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THE EFFECTS OF MAJORITY IN FERMAT-WEBER PROBLEMS WITH ATTRACTION AND REPULSION IN A PSEUDOMETRIC SPACE

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Abstract. The well-known majority theorem for Fermat-Weber location problems states that when all distances are measured by a fixed pseudometric, then any destination with weight at least half of the total weight of all destinations is an optimal site. In this paper we study the implications of such majority when both attracting (positive weight) and repelling (negative weight) destinations are present.

When no constraints are present, and when majority holds at an attracting destination, the classical majority theorem is still valid, while when there is a repelling strict majority in an unbounded space, the objective is unbounded below.

We then consider the constrained case where the location is restricted to lie within a given compact region. When majority is at an attracting destination then an optimal solution exists which is "first-reachable" from this destination, a generalization of visibility to general pseudometric spaces. When majority is at a repelling destination an optimal solution exists which is "last reachable" from this destination.

Key words and phrases: location theory, Weber problem, majority

1. INTRODUCTION

The classical Fermat-Weber location problem asks for determining the point at which the (positively) weighted sum of distances to a finite set of destination points is minimal. It has extensively been studied in many settings and is considered as one of the principal cornerstones of location theory. The most general result in this respect is the majority theorem of Witzgall (1964), stating that when a destination holds a majority, *in the sense of having a weight of at least half of the sumtotal* weight of all destinations, then its location is an optimal site. This result is valid whenever the distance measure used is a pseudometric, which makes it equally applicable to a wide variety of situations, e.g. euclidean spaces, the sphere with great circle distance, undirected networks with shortest path distance, etc.

In almost all work on the Fermat-Weber problem it is assumed that all weights are positive. In practice this means that nearness to the sought central point is advantageous to all destinations, i.e. they are all attracting. In many real world situations this is an oversimplification of the problem. Indeed with the growing attention to environmental effects many types of facilities must also be considered as potentially dangerous and nearness to it as a disadvantage, mainly to populous areas.

One simple way to build this factor into the location model is to consider destinations of two types, attracting and repelling ones, defined by positive and negative weights respectively. The thus obtained extended Fermat-Weber problem has been studied (to our knowledge exclusively) by Tellier and Polanski (1989) in the Euclidean plane.

In this paper, the authors observe without proof that the majority theorem remains valid in this extended case, provided majority is at an attracting destination. In case majority occurs at a repelling destination, "the optimal solution is at infinity".

We first show that Tellier and Polanski's observations hold in any pseudometric space. Then we consider what may be said in the constrained case, where the sought location is restricted to some compact region. We obtain two localization theorems, one for an attracting majority point and one in case of a repelling majority. Both make use of the notion of "reachability" which may be viewed as an extension of "visibility" (introduced by Witzgall (1964) in affine spaces) to general pseudometric spaces.

2. PROBLEM STATEMENT

Let (X, d) be a pseudometric space, i.e. d is a map $X \times X \to \mathbf{R}$ with following properties for any $x, y, z \in X$:

(1) $d(x,y) \geq 0$ (nonnegativity)
(2) $d(x,x) = 0$ (identity)
(3) $d(x,y) = d(y,x)$ (symmetry)
(4) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

When additionally d is definite, i.e.

(5) $d(x,y) = 0$ implies $x = y$,

then d is a metric.

Any pseudometric space has a natural topology defined by the basis consisting of open d-balls $B^o(x,r) = \{y \in X \mid d(x,y) < r\}$, where $x \in X$ and $r > 0$. When d is a metric we obtain a Hausdorff topology. The pseudometric d is always continuous with respect to this topology.

Let A and R be two finite subsets of X. To each point $a \in A$ (resp. $r \in R$) a positive weight w_a (resp. w_r) is associated. The points of A are the *attracting destinations* and those of R the *repelling* ones.

We define the *Fermat-Weber problem with attraction and repulsion* (FWAR) as

$$\underset{x\in X}{\text{MIN}}\, f(x)$$

$$\text{where}\quad f(x) = \sum_{a\in A} w_a d(x,a) - \sum_{r\in R} w_r d(x,r).$$

Since d is continuous, $d(\cdot, a)$ and $d(\cdot, r)$ also are, showing that f is continuous.

Note that for any two *close* points $x, y \in X$, i.e. with $d(x,y) = 0$, we always have $d(x,z) = d(y,z)$ for any $z \in X$, as an easy consequence of symmetry and triangle inequality. It follows that in such a case $f(x) = f(y)$, and that whenever x is optimal y also will be.

In most practical location problems distances may be considered to be determined by way of shortest paths (geodesics): in networks this is the standard way, in euclidean spaces shortest paths are given by straight line segments, in rectilinear spaces (Manhattan distance) shortest paths use the two main directions only, on the sphere shortest paths are great circle arcs no greater than 180°, etc.

In all these "geodesic" spaces there exists the notion of "lying between" : x lies between y and z iff there exists a shortest path from y to z which passes through x. In terms of the underlying pseudometric we may formalize this notion as follows.

The point $x \in X$ is said to *lie between* the points $y, z \in X$ iff

$$d(y,z) = d(y,x) + d(x,z).$$

Note that x always lie between x and x, and that this is also the case for any point x' at distance 0 from x.

LEMMA 1: *Let $x \in X$ lie between the points $b \in A$ and $y \in X$, then*

$$f(y) - f(x) \geq \Big(w_b - \sum_{a\in A\setminus\{b\}} w_a - \sum_{r\in R} w_r\Big) d(x,y).$$

Similarly, when $x \in X$ lies between $s \in R$ and $y \in X$, then

$$f(y) - f(x) \leq \Big(\sum_{a\in A} w_a + \sum_{r\in R\setminus\{s\}} w_r - w_s\Big) d(x,y).$$

PROOF: By the triangle inequality we have

$$d(a,y) - d(a,x) \geq -d(x,y) \qquad \text{for any } a \in A$$

and

$$d(r,y) - d(r,x) \leq d(x,y) \qquad \text{for any } r \in R.$$

Hence

$$f(y) - f(x) = \sum_{a\in A} w_a(d(a,y) - d(a,x)) - \sum_{r\in R} w_r(d(r,y) - d(r,x))$$

$$\geq \sum_{a\in A\setminus\{b\}} w_a(-d(y,x)) - \sum_{r\in R} w_r d(x,y) + w_b(d(b,y) - d(b,x))$$
$$= \left(w_b - \sum_{a\in A\setminus\{b\}} w_a - \sum_{r\in R} w_r\right) d(x,y).$$

The second result follows from the first by sign inversion, thereby interchanging attraction and repulsion. ■

3. MAJORITY AND ITS IMPLICATIONS

A destination $d \in A \cup R$, either attracting or repelling, is said to hold a *majority* iff

$$w_d \geq 0.5 \sum_{c\in A\cup R} w_c$$

or equivalently

$$w_d \geq \sum_{c\in A\cup R\setminus\{d\}} w_c.$$

This majority is said to be strict whenever the defining inequality is a strict one.

THEOREM 2. *If some attracting destination $b \in A$ holds a majority, then b is an optimal solution. If b's majority is strict then the optimal solutions are exactly those points at distance 0 to b. In a metric space b is the unique optimal solution in this latter case.*

If there is a strict majority at some repulsion point $s \in R$, and distance up to s is unbounded, then there is no optimal solution, in the sense that f may be decreased as far as one wishes.

PROOF. Let $b \in A$ hold a majority. Since b lies between b and any point $y \in X$, we may apply the first part of Lemma 1, taking $x = b$. It then follows that $f(y) \geq f(b)$ for any $y \in X$, hence that b is an optimal solution.

If the majority of b is strict, then by the same reasoning we have that $f(y) > f(b)$ for any $y \in X$ such that $d(y,b) > 0$. Since for any point x close to b we have $f(x) = f(b)$, it follows that these points are exactly the optimal solutions. In a metric space b is the only point close to b, so b will be the only optimal solution.

Suppose now that a strict majority occurs at $s \in R$. The second part of Lemma 1 applies with $x = s$, yielding

$$f(y) \leq f(s) + \left(\sum_{a\in A} w_a + \sum_{r\in R\setminus\{s\}} w_r - w_s\right) d(s,y)$$

The factor of $d(s,y)$ being strictly negative, we may reduce $f(y)$ below any level by increasing $d(s,y)$ sufficiently. Thus if distance to s is unbounded on X, there will not exist any optimal solution and the "optimal value will be $-\infty$". ■

4. THE CONSTRAINED CASE

Let us now move to the constrained Fermat-Weber problem with attraction and repulsion. Let S denote the feasible region. In order to guarantee existence of optimal solutions we will assume S to be compact. The problem is then formulated as

$$\text{(CFWAR)} \qquad \underset{x \in S}{\text{MIN}}\, f(x)$$

where $f(x)$ is defined in the same way as in section 2.

In affine spaces, where lines and convexity are defined, we know that constrained optimal solutions to convex objectives may be found by restricting search to the feasible points which are visible from some unconstrained optimum. What we will show here is first that visibility may be generalized to general pseudometric spaces, and secondly that in the particular case of constrained Fermat-Weber problems in which a majority destination exists a similar property may be derived. This may come somewhat as a surprise, since the presence of repelling destinations destroys convexity of the objective.

A point $z \in S$ is said to be *first reachable from* $y \in X$ iff whenever $x \in S$ lies between y and z, this implies that $d(x, z) = 0$.

Note that when d is actually a metric then $z \in S$ is first reachable from y iff no other point from S lies between y and z. In other words, z is the only point of S along any shortest path from y to z. This last property gives a rationale for the term "first reachable".

Similarly, we will say that $z \in S$ is *last reachable from* $y \in X$ iff whenever $x \in S$ is such that z lies between y and x, then $d(y, z) = 0$.

Again, if d is a metric, this is equivalent to stating that z does not lie between y and any other point of S, i.e. there is no shortest path from y to any point of S which passes through z, explaining the term "last reachable".

First reachable points may be considered as an extension to general pseudometric spaces of visible points (see Witzgall (1964)) in affine spaces. Indeed a point z of S is visible from y iff the line segment joining y to z has no points in common with S, except for z. Since this line segment is in fact the set of points between y and z in the context of the euclidean metric (or any other strict metric) we may similarly define the *d-segment* $d[y, z]$ as the set of all points lying between points y and z in the pseudometric space (X, d). In networks $d[y, z]$ will be the union of all shortest paths from y to z, while in the Manhattan plane it will be the rectangle with points y and z as opposite corners. Point $z \in S$ will then be first reachable from y iff the intersection of $d[y, z]$ with S is reduced to z in metric spaces, and, in more general pseudometric spaces, reduced to $d[z, z]$, which is the set of all points close to z, i.e. at distance 0 from z.

Let us similarly introduce the notation $d(y[z)$ as the set of points $x \in X$ such that z lies between y and x. This is a generalization of the halfray starting at z and away from y. Point $z \in S$ is then last reachable from y iff the intersection of $d(y[z)$ with S reduces to $d[z, z]$.

LEMMA 3. *If S is compact, then for any $z \in S$ the d-segment $d[y,z]$ contains a point first reachable from y, and the set $d(y[z)$ contains a point last reachable from y.*

PROOF. We will only prove the first statement, since the second one is proved in analogous way.

The segment $d[y,z]$ is closed. Indeed it is the inverse image of the closed singleton $\{d(y,z)\}$ of $\mathbf{R}$ by the continuous function $d(y,\cdot)+d(\cdot,z)$. Hence if $z \in S$, the (nonvoid) intersection of $d[y,z]$ with the compact set S is also compact. It follows that there exists a point x minimizing the continuous function $d(y,\cdot)$ on this intersection. We proceed to show that this x is indeed first-reachable from y.

Suppose $t \in S$ lies between y and x, then

$$\begin{aligned} d(y,x) &= d(y,t) + d(t,x) \\ &\geq d(y,x) + d(t,x) \qquad \text{by construction of } x \\ &\geq d(y,x) \end{aligned}$$

it follows that $d(t,x) = 0$ as required. ∎

Using these notions we may formulate the main theorem of the paper:

THEOREM 4. *Suppose the attracting destination b holds a majority, then there always exists an optimal solution to CFWAR which is first reachable from b. If b's majority is strict, then any optimal solution is first reachable from b.*

Similarly if there is a majority at some repelling destination s, then there always exists an optimal solution which is last reachable from s. If s's majority is strict then all the optimal solutions are last reachable from s.

PROOF. Suppose that $b \in A$ holds a majority. Let then y be an optimal solution to CFWAR, then for any point x between b and y we have, by Lemma 1 that $f(y) - f(x) \geq 0$, hence $f(y) \geq f(x)$. Since $d[b,y]$ contains a first reachable point by Lemma 3, which then lies between b and y, this is at least as good as y, and thus can serve as an optimal solution.

When the majority of b is strict, for any $y \in S$ which is not first reachable from b, there exists an $x \in S$ between b and y with $d(x,y) > 0$. By the lemma we then have $f(y) > f(x)$, showing that y is not optimal.

The second part of the theorem is proved in similar way. ∎

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