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ON STABILITY IN QUASICONVEX SEMI-INFINITE OPTIMIZATION

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Abstract. The paper gives sufficient conditions for the stability of the parametric semi-infinite problem with the quasiconvex objective and constraint functions. The obtained result generalizes results concerning parametric linear semi-infinite optimization [4], as well as the stability results for convex case [1], [3], [5].

Key words and phrases: Parametric semi-infinite optimization, stability

1. INTRODUCTION

Consider the following parametric semi-infinite optimization problem

(P_θ) : $$\inf f(x,\theta)$$

subject to

$$g_t(x,\theta) \leq 0, \qquad t \in T$$

where:

(1) $x \in E^n$;

(2) T is a compact topological space;

(3) parameter θ belongs to a metric space Θ;

(4) functions $(x,\theta) \mapsto f(x,\theta)$ and $(x,\theta,t) \mapsto g_t(x,\theta,t)$ are continuous;

(5) for every $\theta \in \Theta$ and $t \in T$, functions $x \mapsto f(x,\theta)$ and $x \mapsto g_t(x,\theta)$ are quasiconvex on E^n.

Recall that a function $f : E^n \to R$ is quasiconvex iff for every $x, y \in E^n$ and $\lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\},$$

and strictly quasiconvex iff for every $x, y \in E^n$,

$$f(x) < f(y) \quad \text{implies} \quad f(\lambda x + (1-\lambda)y) < f(y), \quad \forall\lambda \in (0,1).$$

For a fixed parameter $\theta \in \Theta$, (P_θ) is a quasiconvex semi-infinite problem. Denote by

$X(\theta)$ the set of feasible points,
$S(\theta)$ the set of optimal solutions,

of the problem (P_θ). The aim of the paper is to show some sufficient conditions under which the problem (P_θ) is *stable at a point* $\theta_0 \in \Theta$, i.e. the following is valid:

(i) *$S(\theta)$ is not empty for every parameter θ in some neighbourhood $N(\theta_0)$ of the point θ_0;*

(ii) *for every sequence (θ_n), $\theta_n \in N(\theta_0)$ such that $\theta_n \to \theta_0$, the sequence of arbitrary chosen optimal solutions $(s(\theta_n))$ is bounded with accumulation points in $S(\theta_0)$.*

This notion of stability was introduced in Eremin-Astafiev [**1**].

REMARK. It could be easily proved that stability of (P_θ) at $\theta_0 \in \Theta$ implies that the function $\theta \mapsto S(\theta)$ is upper semicontinuous at θ_0 in the sense of Berge. Conversely, upper semicontinuity of the function $\theta \mapsto S(\theta)$ implies stability of (P_θ) under additional conditions such as

(i) *$S(\theta_0)$ is a nonempty and bounded set and*

(ii) *$S(\theta)$ is not empty, for every θ in some neighbourhood of θ_0 (see* [**2**, Th. 7.5]).

We need also the following definitions. We say that a sequence (C_n), $C_n \subset E^m$ is convergent to a $C \subset E^m$, denoted by $C_n \to C$, iff $C = \limsup C_n = \liminf C_n$, where

$$\liminf C_n = \{x \in E^m \mid x = \lim x_n \text{ for some } (x_n),\ x_n \in C_n,\ n \in N\},$$
$$\limsup C_n = \{x \in E^m \mid x \text{ is an accumul. point for some } (x_n),\ x_n \in C_n,\ n \in N\}.$$

For a given function $\theta \mapsto C(\theta)$, $\theta \in \Theta$, $C(\theta) \subset E^m$ and $C \subset E^m$, we say also that $C(\theta) \to C$ as $\theta \to \theta_0$, $(\theta_0 \in \Theta)$ iff $C(\theta_n) \to C$, for every sequence (θ_n), $\theta_n \in \Theta$ such that $\theta_n \to \theta_0$.

2. CONDITIONS FOR STABILITY

The following theorem gives sufficient conditions for the stability of (P_θ).

THEOREM 1. *A quasiconvex parametric semi-infinite problem (P_θ) is stable at $\theta_0 \in \Theta$, if the following conditions are satisfied*

(i) *$S(\theta_0)$ is a nonempty bounded set,*

(ii) *$X(\theta) \to X(\theta_0)$ as $\theta \to \theta_0$.*

PROOF. According to [**3**, Th. I.3.3], the function $\theta \mapsto S(\theta)$ is upper semicontinuous at θ_0. Keeping in mind the results from the Remark, the conclusion follows if we show that $S(\theta) \neq \emptyset$, for every θ in a neighbourhood of θ_0. Suppose the contrary. Then there exists a sequence (θ_n), such that $\theta_n \to \theta_0$ and $S(\theta_n) = \emptyset$, $n \in N$. Let $x_0 \in S(\theta_0)$ and $K(0,r)$ be a ball with the center 0 and the radius r such that $S(\theta_0) \subset K(0,r)$. Since $X(\theta_n) \to X(\theta_0)$, there exists (x_n), $x_n \in X(\theta_n)$ such that $x_n \to x_0$. Let us prove that, for every $n \in N$, there exists a direction $d_n \neq 0$ such that for every $t \geq 0$

$$x_n + td_n \in X(\theta_n) \tag{1}$$

and

$$f(x_n + td_n, \theta_n) \leq f(x_n, \theta_n). \tag{2}$$

Since $S(\theta_n) = \emptyset$, we can find a sequence (y_m), $y_m \in X(\theta_n)$ such that $\|y_m\| \to \infty$ and $f(y_m, \theta_n) \to -\infty$ as $m \to \infty$. Let d_n be an accumulation point of $(y_m/\|y_m\|)$. Without loss of generality, assume that $d_n = \lim_{m\to\infty}(y_m/\|y_m\|)$. Since $X(\theta_n)$ is a convex set, for $t \geq 0$ and $m \geq m_0$, we have

$$\left(1 - \frac{t}{\|y_m\|}\right) x_n + \frac{t}{\|y_m\|} y_m \in X(\theta_n).$$

Putting $m \to \infty$ yields (1). Quasiconvexity of f yields

$$f\left(\left(1 - \frac{t}{\|y_m\|}\right) x_n + \frac{t}{\|y_m\|} y_m, \theta_n\right) \leq \max\{f(x_n, \theta_n), f(y_m, \theta_n)\} = f(x_n, \theta_n)$$

and $m \to \infty$ implies (2). Choosing $t_n \geq 0$ such that $y_n = x_n + t_n d_n \in K(0, 2r) \setminus K(0,r)$, we find a bounded sequence (y_n), which has an accumulation point y_0 in $X(\theta_0) \setminus S(\theta_0)$. From $f(y_n, \theta_n) \leq f(x_n, \theta_n)$ we obtain $f(y_0, \theta_0) \leq f(x_0, \theta_0)$, contradicting $y_0 \notin S(\theta_0)$.

The condition (ii) in Theorem 1 is hard to check. Hence, the following result is of interest:

THEOREM 2. *Assume that the problem (P_θ) from the introduction satisfies:*

(i) *the functions $x \mapsto g_t(x, \theta_0)$, $t \in T$ are strictly quasiconvex and*

(ii) *there exists $x_0 \in E^n$ such that, for all $t \in T$, $g_t(x_0, \theta_0) < 0$ (Slater's condition).*

Then $X(\theta) \to X(\theta_0)$ as $\theta \to \theta_0$.

In the proof we use the following lemma which has the interest of its own.

LEMMA. *Let T be a compact set and let $(x,t) \mapsto g_t(x)$ be a continuous function on $E^n \times T$ which is strictly quasiconvex in x for every $t \in T$. If the Slater's condition*

$$\exists x_0 \in E^n \ \forall t \in T \ g_t(x_0) < 0$$

is satisfied, then

$$\operatorname{int}\{x \mid g_t(x) \leq 0, t \in T\} = \{x \mid g_t(x) < 0, \ t \in T\}.$$

Proof. Let $a \in E^n$ be such that $g_t(a) < 0$ for all $t \in T$. Assume $a \notin \text{int}\{x \mid g_t(x) \leq 0,\ t \in T\}$. Then, there exist sequences (x_n) and (t_n) such that $x_n \to a$ and $g_{t_n}(x_n) > 0$, for $n \in N$. Since T is compact, there is a subsequence (t_{n_k}), $t_{n_k} \to t_0$, $t_0 \in T$ as $k \to \infty$. Using the continuity, from $g_{t_{n_k}}(x_{n_k}) > 0$, if $k \to \infty$, we obtain $g_{t_0}(a) \geq 0$, which is a contradiction. On the other hand, let $a \in \text{int}\{x \mid g_t(x) \leq 0, t \in T\}$. Then, for each x in some neighbourhood $K(a,r)$, we have $g_t(x) \leq 0$, $t \in T$. Assume that for some $t \in T$, $g_t(x) = 0$ and let $b \in K(a,r)$ be such that a is a convex combination of x_0 and b. Since $0 = g_t(a) \leq \max\{g_t(x_0), g_t(b)\} \leq 0$, it must be $g_t(b) = 0$. Using the strict quasiconvexity of g_t, we obtain a contradiction $g_t(a) < g_t(b) = 0$. Hence, $g_t(a) < 0$, which completes the proof.

Proof of Theorem 2. Let $\theta_n \to \theta_0$. We show first that x_0 is a Slater's point of the sets $X(\theta_n)$ for all sufficiently large n. Assume the contrary. Then, for some subsequence (t_{n_k}), $t_{n_k} \in T$ we have $g_{t_{n_k}}(x_0, \theta_{n_k}) \geq 0$. Since T is compact, there exists an accumulation point $t_0 \in T$ of (t_{n_k}). Using the continuity, we get $g_{t_0}(x_0, \theta_0) \geq 0$, which is a contradiction. Hence, $X(\theta_n) \neq \varnothing$ for all sufficiently large n. Assume that $x \in \limsup X(\theta_n)$. Then, there exists (x_n), $x_n \in X(\theta_n)$, $n \in N$, such that x is an accumulation point of (x_n). Using the continuity, from $g_t(x_n, \theta_n) \leq 0$, $t \in T$, we find $g_t(x_0, \theta_0) \leq 0$, $t \in T$. Hence,

$$\limsup X(\theta_n) \subset X(\theta_0).$$

The proof will be completed if we show that

$$X(\theta_0) \subset \liminf X(\theta_n).$$

Since, $X(\theta_0) = \text{cl int} X(\theta_0)$, it is sufficient to show that

$$\varnothing \neq \text{int} X(\theta_0) \subset \liminf X(\theta_n). \tag{3}$$

According to the Lemma, we have

$$\text{int} X(\theta_0) = \{x \mid g_t(x, \theta_0) < 0, t \in T\} \neq \varnothing.$$

Let $a \in \text{int} X(\theta_0)$. As in the proof of the Lemma, we could show that a is a Slater's point of $X(\theta_n)$, for all $n \geq n_0$. The sequence (x_n), $x_n = a$ for $n \geq n_0$, is convergent to a and satisfies $x_n \in X(\theta_n)$. This yields (3). Hence, $X(\theta) \to X(\theta_0)$ as $\theta \to \theta_0$.

Combining results of Theorems 1 and 2, we have an operative result concerning stability of a quasiconvex semi-infinite problem. We show by examples that our assumptions are essential.

Example 1. Let $M = R$, $\theta_0 = 0$ and

$$(P_\theta): \qquad \inf \ -\theta x + y$$

subject to

$$t\theta^2 x - y \leq 1, \qquad t \in [0,1]$$
$$y \geq 0.$$

Slater's condition is satisfied, $X(\theta_0)$ is not bounded. The problem is unstable at θ_0, since, for $\theta_n = 1/n$, the sequence $s(\theta_n) = (n^2, 0)$ is unbounded.

EXAMPLE 2. Let $M = C[0,1] \times C[0,1]$, $\theta = (a(t), b(t)) \in M$, $\theta_0 = (t,t) \in M$ and

$$(P_\theta): \quad \inf \ -y$$

subject to

$$x + a(t)y \le b(t), \qquad t \in [0,1]$$
$$x + y \le 2$$
$$x, y \ge 0$$

(P_θ) is unstable at θ_0, since for $\theta_n = (a_n(t), b_n(t))$

$$a_n(t) = b_n(t) = \begin{cases} t, & \text{if } 1/n \le t \le 1; \\ 1/n, & \text{if } 0 \le t \le 1/n \end{cases}$$

$\theta_n \to \theta_0$, but the sequence $s(\theta_n) = (0,1)$ does not have accumulation points in $S(\theta_0) = \{(0,2)\}$. Slater's condition is not satisfied.

EXAMPLE 3. Let $M = R$, $\theta_0 = 0$ and

$$(P_\theta): \quad \inf \ -x$$

subject to

$$g(x) + \theta^2 \le 0$$
$$x \ge 0,$$

where

$$g(x) = \begin{cases} x-1, & x-1 \le 0; \\ 0, & 1 \le x \le 2, \\ x-2, & x \ge 2. \end{cases}$$

Problem (P_θ) is not stable at θ_0, since for $\theta \to 0, s(\theta) = 1-\theta^2 \to 1$, but $S(\theta_0) = \{2\}$. The reason lies in the fact that g is not strictly quasiconvex, although all the other conditions are satisfied.

3. CONCLUSION

Theorems 1 and 2 extend the results of Colgen and Schnatz ([**4**, p. 113–116, 216–218]) concerning parametric linear semi-infinite problem

$$(P_\theta): \quad \inf \ \langle c, x\rangle$$

subject to

$$\langle a(t), x\rangle \ge b(t), \qquad t \in T$$

where T is a compact set, $M = C(T, E^n) \times C(T) \times E^n$, $\theta = (a(t), b(t), c) \in M$. Theorems 1 and 2 extend also the result of Brosowski ([**4**, Th. 11, p. 213]), where the sublinearity of the objective function is supposed, as well as the well known stability results for the convex case ([1], [3], [5]).

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