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ABOUT THE PROBLEM OF DISJUNCTIVE PROGRAMMING*

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Abstract: In this paper we investigate the problems of disjunctive programming with an infinite array of components forming a feasible set (as their union). The investigation continues the theme of the author's earlier work and describes an original conceptual approach to a) the analysis of saddle point problems for disjunctive Lagrangian functions, b) the analysis of dual relations for disjunctive programming problems and c) the technique of equivalent (on argument) reduction of such problems to the problems of unconstrained optimization.

Keywords: Disjunctive programming, Lagrangian duality, piece-wise linear functions.

1. INTRODUCTION

It is convenient to start with some definitions. Let $\{F_a(x)\}_{a\in\Omega}$ be a given set of vector-functions defined on $\mathbf{R}^n : x \xrightarrow{F_a} F_a(x) \in \mathbf{R}^{m_a}$. Introduce discrete maximum operation $|\cdot|_{\max}$: if $z^T = [z_1, \dots, z_k]$, then $|z|_{\max} = \max_{i=1,\dots,k} z_i$. Let us call the inequality

$$\sup_{a\in\Omega} |F_a(x)|_{\max} \le 0, \quad x \ge 0 \tag{1.1}$$

conjunctive and the inequality

$$\inf_{a\in\Omega} |F_a(x)|_{\max} \le 0, \quad x \ge 0 \tag{1.2}$$

disjunctive (constraint $x \ge 0$ in both cases being included for convenience, in particular, for the convenience of the dual framework).

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Let us denote $M_a := \{x \ge 0 \mid F_a(x) \le 0\}$. The solution sets $M_\cap$ and $M_\cup$ for the systems (1.1) and (1.2) take the forms: $M_\cap = \bigcap_{a\in\Omega} M_a$, $M_\cup = \bigcup_{a\in\Omega} M_a$. The problem

$$P_\cap : \sup\{f(x) \mid x \in M_\cap\} \tag{1.3}$$

is very standard for mathematical programming (MP) theory. The problem $\sup\{f(x) \mid x \in M_\cup\}$, i.e.

$$P_\cup : \sup\{f(x) \mid \inf_{(a)} |F_a(x)|_{\max} \le 0,\ x \ge 0\} \quad (=: g) \tag{1.4}$$

is well-known as the problem of disjunctive programming. We are interested in a special case of (1.4), namely:

$$\sup\{(c, x) \mid \inf_{a\in\Omega} |A_a x - b_a|_{\max} \le 0,\ x \ge 0\}. \tag{1.5}$$

In what follows we shall use Lagrangian functions associated with (1.3) and (1.4). If the set Ω is finite, i.e. $\Omega = \{1,\dots,m\}$, then the classical Lagrangian for (1.3) takes the form $f(x) - \sum_{j=1}^{m} (u_j, F_j(x))$, where $u^T := [u_1,\dots,u_m] \ge 0$. Along with this it may be convenient to operate with the function $F_0(x,u) := f(x) - \max_{(j)}(u_j, F_j(x))$ too; the character of the connection between this function and the problem

$$\sup\{f(x) \mid F_j(x) \le 0,\ j = 1,\dots,m,\ x \ge 0\} \tag{1.6}$$

being the same as between it and the classical Lagrangian, e.g. if pair $[\bar{x}, \bar{u}]$ denotes a saddle point of $F_0(x,u)$, then $\bar{x} \in \arg$ (1.6). This is a good reason to define the Lagrangian function for the general conjunctive instance (1.3) similarly:

$$F_\cap(x,u) := f(x) - \sup_{a\in\Omega}(u_a, F_a(x)), \quad u_a \ge 0. \tag{1.7}$$

Analogously, with the disjunctive problem (1.4) we shall associate the Lagrangian

$$F_\cup(x,u) := f(x) - \inf_{a\in\Omega}(u_a, F_a(x)), \quad u_a \ge 0. \tag{1.8}$$

In a couple with (1.7) and (1.8) we shall exploit modified Lagrangians:

$$F_\cap^\oplus(x,u) := f(x) - \sup_{a\in\Omega}(u_a, F_a^+(x)), \quad u_a \ge 0, \qquad (1.7)_+$$

$$F_\cup^\oplus(x,u) := f(x) - \inf_{a\in\Omega}(u_a, F_a^+(x)), \quad u_a \ge 0, \qquad (1.8)_+$$

where super-index "+" means a positive cut, i.e. if $z^T = [z_1, \dots, z_k]$, then $(z^T)^+ = [z_1^+, \dots, z_k^+]$, $z_i^+ = \max\{0, z_i\}$. Let us emphasize that in $(1.7)_+$ and $(1.8)_+$ $\{u_a \geq 0\}$ denotes a system of Lagrange multipliers, which under some conditions can be fixed at a level $R_a > 0$ such that the problem (1.1) is equivalent to

$$\sup_{x \leq 0} F_{\cap}^{\oplus}(x, R), \tag{1.9}$$

and the problem (1.2) is equivalent to

$$\sup_{x \leq 0} F_{\cup}^{\oplus}(x, R). \tag{1.10}$$

This follows from the well-known exact penalty function framework [4, 14].

Next let us turn our attention to the different notions of solvability of (1.4).

Definition 1.1. Problem (1.4) is value-solvable, if the value g is finite (see (1.4)).

Definition 1.2. Value-solvable problem (1.4) is arg-attainable, if $\exists \overline{a} : g = = \max\{f(x) \mid F_{\overline{a}}(x) \leq 0,\ x \geq 0\}$. In this case some vector $\overline{x} \in \arg\max\{f(x) \mid x \in M_{\overline{a}}\}$ will be optimal for (1.4) in the standard sense.

Definition 1.3. Value-solvable problem (1.4) is arg-solvable in general, if $\exists \{x_{a_k} \in \arg P_{a_k}\} \to \overline{x}$ and $f(\overline{x}) = g$, where $P_a : \sup\{f(x) \mid x \in M_a\}$. Clearly, the vector $\overline{x}$ may belong or not to a feasible set $M_{\cup}$.

Thus, in this paper we consider

1. a saddle point framework for disjunctive Lagrangians associated with (1.4) and $(1.8)_+$;
2. a duality framework for disjunctive programming problems;
3. the technique of exact penalty functions for (1.4).

2. SADDLE POINTS OF DISJUNCTIVE LAGRANGIANS

Let us define a saddle point $[\overline{x}, \overline{u}] \geq 0$ for the function (1.8) by a pair of inequalities:

$$F_{\cup}(x, \overline{u}) \underset{\forall x \geq 0}{\leq} F_{\cup}(\overline{x}, \overline{u}) \underset{\forall u \geq 0}{\leq} F_{\cup}(\overline{x}, u); \tag{2.1}$$

in the same way we define a saddle point for $F_{\cap}(x, u)$.

Theorem 2.1. Let $[\bar{x},\bar{u}]\geq 0$ be a saddle point for $F_{\cup}(x,u)$. Then $\bar{x}\in \arg$ (1.4), and

$$\inf_{a\in\Omega}(\bar{u}_a, F_a(\bar{x}))=0 . \tag{2.2}$$

Proof: 1) Let us show that $\bar{x}\in M_{\cup}=\bigcup_{(a)} M_a$. Assume, on the contrary, that $\bar{x}\notin M_{\cup}$, i.e.

$$\forall a: F_a(\bar{x})\nleq 0 . \tag{2.3}$$

Take the right-hand side inequality from (2.1):

$$\bar{g}:=\inf_{(a)}(\bar{u}_a, F_a(\bar{x})) \underset{\forall u\geq 0}{\geq} \inf_{(a)}(u_a, F_a(\bar{x})) . \tag{2.4}$$

The value $\bar{g}$ is finite. Indeed, if $\bar{g}=-\infty$, then taking all $u_a=0$ in relation (2.4), one gets a contradiction $-\infty>0$. Next take arbitrary positive number $g>0$. By (2.3) for any a there exists a positive coordinate $g_{j(a)}$ of the vector $F_a(\bar{x})$. Let us denote the coordinate of the vector u_a corresponding to it by $d_{j(a)}$. Because $u_a\geq 0$ is arbitrary, it is possible to guarantee the inequality $d_{j(a)}\cdot g_{j(a)}>e>0$, $\forall a$. Taking all other coordinates of the vectors u_a as zero and $t>0$ as sufficiently large, we can obtain the inequality $(u_a^t, F_a(\bar{x}))>g$ for all a, where $u_a^t=tu_a$. Due to the arbitrariness of $g>0$ the last inequality contradicts (2.4). Thus, $\bar{x}\in M_{\cup}$.

2) Next let us prove that $\bar{g}:=\inf_{(a)}(\bar{u}_a, F_a(\bar{x}))=0$, i.e. (2.2) is valid. Indeed, since $\bar{x}\in M_{\cup}$, one has $g\leq 0$. But if $\bar{g}<0$, then taking $u_a=0$ in (2.4) we have $0>\bar{g}\geq 0$, i.e. a contradiction.

3) Let us show finally that $\bar{x}\in\arg$ (1.4). Since $\bar{g}=0$, we can rewrite the left-hand side of the inequality from (2.1) as:

$$f(x) \underset{\forall x\geq 0}{\leq} f(\bar{x})+\inf_{(a)}(\bar{u}_a, F_a(x)) . \tag{2.5}$$

For an arbitrary $x\in M_{\cup}$ the second term in (2.5) will be non-positive, therefore $f(x)\leq f(\bar{x})$ for all $x\in M_{\cup}$, and $\bar{x}\in\arg$ (1.4). ❑

One can prove the following theorem by repeating the reasoning above.

Theorem 2.2. Let $[\bar{x},\bar{u}]\geq 0$ be a saddle point for $F_{\cap}(x,u)$. Then $\bar{x}\in\arg$ (1.3), and

$$\sup_{(a)}(\bar{u}_a, F_a(\bar{x}))=0 . \tag{2.6}$$

Remark 2.1. Analogs of Theorems 2.1, 2.2 are also valid for modified Lagrangians $F_{\cup}^{\oplus}(x,u)$ and $F_{\cap}^{\oplus}(x,u)$. To verify these facts one can follow the scheme of the proof of Theorem 1.

As it is known, for a standard MP problem the existence of a saddle point of its Lagrangian is connected with Kuhn-Tucker's theorem, under some appropriate conditions, in particular, under the conditions of convexity and constraint qualifications of some kind.

In order to establish similar results for problem (1.4), it may be expedient to use the same conditions for each of the sub-problems

$$\sup\{f(x) \mid F_{a}(x) \le 0,\ x \ge 0\}\ . \qquad (2.7)_{a}$$

Instead of a long discussion about conditions, guaranteeing the existence of a saddle point for all $F_{a}(x,u) := f(x) - (u_{a}, F_{a}(x))$, let us take such existence as the initial point of our way to get results on the existence of a saddle point for $F_{\cup}(x,u)$ ($\min F_{\cup}^{\oplus}(x,u)$).

We shall assume the following condition: let a ball $S \in R^n$ exist such that

$$\forall a : S \cap \arg(2.7)_{a} \ne \emptyset\ . \qquad (2.8)$$

If $f(x)$ is continuous and (2.8) holds, then $\tilde{f} := \sup\{f(x) \mid x \in M_{\cup}\} < +\infty$, i.e. problem (1.4) is value-solvable. Nevertheless it may not be arg-attainable, i.e. the property $\exists\, \bar{x} \in M_{\cup} : f(\bar{x}) = \tilde{f}$ may be wrong. That is why we shall define below the optimal vector of problem (1.4) as a limit point $\bar{x}$ of a convergent sequence $\{\bar{x}_{a_j} \in \arg(2.7)_{a}\}$. Such limit point may belong or not to $M_{\cup}$. But in any case $\bar{x} \in \tilde{M} := \overline{\bigcup_{(a)} \arg(2.7)_{a}}$, where a bar over a set denotes its closure. Thus we define the optimal set of problem (1.4) as

$$\{x \in \tilde{M} \mid f(x) = \tilde{f}\} \quad (\equiv \arg(1.4))\ .$$

Let us present the union of all the conditions introduced above:

1. $f(x)$ is continuous over $\overline{M_{\cup}}$;
2. $\forall a\ \exists\, [\bar{x}, \bar{y}] \ge 0$ which is a saddle point for $F_{\cup}(x,y)$; (2.9)
3. (2.8) holds;

and prove the following results.

Theorem 2.3. Let all the conditions (2.9) hold. If one has $\{\bar{x}_{a_j}\} \to \bar{x},\ x_{a_j} \in \arg(2.7)_{a_j}$, $f(\bar{x}) = \tilde{f}$ $(= \mathrm{opt}(1.4))$, then

$$F_{\cup}(x,\overline{u}) \underset{\forall x\geq 0}{\leq} f(\overline{x}), \quad \overline{u}=\{\overline{u}_a\}. \tag{2.10}$$

Proof: We have $(\overline{u}_a, F_a(\overline{x}_a))=0, \forall a$ and $f(x)-(\overline{u}_a, F_a(x)) \leq f(\overline{x}_a) \leq f(\overline{x}), \forall x \geq 0$. Consequently, $\sup\limits_{(a)}[f(x)-(\overline{u}_a, F_a(x))] \leq f(\overline{x})$. But the left part of this inequality is equal to $f(x)-\inf\limits_{(a)}(\overline{u}_a, F_a(x))$, therefore (2.10) is valid. ❑

Remark 2.2. The proved theorem is an analog of Theorem 47.2 [12], known for a finite index set $\Omega=\{1,\ldots,m\}$.

Theorem 2.4. Let only the conditions 1) and 3) from (2.9) hold, as well as condition 2) for the function $F_{\cup}^{\oplus}(x,u)$, i.e. for $(1.8)_+$. If $\overline{x}$ and $\overline{u}$ are the same as in Theorem 2.3, then $[\overline{x},\overline{u}]$ is a saddle point for $F_{\cup}^{\oplus}(x,u)$.

Proof: Let us prove the right-hand side inequality in the relation

$$F_{\cup}^{\oplus}(x,\overline{u}) \underset{\forall x\geq 0}{\leq} F_{\cup}^{\oplus}(\overline{x},\overline{u}) \underset{\forall u\geq 0}{\leq} F_{\cup}^{\oplus}(\overline{x},u), \tag{2.11}$$

which defines a saddle point $[\overline{x},\overline{u}]$. Using $(1.8)_+$ (i.e. using the definition of $F_{\cup}^{\oplus}(x,u)$) one can write this inequality as:

$$\inf_{(a)}(\overline{u}_a, F_a^+(\overline{x})) \underset{\forall u_a\geq 0}{\geq} \inf_{(a)}(u_a, F_a^+(\overline{x})). \tag{2.12}$$

The right-hand side of (2.12) is obviously equal to zero. Since $\{\overline{x}_{a_j}\} \to \overline{x}$, and $(\overline{u}_{a_j}, F_{a_j}^+(\overline{x}_{a_j}))=0, \forall a_j$, the left-hand side is equal to zero too. Therefore the inequality is valid.

Let us pass to the left-hand side inequality from (2.11) and rewrite it in detail:

$$f(x)-\inf_{(a)}(\overline{u}_a, F_a^+(x)) \underset{\forall x\geq 0}{\leq} f(\overline{x})-\inf_{(a)}(\overline{u}_a, F_a^+(\overline{x})). \tag{2.13}$$

When proving (2.12), we already demonstrated that $\inf\limits_{(a)}(\overline{u}_a, F_a^+(\overline{x}))=0$. Therefore (2.13) takes the form of already proved relation (2.5), which is valid for the function $F_a^{\oplus}(x)$ too (see remarks to Theorems 2.1 and 2.2). The proof is complete. ❑

Remark 2.3. Parameter a, which plays the role of index, enumerating the components of disjunctive or conjunctive inequalities (1.1), (1.2) (or of systems of such inequalities), may be of a diverse nature, e.g. be a vector. In particular, instead of function $F_a(x)$ one can consider a vector function $F(x,y)$ of two vector arguments $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, where y plays the role of index a, and consequently Ω may be a subset of $\mathbf{R}^m$, e.g. be a convex compact set. Then inequalities (1.1) and (1.2) may take the form

$$\sup_{y\in\Omega} |F(x, y)|_{\max} \le 0, \quad \inf_{y\in\Omega} |F(x, y)|_{\max} \le 0. \tag{2.14}$$

These inequalities have deep meaning. For example, the general game of two persons with a zero sum may be re-formulated as the first of them. A whole array of optimal control problems can be reduced to the second inequality; y being interpreted as a time parameter t. One can prove his own variants of Theorems 2.1-2.4 for optimization problems with constraints of the type (2.14) under appropriate conditions, e.g. conditions of continuity of $F(x, y)$ on $z=[x, y]$, compactness of Ω, etc. From here a large area of research may emerge.

3. DUALITY FRAMEWORK

We shall construct a duality framework for disjunctive programming problems by means of a general scheme, namely, using the Lagrangian function, in our case - the disjunctive Lagrangian function (1.8).

Let us consider the problems

$$P_\cup : \sup_{x\ge 0} \inf_{u\ge 0} F_\cup(x,u) \quad (=: g), \tag{3.1}$$

$$P_\cup^* : \inf_{u\ge 0} \sup_{x\ge 0} F_\cup(x,u) \quad (=: g^*), \tag{3.2}$$

and their analogs for the linear case

$$L_\cup : \sup_{x\ge 0} \inf_{u\ge 0} L_\cup(x,u), \tag{3.3}$$

$$L_\cup^* : \inf_{u\ge 0} \sup_{x\ge 0} L_\cup(x,u), \tag{3.4}$$

where $L_\cup(x,u)=(c,x)-\inf_{(a)}(u_a, A_a x-b_a), \quad u_a \ge 0$.

Lemma 3.1. Problem (3.1) has the same optimal value as the problem (1.4), where $M_\cup=\{x\ge 0 \mid \inf_{(a)} |F_a(x)|_{\max} \le 0\}$.

Indeed, consider the internal sub-problem from (3.1) and calculate its optimal value:

$$\inf_{u\ge 0}[f(x)-\inf_{(a)}(u_a, F_a(x))] = \begin{cases} f(x), & x\in M_\cup, \\ -\infty, & x\notin M_\cup. \end{cases}$$

This immediately implies

$$g = \sup\{f(x) \mid x \in M_{\cup}\},$$

which proves the lemma. ❑

We shall call the problem $P_{\cup}^{*}$, i.e. (3.2), the dual one for $P_{\cup}$, i.e. (3.1); similarly $L_{\cup}^{*}$ will be called the dual for $L_{\cup}$. According to Lemma 3.1, problem $L_{\cup}$ is equivalent to problem (1.5), which is a linear disjunctive problem.

Lemma 3.2. Problem $L_{\cup}^{*}$, i.e. (3.4), has the same optimal value as the problem

$$\inf_{(u)}\{\sup_{(a)}(b_a, u_a) \mid A_a^T u_a \geq c,\ u_a \geq 0,\ a \in \Omega\}. \tag{3.5}$$

Proof: Let us rewrite problem $L_{\cup}^{*}$ in another form. At first rewrite its Lagrangian $L_{\cup}(x,u)$: $L_{\cup}(x,u) = (c,x) - \inf_{(a)}(u_a, A_a x - b_a) = \sup_{(a)}[(c,x) - (u_a, A_a x - b_a)] = \sup_{(a)}[(b_a, u_a) + (c - A_a^T u_a, x)]$. Then, taking the internal operation $\sup_{x \geq 0}$ in (3.4), we get:

$$\sup_{x\geq 0}[(b_a, u_a) + (c - A_a^T u_a, x)] = \begin{cases} \sup_{(a)}(b_a, u_a), & \text{if } A_a^T u_a \geq c,\ \forall a, \\ +\infty, & \text{if } \exists a : A_a^T u_a \ngeq c. \end{cases}$$

This implies

$$\inf_{u\geq 0}\sup_{x\geq 0} L_{\cup}(x,u) = \operatorname{opt}(3.5).$$

❑

Thus, we get the dual for problem (1.5) in the form (3.5). It is interesting to see that the original problem $L_{\cup}$, i.e. (1.5), can be written in equivalent form

$$\sup_{(x)}\{\sup_{(a)}(c, x_a) \mid A_a x_a \leq b_a,\ x_a \geq 0,\ a \in \Omega\}. \tag{3.6}$$

The above makes it possible to recover the symmetry in the instances $L_{\cup}$ and $L_{\cup}^{*}$. Operations $\inf_{(u)}$ and $\sup_{(a)}$ in (3.5), as well as $\sup_{(x)}$ and $\sup_{(a)}$ in (3.6) are commutative. Consequently, problem (3.6) can be reduced to the problem of seeking an exact upper bound, i.e. $\sup_{(a)}$, for the set of optimal values of the problems

$$L_a : \max\{(c,x) \mid x \in \mathbf{N}_a\}, \tag{3.7}_a$$

where $\mathbf{N}_a := \{x \geq 0 \mid A_a x \leq b_a\}$. Analogously, problem (3.5) can be reduced to the problem of seeking an exact upper bound for optimal values of the problems

$$L_a^* : \min\{(b_a, u_a) \mid u_a \in \mathbf{N}_a^*\}, \qquad (3.7)_a^*$$

where $\mathbf{N}_a^* := \{u_a \geq 0 \mid A_a^T u_a \geq c\}$. If we assume that all problems $(3.7)_a$ are solvable, then according to the duality theorem in linear programming one has $\operatorname{opt} L_a = \operatorname{opt} L_a^*$, and therefore $\sup_{(a)} \operatorname{opt} L_a = \sup_{(a)} \operatorname{opt} L_a^*$.

Theorem 3.3. Let problem (1.5) (i.e. (3.6)) be value-solvable and assume that there are no improper problems of the 3rd kind among L_a. Then problem (3.5) is also value-solvable and opt (3.6) = opt (3.5).

Proof: Any of the problems L_a and L_a^* may be solvable or not. Unsolvable (improper) problems may be classified [12]:

1) $\mathbf{N}_a = \emptyset$, $\mathbf{N}_a^* \neq \emptyset$ corresponds to improper problems of the 1st kind;

2) $\mathbf{N}_a \neq \emptyset$, $\mathbf{N}_a^* = \emptyset$ - the 2nd kind;

3) $\mathbf{N}_a = \emptyset$, $\mathbf{N}_a^* = \emptyset$ - the 3rd kind.

In mathematical programming, if the feasible set of a problem is empty, its optimal value is usually put equal to $-\infty$ for sup -problems and $+\infty$ for inf -problems. In our case it will be considered (according to the classification above):

1. opt $L_a = -\infty$, opt $L_a^* = -\infty$;
2. opt $L_a = +\infty$, opt $L_a^* = +\infty$;
3. opt $L_a = -\infty$, opt $L_a^* = +\infty$.

Since the original problem has a finite optimal value, there are no improper problems of the 2nd kind among $\{L_a\}$. As for improper problems of the 3rd kind, they are forbidden by the assumptions. Thus, the real situation is as follows: each of the problems L_a is either solvable (with L_a^*), or unsolvable, and then opt $L_a =$ opt $L_a^* = -\infty$. Consequently, the sets of optimal values for $\{L_a\}_{a\in\Omega}$ and $\{L_a^*\}_{a\in\Omega}$ coincide, as do their exact upper bounds. The proof is complete. ❑

Theorem 3.4. Let some analog of condition (2.8) hold, namely: there exists a ball $S \subset \mathbf{R}^n$ such that

$$\forall a : S_a \coloneqq S \cup \arg L_a \neq \emptyset .$$

Then problem (1.5) is arg-solvable in general, problem (3.5) is value-solvable and opt (1.5) = opt (3.5).

Proof: From (1.9) the value-solvability of (1.5) is immediate. Indeed, if $g = \text{opt}$ (1.5), then one can take a convergent sequence $\{\bar{x}_{a_k} \in S_{a_k}\} \to \bar{x}$ and $\{(c, \bar{x}_{a_k})\} \to (c, \bar{x}) = g$. That is what we mean when we say that problem (1.5) is arg-solvable in general. The value-solvability of the dual problem to (3.5) is evident. ❑

Problems (1.5) and (3.5) may be written more compactly and symmetrically, when set Ω is ordered, i.e.

$$\Omega = \{a_1, a_2, \ldots, a_w, a_w + 1, \ldots\} .$$

Let us introduce the transfinite matrix $\mathbf{A}$ and vectors $\tilde{x}, \tilde{u}, \tilde{b}$ and $\tilde{c}$:

$$\mathbf{A} = \begin{bmatrix} A_{a_1} \\ A_{a_2} \\ \vdots \end{bmatrix}, \ \tilde{x} = \begin{bmatrix} x_{a_1} = x \\ x_{a_2} = x \\ \vdots \end{bmatrix}, \ \tilde{u} = \begin{bmatrix} u_{a_1} \\ u_{a_2} \\ \vdots \end{bmatrix}, \ \tilde{b} = \begin{bmatrix} b_{a_1} \\ b_{a_2} \\ \vdots \end{bmatrix}, \ \tilde{c} = \begin{bmatrix} c_{a_1} = c \\ c_{a_2} = c \\ \vdots \end{bmatrix},$$

and functions $\Psi(\tilde{x}) = \sup_{(k)}(c_{a_k}, x_{a_k})$, $\Psi^*(\tilde{u}) = \sup_{(k)}(b_{a_k}, u_{a_k})$. Then problems (1.5) and (3.5) may be written as

$$\sup\{\Psi(\tilde{x}) \mid \mathbf{A}\tilde{x} \le \tilde{b}, \tilde{x} \ge 0\},$$

$$\inf\{\Psi^*(\tilde{u}) \mid \mathbf{A}^T \tilde{u} \ge \tilde{c}, \tilde{u} \ge 0\} .$$

Let us note that the first of these problems is not convex but the second one is.

The scheme of dual construction for the general problem of disjunctive programming (1.4) may be the same as for problem (1.5). The dual objects take the form (1.5) and (3.2), i.e. $P_\cup$ and $P_\cup^*$ (in our notation). The required dual relation $\text{opt}\, P_\cup = \text{opt}\, P_\cup^*$ is usually connected (and often coincides) with the existence of saddle point of the function $F_\cup(x, u)$. If we assume the existence of a saddle point for $F_\cup(x, u)$, then this dual relation will hold. Of course, one can apply any condition guaranteeing the existence of a saddle point for $F_\cup(x, u)$ (as in Theorem 2.4 for modified function $F_\cup^\oplus(x, u)$). This surely makes it possible to formulate duality theorems in the general case too. We omit the details here, restricting ourselves to methodological considerations.

4. EXACT PENALTY FUNCTION METHOD

Let us consider the question of the equivalence of problems (1.4) and (1.10) under appropriate value of parameter R .

Theorem 4.1. Let the Lagrangian $F_{\cup}(x,u)$, associated with problem (1.4), has a saddle point $[\bar{x},\bar{u}]$, $\bar{u}=\{\bar{u}_a \geq 0\}$. If $R_a \geq \bar{u}_a$ for all $a \in \Omega$, then

$$\text{opt (1.4) = opt (1.10)} \tag{4.1}$$

Proof: According to Theorem 2.1 the following relations hold: $\bar{x} \in \arg(1.4)$ and $\bar{g} := \inf_{a\in\Omega}(\bar{u}_a, F_a(\bar{x})) = 0$. Using the definition of a saddle point and equality $\bar{g}=0$ one has

$$f(x) - \inf_{(a)}(\bar{u}_a, F_a(x)) \leq f(\bar{x}), \quad \forall x \geq 0. \tag{4.2}$$

Hence $F_{\cup}^{\oplus}(x,R) = f(x) - \inf_{(a)}(R_a, F_a^{+}(x)) \leq f(\bar{x}) + \inf_{(a)}(\bar{u}_a, F_a(x)) - \inf_{(a)}(R_a, F_a^{+}(x)) \leq$ $\leq f(\bar{x}) + \inf_{(a)}(\bar{u}_a, F_a^{+}(x)) - \inf_{(a)}(R_a, F_a^{+}(x)) \leq f(\bar{x})$. Since this inequality holds for any $x \geq 0$, we get

$$\sup_{x\geq 0} F_{\cup}^{\oplus}(x,R) \leq f(\bar{x}) \quad (= \text{opt (1.4)}).$$

Since the inverse inequality is obvious, the proof is complete. ❑

Stronger theorems connecting problems (1.4) and (1.10) can be formulated under the following assumptions on the set Ω and the functions $F_a(x) =: j(z)$, $z=[a,x]$:

$$\left.\begin{array}{l} 1)\ F_a : \mathbf{R}^n \xrightarrow{F_a} \mathbf{R}^m, \\ 2)\ \Omega \text{ is a convex compact subset of } \mathbf{R}^k, \\ 3)\ j(z) \text{ is continuous in } z. \end{array}\right\} \tag{4.3}$$

Since $j(a,x) \in \mathbf{R}^m$ for any a, then Lagrange vectors u_a, used above in the text, are vectors of dimension m. Let us take the vector $\vec{d} = [d,\ldots,d] \in \mathbf{R}^m$, $d>0$ and choose the penalty constants R_a so that $R_a \geq \bar{u}_a + \vec{d}$.

Theorem 4.2. Let all the assumptions of Theorems 4.1, 4.3 and condition $R_a \geq \bar{u}_a + \vec{d}$ hold. Then

$$\text{arg (1.4) = arg (1.10).} \tag{4.4}$$

Proof: According to the assumptions made, equality (4.1) holds. Recall the inequality obtained in Theorem 4.1:

$$F_{\cup}^{+}(x, R) \underset{\forall x\geq 0}{\leq} f(\overline{x}) + \inf_{(a)}(\overline{u}_a, F_a^{+}(x)) - \inf(R_a, F_a^{+}(x)) \leq f(\overline{x}), \tag{4.5}$$

where $\overline{x} \in \arg$ (1.4). From this relation it follows that

opt (1.10) = $\sup F_{\cup}^{+}(x, R) \leq f(\overline{x})$ = opt (1.4).

But for $x = \overline{x}$ one has $F_{\cup}^{+}(\overline{x}, R) = f(\overline{x})$, therefore $\overline{x} \in \arg$ (1.10). Consequently, the inclusion arg (1.4) $\subset$ arg (1.10) is valid.

Let us prove the inverse inclusion. Take any $\tilde{x} \in \arg$ (1.10) and substitute it in (4.5):

$$F_{\cup}^{+}(\tilde{x}, R) \leq f(\overline{x}) + \inf_{(a)}(\overline{u}_a, F_a^{+}(\tilde{x})) - \inf_{(a)}(R_a, F_a^{+}(\tilde{x})) \leq \text{opt (1.4)}. \tag{4.6}$$

Since $F_{\cup}^{+}(\tilde{x}, R)$ = opt (1.4), one has

$$\inf_{(a)}(\overline{u}_a, F_a^{+}(\tilde{x})) = \inf_{(a)}(R_a, F_a^{+}(\tilde{x})). \tag{4.7}$$

Taking appropriate sequence $\{a_k\} \to \overline{a}$ we can rewrite (4.7) as

$$(\overline{u}_{a_k}, F_{a_k}^{+}(\tilde{x})) + e_k = (R_{a_k}, F_{a_k}^{+}(\tilde{x})), \quad \{e_k > 0\} \to 0. \tag{4.8}$$

Because

$$(R_{a_k}, F_{a_k}^{+}(\tilde{x})) \geq (\overline{u}_{a_k}, F_{a_k}^{+}(\tilde{x})) + (\vec{d}, F_{a_k}^{+}(\tilde{x})) \geq (\overline{u}_{a_k}, F_{a_k}^{+}(\tilde{x})) + d\,|\,F_{a_k}^{+}(\tilde{x})|_{\max},$$

then from (4.8) it follows that $|\,F_{a_k}^{+}(\tilde{x})|_{\max} \leq \dfrac{e_k}{d} \to 0$. Therefore we have $|\,F_{\overline{a}}^{+}(\tilde{x})|_{\max} = 0$ and $F_{\overline{a}}(\tilde{x})| \leq 0$, i.e.

$$\tilde{x} \in M_a = \{x \geq 0 \mid F_a(x) \leq 0\} \subset M = \bigcup_{(a)} M_a.$$

Thus, we proved that $\tilde{x}$ from arg (1.10) is feasible for problem (1.4) and provides an optimal value for $f(x)$, i.e. $\tilde{x} \in \arg$ (1.4). Consequently, arg (1.10) $\subset$ arg (1.4) and (4.4) is valid. ❑

The analogs of Theorems 4.1 and 4.2 for the conjunctive problem (1.3) are valid too. Let us write problem (1.3) in detail:

$$\sup\{f(x) \mid \sup |\,F_a(x)|_{\max} \leq 0,\ x \geq 0\} \tag{4.9}$$

and consider the associated problem (1.9), i.e.

$$\sup_{x \geq 0}\{ f(x) - \sup_{a \in \Omega}(R_a, F_a^+(x))\}. \qquad (4.10)$$

Theorem 4.3. Let the function $F_{\cap}(x,u)$, i.e. (1.7), have a saddle point $[\bar{x},\bar{u}]$, $\bar{u} = \{\bar{u}_a \geq 0\}$ and the conditions (4.3) hold. Then:

1. If $\forall a \in \Omega: R_a \geq \bar{u}_a$,then

opt (4.9) = opt (4.10).

2. If $\forall a \in \Omega: R_a \geq \bar{u}_a + \vec{d}$ ($\vec{d}$ as in Theorem 4.2), then

arg (4.9) = arg (4.10).

The proof of this statement is very similar to the proof of Theorems 4.1 or 4.2. It is omitted, since in this article the main accent is placed on disjunctive problems.

REFERENCES

[1] Balas, E., "Disjunctive programming", Annals of Discrete Mathematics, 5 (1979).

[2] Balas, E., "Disjunctive programming: Properties of the convex hull of feasible points", MSSR N 348, Carnegie Mellon University, 1974.

[3] Balas, E., Tama, J.M., and Tind, J., "Sequential convexification in reverse convex and disjunctive programming", Publication N 87 (1), Matematisk Institut, Afdeling for Operationsanalyse, University of Aaarhus, Denmark, 1986.

[4] Eremin, I.I., "About penalty methods in convex programming", Kibernetika, 4 (1967).

[5] Eremin, I.I., "About some problems of disjunctive programming", Yugoslav Journal of Operations Research, 8 (1) (1998) 25-43.

[6] Eremin, I.I., Theory of Linear Optimization, Ekaterinburg Publisher, Ekaterinburg, 1999.

[7] Helbig, S., "Optimality criteria in disjunctive optimization and some applications", in: Methods of Operations Research, Proceedings of the XIV Symposium on Operations Research, Ulm, September 6-8, 1989, 62 (1989) 67-78.

[8] Helbig, S., "Stability in disjunctive programming I: Continuity of the feasible set", Optimization, 21 (1990) 855-869.

[9] Helbig, S., "Duality in disjunctive programming via vector optimization", Mathematical Programming, 65 (1994).

[10] Helbig, S., "Stability in disjunctive programming II: Continuity of the feasible and optimal set", Optimization, 31 (1994) 63-93.

[11] Jongen, H.Th., R•ckmann, J.-J., and Stein, O., "Disjunctive optimization: Critical point theory", Opt. Theory Appl., 93 (2) (1997) 321-336.

[12] Sherali, H.D., and Shetty, C.M., Optimization with Disjunctive Constraints, Springer, Berlin-Heidelberg-New York, 1980.

[13] Zafriou, F., "Parametrische disjunctive lineare optimierung", Diplom-Thesis, Frankfurt/Main, 1988.

[14] Zangwill, W.I., "Non-linear programming via penalty functions", Management Sci., 13 (5) (1967).