set function (logical and/or pseudo-logical).

The application of [0,1]-valued logic to different problems in OR (MADM, classification of elements, analysis of system reliability, data mining etc.) will be the subject of a forthcoming paper.

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