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# STOCHASTIC INVENTORY MODELS INVOLVING VARIABLE LEAD TIME WITH A SERVICE LEVEL CONSTRAINT*

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**Abstract:** The stochastic inventory models analyzed in this study involve two models that are continuous review and periodic review. Instead of having a stockout cost term in the objective function, a service level constraint is added to each model. For both these models with a mixture of backorders and lost sales, we first assume that the lead time ( $L$ )/protection interval ( $T + L$ ) demand follows a normal distribution, and then relax this assumption by only assuming that the mean and variance are known. For each case, we develop a procedure to find the optimal solution, and then an illustrative numerical example is given.

**Keywords:** Inventory, lead time, service level, minimax distribution free procedure.

## 1. INTRODUCTION

In most of the early literature dealing with inventory problems, in both deterministic and probabilistic models, lead time is viewed as a prescribed constant or a stochastic variable, which therefore is not subject to control (see, e.g., Naddor [8] and Silver and Peterson [11]). In 1983, Monden [5] studied the Toyota production system, and pointed out that shortening lead time is a crux of elevating productivity. The successful Japanese experiences using Just-In-Time (JIT) production show that the advantages and benefits associated with efforts to control the lead time can be clearly perceived.

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Recently, several continuous review inventory models have been developed to consider lead time as a decision variable. Liao and Shyu [4] first presented a continuous review inventory model in which the order quantity was predetermined and lead time was a unique decision variable. Later, Ben-Daya and Raouf [1] extended Liao and Shyu's [4] model by considering both lead time and order quantity as decision variables. Ouyang et al. [9] allowed both the lead time and order quantity as decision variables and considered a stockout case. Ouyang and Wu [10] utilized the minimax decision criterion to solve the distribution free model. However, in the models previously mentioned [1, 9, 10], reorder point was not taken into account, and they merely focused on the relationship between lead time and order quantity. In other words, they neglected the possible impact of the reorder point on the economic ordering strategy. Such a phenomenon is usually not perfect in a real inventory situation. In a recent research article, Moon and Choi [7] revised Ouyang et al.'s [9] model by considering the reorder point to be another decision variable. We note that the stockout cost in their paper is an exact value. However, in many practices, it is difficult to determine an exact value for the stockout cost, hence, we here replace the stockout cost term in the objective function by a service level constraint.

The objective of this paper is to extend Ouyang and Wu's [10] continuous review models to accommodate a more realistic situation. That is, our goal is to establish a $(Q, r, L)$ inventory model with a service level constraint. From the numerical example provided, we can show that our new model is better than that of Ouyang and Wu [10]. On the other hand, we also propose a new $(T, R, L)$ inventory model for periodic review. For both of these models, we first assume that the lead time/protection interval demand follows a normal distribution, and then try to find the optimal ordering policy. We next relax this assumption and merely assume that the first and second moments of the probability distribution of lead time/protection interval demand are known and finite, and then solve this inventory model by using the minimax distribution free approach. An illustrative numerical example is provided in each case.

## 2. NOTATIONS AND ASSUMPTIONS

To develop the mathematical models, we utilize the following notations and assumptions in our presentation.

Notations:

$D$ = expected demand per year

$A$ = ordering cost per order

$h$ = holding cost per unit per year

$\sigma^2$ = variance of the demand per unit time

$\alpha$ = proportion of demand that is not met from stock. Hence, $1-\alpha$ is the service level, $\alpha$ takes a suitable value

$\beta$ = the fraction of the demand during the stockout period that will be backordered, $0 \le \beta \le 1$

$Q$ = order quantity, a decision variable in the continuous review case

$r$ = reorder point, a decision variable in the continuous review case

$R$ = target inventory level, a decision variable in the periodic review case

$T$ = length of a review period, a decision variable in the periodic review case

$L$ = length of lead time, a decision variable

$X$ = the lead time/protection interval demand which has a probability density function (p.d.f.) $f_X$

$E(\cdot)$ = mathematical expectation

$x^+$ = maximum value of $x$ and 0, i.e., $x^+ = \max\{x, 0\}$.

Assumptions:

1. The lead time $L$ consists of $n$ mutually independent components. The $i$-th component has a minimum duration $a_i$, a normal duration $b_i$, and a crashing cost per unit time $c_i$. Further, for convenience, we rearrange $c_i$ such that $c_1 \le c_2 \le \cdots \le c_n$. Then, it is clear that the reduction of lead time should be first on component 1 because it has the minimum unit crashing cost, and then on component 2, and so on.
2. If we let $L_0 = \sum_{j=1}^{n} b_j$ and $L_i$ be the length of lead time with components 1,2,..., $i$ crashed to their minimum duration, then $L_i$ can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i} (b_j - a_j)$, $i = 1,2,\ldots,n$; and the lead time crashing cost $C(L)$ per cycle for $L \in [L_i, L_{i-1}]$ is given by $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.
3. The demand per unit time has a probability distribution with mean $D$ and standard deviation $\sigma$, then the distribution of the demand during the length of time $t$ is $t$-fold, that is, the demand $X$ follows p.d.f. $f_X(x)$ with mean $Dt$ and standard deviation $\sigma\sqrt{t}$.

## 3. CONTINUOUS REVIEW MODEL

In this section, we assume that inventory is continuously reviewed. An order quantity of size $Q$ is ordered whenever the inventory level drops to the reorder point $r$. The reorder point $r$ is given by $r = DL + k\sigma\sqrt{L}$, where $k$ is a safety factor. The expected demand shortage at the end of the cycle is given by $E(X-r)^+ = \int_r^\infty (x-r) f_X(x)\,dx$, and thus the expected number of backorders per cycle is $\beta E(X-r)^+$, and the expected number of lost sales per cycle is $(1-\beta)E(X-r)^+$, where $0 \le \beta \le 1$.

We further suppose that the length of lead time does not exceed an inventory cycle time so that there is never more than a single order outstanding in any cycle. The expected net inventory level just before the order arrives is $r - DL + (1-\beta)E(X-r)^+$, and the expected net inventory level at the beginning of the cycle is $Q + r - DL + (1-\beta)E(X-r)^+$ (it can be verified that this quantity is greater than the reorder point $r$). Thus, the expected annual holding cost is approximately $h[\frac{Q}{2} + r - DL + (1-\beta)E(X-r)^+]$.

On the other hand, in many practices, the stockout cost often includes intangible components such as loss of goodwill and potential delay to the other parts of the inventory system, and hence it is difficult to determine an exact value for the stockout cost. Therefore, in this study, we replace the stockout cost by a constraint on the service level. That is, our objective is to minimize the expected total annual inventory cost which is the sum of ordering cost, holding cost, and lead time crashing cost, subject to a constraint on service level. Symbolically, the mathematical model of this problem can be formulated as

$$\min EAC(Q, r, L) = \frac{D}{Q}A + h\left[\frac{Q}{2} + r - DL + (1-\beta)E(X-r)^+\right] + \frac{D}{Q}C(L)$$

subject to

$$\frac{E(X-r)^+}{Q} \le \alpha . \tag{1}$$

### 3.1. Normal distribution case

In this subsection, we assume the lead time demand $X$ follows a normal distribution with mean $DL$ and standard deviation $\sigma\sqrt{L}$. Given that $r = DL + k\sigma\sqrt{L}$, we can also consider the safety factor k as a decision variable instead of $r$. Thus, the expected shortage quantity $E(X-r)^+$ at the end of the cycle can be expressed as a function of $k$; that is,

$$E(X-r)^+ = \int_r^\infty (x-r) f_X(x)\,dx = \int_k^\infty \sigma\sqrt{L}(z-k) f_z(z)\,dz = \sigma\sqrt{L}G(k) > 0, \tag{2}$$

where $f_z(z)$ is the p.d.f. of the standard normal random variable $Z$ and $G(k) = \int_k^\infty (z-k) f_z(z)\,dz$. Therefore, model (1) can be rewritten as

$$\min \mathrm{EAC}_N(Q,k,L) = \frac{D\,[A+C(L)]}{Q} + \frac{hQ}{2} + h[k\sigma\sqrt{L} + (1-\beta)\sigma\sqrt{L}G(k)]$$

subject to

$$\sigma\sqrt{L}G(k) \le \alpha Q\ , \tag{3}$$

where the subscript $N$ in $EAC$ denotes the normal distribution case.

The inequality constraint in model (3) can be converted into equality by adding a nonnegative slack variable, $S^2$. Thus, the Lagrangean function is given by

$$\mathrm{EAC}_N(Q,k,L,\lambda,S) = \mathrm{EAC}_N(Q,k,L) + \lambda[\sigma\sqrt{L}G(k) + S^2 - \alpha Q]$$

$$= \frac{D\,[A+C(L)]}{Q} + \frac{hQ}{2} + h\sigma\sqrt{L}[k + (1-\beta)G(k)] + \lambda[\sigma\sqrt{L}G(k) + S^2 - \alpha Q]\,, \tag{4}$$

where $\lambda$ is a Lagrange multiplier.

For any given $(Q,k,\lambda,S)$, $\mathrm{EAC}_N(Q,k,L,\lambda,S)$ is a concave function in $L \in [L_i, L_{i-1}]$, because

$$\frac{\partial^2 \mathrm{EAC}_N(Q,k,L,\lambda,S)}{\partial L^2} = -\frac{1}{4}hk\sigma L^{-3/2} - \frac{1}{4}\sigma L^{-3/2}G(k)[h(1-\beta)+\lambda] < 0\,.$$

Hence, for fixed $(Q,k,\lambda,S)$, the minimum expected total annual cost will occur at the end points of the interval $[L_i, L_{i-1}]$. On the other hand, we can further prove that, for any given $L \in [L_i, L_{i-1}]$, by the Kuhn-Tucker necessary conditions for minimization problem, we can obtain the slack variable $S^2 = 0$ (hence, the inequality constraint in model (3) will become an equality). Therefore, for fixed $L \in [L_i, L_{i-1}]$, the minimum value of Eq. (4) (in which the variable $S = 0$) will occur at the point $(Q,k,\lambda)$ which satisfies

$$0 = \frac{\partial\, \mathrm{EAC}_N(Q,k,L,\lambda)}{\partial Q} = -\frac{D\,[A+C(L)]}{Q^2} + \frac{h}{2} - \lambda\alpha\,, \tag{5}$$

$$0 = \frac{\partial\, \mathrm{EAC}_N(Q,k,L,\lambda)}{\partial k} = h\sigma\sqrt{L}\left[1 - (1-\beta)P_z(k) - \frac{\lambda}{h}P_z(k)\right], \tag{6}$$

and

$$0 = \frac{\partial\, \mathrm{EAC}_N(Q,k,L,\lambda)}{\partial \lambda} = \sigma\sqrt{L}G(k) - \alpha Q\,, \tag{7}$$

where $P_z(k) \equiv P(Z \geq k)$, and Z is the standard normal random variable.

Solving (5), (6), and (7) for $Q$, $\lambda$, and $G(k)$, respectively, leads to

$$Q = \left\{ \frac{2D[A + C(L)]}{h - 2\lambda\alpha} \right\}^{1/2}, \tag{8}$$

$$\lambda = \frac{h}{P_z(k)}[1 - (1-\beta)P_z(k)], \tag{9}$$

and

$$G(k) = \frac{\alpha Q}{\sigma\sqrt{L}}. \tag{10}$$

Substituting (9) into (8) yields

$$Q = \left\{ \frac{2D[A + C(L)]/h}{1 - 2\alpha \dfrac{1 - (1-\beta)P_z(k)}{P_z(k)}} \right\}^{1/2}, \tag{11}$$

where $\alpha < 1/4$.

We cannot obtain the explicit general solutions for $Q$ and $k$ by solving Eqs. (10) and (11) because the evaluation of each of the expressions requires knowledge of the value of the other. Therefore, we can establish the following iterative algorithm to find the optimal ($Q$, $k$, $L$).

**Algorithm 1**

Step 1. For each $L_i$, $i = 0,1,2,\ldots,n$, perform (i) to (v).

(i) Start with $k_{i1} = 0$ and get $P_z(k_{i1}) = 0.5$.

(ii) Substituting $P_z(k_{i1})$ into (11) evaluate $Q_{i1}$.

(iii) Using $Q_{i1}$ determine $G(k_{i2})$ from (10).

(iv) Check $G(k_{i2})$ from Silver and Peterson [11, pp.779-786] or Brown [2, pp. 95-103] to find $k_{i2}$, and then $P_z(k_{i2})$.

(v) Repeat (ii) to (iv) until no change occurs in the values of $Q_i$ and $k_i$.

Step 2. For each $(Q_i, k_i, L_i)$, compute the corresponding expected total annual cost $EAC_N(Q_i, k_i, L_i)$, $i = 0,1,2,\ldots,n$.

Step 3. Find $\min_{i=0,1,2,\ldots,n} EAC_N(Q_i, k_i, L_i)$. If $EAC_N(Q^*, k^*, L^*) = \min_{i=0,1,2,\ldots,n} EAC_N(Q_i, k_i, L_i)$, then $(Q^*, k^*, L^*)$ is the optimal solution. And hence, the optimal reorder point is $r^* = DL^* + k^*\sigma\sqrt{L^*}$.

**Example 1.** In order to illustrate the above solution procedure, let us consider an inventory system with the data used in Ouyang and Wu [10]: $D = 600$ units/year, $A =$ \$ 200 per order, $h =$ \$ 20/unit/year, $\sigma = 7$units/week, the service level $1-\alpha$ = 0.985, i.e., the proportion of demand that is not met from stock is $\alpha = 0.015$. The lead time has three components with data as shown in Table 1.

**Table 1:** Lead time data

| Lead time component $i$ | Normal duration $b_i$ (days) | Minimum duration $a_i$ (days) | Unit crashing cost $c_i$ (\$/day) |
|---|---|---|---|
| 1 | 20 | 6 | 0.4 |
| 2 | 20 | 6 | 1.2 |
| 3 | 16 | 9 | 5.0 |

We assume that the lead time demand follows a normal distribution and consider the cases when $\beta$ = 0, 0.5, 0.8 and 1. Applying the Algorithm 1 procedure yields the results as shown in Table 2. From this table, the optimal inventory policy can be found by comparing $EAC_N(Q_i, r_i, L_i)$, for $i = 0,1,2,3$, and thus we summarize these in Table 3.

**Table 2:** Solution procedure of Algorithm 1 ( $L_i$ in week )

| $i$ | $L_i$ | $C(L_i)$ | $\beta=0$ | | | $\beta=0.5$ | | | $\beta=0.8$ | | | $\beta=1$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $r_i$ | $Q_i$ | $EAC_N(Q_i,r_i,L_i)$ | $r_i$ | $Q_i$ | $EAC_N(Q_i,r_i,L_i)$ | $r_i$ | $Q_i$ | $EAC_N(Q_i,r_i,L_i)$ | $r_i$ | $Q_i$ | $EAC_N(Q_i,r_i,L_i)$ |
| 0 | 8 | 0 | 107 | 119 | \$ 2,613.54 | 107 | 120 | \$ 2,595.67 | 107 | 120 | \$ 2,584.87 | 107 | 121 | \$ 2,577.65 |
| 1 | 6 | 5.6 | 81 | 119 | \$ 2,564.23 | 81 | 120 | \$ 2,546.31 | 81 | 121 | \$ 2,535.51 | 81 | 121 | \$ 2,528.25 |
| 2 | 4 | 22.4 | 55 | 122 | \$ 2,560.93 | 54 | 123 | \$ 2,542.57 | 54 | 124 | \$ 2,531.49 | 54 | 124 | \$ 2,524.05 |
| 3 | 3 | 57.4 | 41 | 130 | \$ 2,679.55 | 41 | 131 | \$ 2,660.00 | 41 | 131 | \$ 2,648.21 | 41 | 132 | \$ 2,640.29 |

**Table 3:** Summary of the optimal procedure solution for Algorithm 1 ( $L_i$ in week)

| $\beta$ | $(Q^*, r^*, L^*)$ | $EAC_N(Q^*, r^*, L^*)$ |
|---|---|---|
| 0.0 | (122, 55, 4) | \$ 2,560.93 |
| 0.5 | (123, 54, 4) | 2,542.57 |
| 0.8 | (124, 54, 4) | 2,531.49 |
| 1.0 | (124, 54, 4) | 2,524.05 |

Observing Table 2, shows that benefit can be achieved by crashing. For example, under $\beta = 1$ comparing the difference between the cases of $L = 8$ weeks (no crashing case) and $L = 4$ weeks (the minimum cost case), benefit = \$ 2,577.65–2,524.05 = \$ 53.60.

**Remark 1:** In Ouyang and Wu's [10] model, which considered a fixed reorder point $r$ (i.e., they let $P(X > r) = 0.2$). With the normal distribution demand case and $\beta = 1$, they obtained the optimal $(Q, L) = (116, 4)$ and the minimum expected total annual cost $EAC_N(116, 4) = \$\ 2{,}546.94$. In our model, for the same case $\beta = 1$, the optimal $(Q^*, r^*, L^*) = (124, 54, 4)$ and $EAC_N(124, 54, 4) = \$\, 2{,}524.05$. Thus, we find that our model savings $EAC_N(Q, L) - EAC_N(Q^*, r^*, L^*) = EAC_N(116, 4) - EAC_N(124, 54, 4)$ $= \$\ 2{,}546.94 - 2{,}524.05$ = \$ 22.89, which can be viewed as the reward due to controlling the reorder point as a decision variable.

**Remark 2:** As in Example 1, we further perform a sensitivity analysis by considering the change of values of $h$, $D$, $A$ and $\sigma$ which range from +50% to -50%. For the case $\beta = 1$, computed results are shown in Table 4.

**Table 4:** Effect of parameters on the total cost and order strategy

| Change of the parameter (%) | Order policy $(Q^*, r^*, L^*)$ | Expected total annual cost $EAC_N(Q^*, r^*, L^*)$ | Change of percentage of total cost |
|---|---|---|---|
| $h = 30$ (+50%) | (102, 56, 4) | \$ 3,196.70 | +26.65% |
| $= 25$ (+25%) | (112, 55, 4) | 2,871.77 | +13.78% |
| $= 20$ (0 %) | (124, 54, 4) | 2,524.05 | 0% |
| $= 15$ (-25%) | (138, 80, 6) | 2,134.04 | –15.45% |
| $= 10$ (-50%) | (168, 78, 6) | 1,691.44 | –32.99% |
| $D = 900$ (+50%) | (146, 79, 6) | \$ 2,989.86 | +18.46% |
| $= 750$ (+25%) | (134, 80, 6) | 2,769.96 | + 9.74% |
| $= 600$ (0 %) | (124, 54, 4) | 2,524.05 | 0% |
| $= 450$ (-25%) | (108, 56, 4) | 2,236.69 | –11.38% |
| $= 300$ (-50%) | ( 90, 57, 4) | 1,899.52 | –24.74% |
| $A = 300$ (+50%) | (148, 53, 4) | \$ 2,965.58 | +17.49% |
| $= 250$ (+25%) | (136, 54, 4) | 2,754.51 | + 9.13% |
| $= 200$ (0 %) | (124, 54, 4) | 2,524.05 | 0% |
| $= 150$ (-25%) | (106, 82, 6) | 2,264.19 | –10.00% |
| $= 100$ (-50%) | ( 89, 84, 6) | 1,956.96 | –22.47% |
| $\sigma = 10.5$ (+50%) | (127, 64, 4) | \$ 2,721.78 | + 7.83% |
| $= 8.75$(+25%) | (126, 59, 4) | 2,620.06 | + 3.80% |
| $= 7.00$ (0 %) | (124, 54, 4) | 2,524.05 | 0% |
| $= 5.25$ (-25%) | (119, 75, 6) | 2,411.06 | – 4.48% |
| $= 3.50$ (-50%) | (117, 70, 6) | 2,306.30 | – 8.63% |

From Table 4, we can easily observe that the holding cost is the most important parameter of cost savings.

### 3.2. Distribution free case

In many practical situations, the distributional information of lead time demand is often quite limited. Hence, in this subsection, we relax the assumption about the normal distribution of the lead time demand by only assuming that the probability distribution of the lead time demand X has given finite first two moments (and hence, mean and variance are also known and finite); i.e., the p.d.f. $f_X$ of X belongs to the class **F.** of p.d.f.'s with finite mean $DL$ and variance $\sigma^2 L$.

Since the form of the probability distribution of lead time demand X is unknown, we cannot find the exact value of $E(X-r)^+$. Hence, we use the minimax distribution free procedure to solve our problem. The minimax distribution free approach for this problem is to find the most unfavorable p.d.f. $f_X$ in **F** for each $(Q, r, L)$ and then minimize over $(Q, r, L)$; that is, our problem is to solve

$$\min_{Q,r,L} \max_{f_X \in \mathbf{F}} EAC(Q,r,L)$$

subject to

$$\frac{E(X-r)^+}{Q} \le \alpha \,. \tag{12}$$

For this purpose, we need the following proposition which was asserted by Gallego and Moon [3].

**Proposition.**

For any $f_X \in \mathbf{F}$,

$$E(X-r)^+ \le \frac{1}{2}\left[\sqrt{\sigma^2 L + (r-DL)^2} - (r-DL)\right]. \tag{13}$$

Moreover, the upper bound (13) is tight.

Because we have $r = DL + k\sigma\sqrt{L}$, and demand X for any probability distribution of the lead time, the above inequality always holds. Then, using model (1) and inequality (13) and considering the safety factor k as a decision variable instead of $r$, model (12) is reduced to

$$\min EAC_U(Q,k,L) = \frac{D[A+C(L)]}{Q} + \frac{hQ}{2} + h\sigma\sqrt{L}\left[k + \frac{1}{2}(1-\beta)(\sqrt{1+k^2} - k)\right]$$

subject to

$$\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right)\le 2\alpha Q\,, \tag{14}$$

where the subscript U in EAC denotes the distribution free case.

Therefore, the Lagrangian function of model (14) can be formulated as

$$EAC_U(Q,k,L,\lambda,S) = EAC_U(Q,k,L)+\lambda\left[\sigma\sqrt{L}(\sqrt{1+k^2}-k)+S^2-2\alpha Q\right]$$

$$=\frac{D\,[A+C(L)]}{Q}+\frac{hQ}{2}+h\sigma\sqrt{L}\left[k+\frac{1}{2}(1-\beta)(\sqrt{1+k^2}-k)\right]$$

$$+\lambda\left[\sigma\sqrt{L}(\sqrt{1+k^2}-k)+S^2-2\alpha Q\right], \tag{15}$$

where $\lambda$ is a Lagrange multiplier and $S^2$ is a nonnegative slack variable.

By analogous arguments in the previous normal distribution demand case, it can be verified that for fixed $(Q,k,\lambda,S)$, $EAC_U(Q,k,L,\lambda,S)$ is a concave function of $L\in[L_i,L_{i-1}]$. Thus, for fixed $(Q,k,\lambda,S)$, the minimum value of $EAC_U(Q,k,L,\lambda,S)$ will occur at the end points of the interval $[L_i,L_{i-1}]$. Furthermore, for fixed $L\in[L_i,L_{i-1}]$, by the Kuhn-Tucker necessary conditions for minimization problem, we can get the slack variable $S^2=0$. Therefore, for fixed $L\in[L_i,L_{i-1}]$, the minimum value of equation (15) (in which the variable $S=0$) will occur at the point $(Q,k,\lambda)$ which satisfies $\frac{\partial EAC_U(Q,k,L,\lambda)}{\partial Q}=0$, $\frac{\partial EAC_U(Q,k,L,\lambda)}{\partial k}=0$ and $\frac{\partial EAC_U(Q,k,L,\lambda)}{\partial \lambda}=0$, simultaneously. The resulting solutions are

$$Q=\left\{\frac{2D[A+C(L)]}{h-4\lambda\alpha}\right\}^{1/2}, \tag{16}$$

$$\lambda=h\left[\frac{\sqrt{1+k^2}}{\sqrt{1+k^2}-k}-\frac{1}{2}(1-\beta)\right], \tag{17}$$

and

$$\sqrt{1+k^2}-k=\frac{2\alpha Q}{\sigma\sqrt{L}}\,. \tag{18}$$

Combining Eqs. (16), (17), and (18), the order quantity

$$Q = \left\{ \frac{4\alpha D[A + C(L)] + h\sigma^2 L}{2\alpha h(1 - 2\alpha\beta)} \right\}^{1/2}, \tag{19}$$

where $\alpha < 1/2$.

Consequently, we can establish the following algorithm to find the optimal ( $Q$ , $k$ , $L$ ).

**Algorithm 2**

Step 1. For each $L_i$, $i = 0,1,2,..., n$, we use Eq. (19) to evaluate $Q_i$, and then use Eq. (18) to determine $k_i$.

Step 2. For each $(Q_i, k_i, L_i)$, compute the corresponding expected total annual cost $EAC_U(Q_i, k_i, L_i)$, $i = 0,1,2,..., n$.

Step 3. Find $\min_{i=0,1,2,....,n} EAC_U(Q_i, k_i, L_i)$. If $EAC_U(Q_*, k_*, L_*) = \min_{i=0,1,2,....,n} EAC_U(Q_i, k_i, L_i)$, then $(Q_*, k_*, L_*)$ is the optimal solution. And hence, the optimal reorder point is $r_* = DL_* + k_*\sigma\sqrt{L_*}$.

**Example 2.** Using the same data in Example 1, we assume that the distribution of the lead time demand is free except that its first and second moments are given. Applying Algorithm 2, the summarized optimal values are presented in Table 5.

**Table 5:** Summary of the optimal procedure solution for Algorithm 2 ( $L_i$ in week )

| $\beta$ | $(Q_*, r_*, L_*)$ | $EAC_U(Q_*, r_*, L_*)$ |
|---|---|---|
| 0.0 | (141, 65, 4) | $ 2,818.77 |
| 0.5 | (142, 65, 4) | 2,798.23 |
| 0.8 | (143, 65, 4) | 2,786.12 |
| 1.0 | (143, 65, 4) | 2,777.55 |

The expected total annual cost $EAC_N(Q_*, r_*, L_*)$ is obtained by substituting $(Q_*, r_*, L_*)$ into model (3); and thus, $EAC_N(Q_*, r_*, L_*) - EAC_N(Q^*, r^*, L^*)$ is the largest amount that we would be willing to pay for the knowledge of the probability distribution of demand. This quantity can be regarded as the expected value of additional information (EVAI), i.e., $EVAI = EAC_N(Q_*, r_*, L_*) - EAC_N(Q^*, r^*, L^*)$. Moreover, it can be observed from Table 6 that the amount of EVAI increases as $\beta$ increases.

**Table 6:** Calculation of EVAI for the continuous review model

| $\beta$ | $EAC_N(Q_*, r_*, L_*)$ | $EAC_N(Q^*, r^*, L^*)$ | EVAI |
|---|---|---|---|
| 0.0 | $ 2,784.59 | $ 2,560.93 | $ 223.66 |
| 0.5 | 2,781.12 | 2,542.57 | 238.55 |
| 0.8 | 2,779.26 | 2,531.49 | 247.77 |
| 1.0 | 2,777.55 | 2,524.05 | 253.50 |

**Remark 3.** Analogous to the argument in Remark 1, we discuss the case that the distribution of the lead time demand is free and $\beta = 1$. Ouyang and Wu [10] obtained the optimal $(Q, L) = (116,4)$ and the expected total annual cost $EAC_U(116,4) = \$\ 2,839.06$. In our model, the optimal $(Q_*, r_*, L_*) = (143,65,4)$ and $EAC_U(143,65,4) = \$\ 2,777.55$. Thus, observe that our model savings $EAC_U(Q, L) - EAC_U(Q_*, r_*, L_*) = EAC_U(116,4) - EAC_U(143,65,4) = \$\ 2,839.06 - 2,777.55 = \$\ 61.51$, which is the reward due to controlling the reorder point.

**Remark 4:** From Table 3 and Table 5, it is interesting to observe that, regardless of the normal distribution or distribution free model in the continuous review case, increasing the backorder rate $\beta$ results in a decrease in the minimum expected total annual cost, but it results in an increase in the order quantity. On the other hand, there is a robustness property for the optimal reorder point and lead time as $\beta$ varies.

## 4. PERIODIC REVIEW MODEL

In this section, we consider a periodic review inventory model. We first assume that the inventory level is reviewed every $T$ units of time, a sufficient ordering quantity is ordered up to the target inventory level $R$, and the ordering quantity is arrived after $L$ units of time, where $L < T$ so that at most one order is outstanding in any cycle. Again, we suppose that the protection interval is $T + L$, demand $X$ has a p.d.f. $f_X(x)$ with mean $D(T + L)$ and standard deviation $\sigma\sqrt{T+L}$, and the target inventory level $R$ is given by $R = D(T + L) + \delta\sigma\sqrt{T+L}$, where $\delta$ is a safety factor. Therefore, the expected demand shortage at the end of the cycle is given by $E(X - R)^+ = \int_R^\infty (x-R) f_X(x)\,dx$, and the expected number of lost sales per cycle is $(1-\beta)E(X - R)^+$.

For this new model, we attempt to utilize some results in Montgomery *et al.* [6], and get the mathematical expression of this problem as follows:

$$\min EAC(T, R, L) = \frac{A}{T} + h\left[R - DL - \frac{DT}{2} + (1-\beta)E(X-R)^+\right] + \frac{C(L)}{T}$$

subject to

$$\frac{E(X-R)^+}{D(T+L)} \le \alpha . \tag{20}$$

### 4.1. Normal distribution case

In this subsection we assume that the protection interval demand for the $(T, R, L)$ periodic review model follows a normal p.d.f. $f_X(x)$ with mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$. By applying $R = D(T+L) + \delta\sigma\sqrt{T+L}$ and further allowing the safety factor $\delta$ as a decision variable instead of $R$, the expected shortage quantity $E(X-R)^+$ at the end of the cycle can be expressed as a function of $\delta$ as

$$E(X-R)^+ = \int_R^\infty (x-R) f_X(x)\, dx = \int_\delta^\infty \sigma\sqrt{T+L}(z-\delta) f_z(z)\, dz = \sigma\sqrt{T+L}G(\delta) > 0 ,$$

where $f_z(z)$ and $G(\delta)$ are defined in Section 3.1. Hence, model (23) is reduced to

$$\min EAC_N(T, \delta, L) = \frac{A + C(L)}{T} + h\left[DT + \delta\sigma\sqrt{T+L} - \frac{DT}{2} + (1-\beta)\sigma\sqrt{T+L}G(\delta)\right]$$

subject to

$$\sigma\sqrt{T+L}G(\delta) \le \alpha D(T+L) . \tag{21}$$

The Lagrangian function is thus given by

$$\begin{aligned} EAC_N(T, \delta, L, \lambda, S) = \frac{A + C(L)}{T} + \frac{hDT}{2} + h\sigma\sqrt{T+L}[\delta + (1-\beta)G(\delta)] \\ + \lambda[\sigma\sqrt{T+L}G(\delta) + S^2 - \alpha D(T+L)] , \end{aligned} \tag{22}$$

where $\lambda$ is a Lagrange multiplier and $S^2$ is a nonnegative slack variable.

Similar to the arguments in Section 3, we obtain the slack variable $S^2 = 0$. And, for fixed $(T, \delta, \lambda)$, $EAC_N(T, \delta, L, \lambda)$ has a minimum value at the points of $[L_i, L_{i-1}]$. On the other hand, for fixed $L \in [L_i, L_{i-1}]$, the minimum value of (22) will occur at the point $(T, \delta, \lambda)$ which satisfies $\partial EAC_N(T, \delta, L, \lambda)/\partial T = 0$, $\partial EAC_N(T, \delta, L, \lambda)/\partial \delta = 0$, and $\partial EAC_N(T, \delta, L, \lambda)/\partial \lambda = 0$. Thus, we obtain the following equations:

$$\left[\frac{A + C(L)}{T^2} - \frac{hD}{2} + \lambda\alpha D\right](T+L)^{1/2} = \frac{h\sigma\delta}{2} + \frac{\sigma}{2}G(\delta)[h(1-\beta) + \lambda] , \tag{23}$$

$$\lambda = \frac{h}{P_z(\delta)}[1-(1-\beta)P_z(\delta)], \tag{24}$$

and

$$G(\delta) = \frac{\alpha D}{\sigma}\sqrt{T+L}, \tag{25}$$

where $P_z(\delta)$ is defined as in Section 3.1.

Substituting (24) into (23) yields

$$\left\{\frac{A+C(L)}{hT^2} - \frac{D}{2} + \frac{\alpha D}{P_z(\delta)}[1-(1-\beta)P_z(\delta)]\right\}(T+L)^{1/2} = \frac{\sigma}{2}\left[\delta + \frac{G(\delta)}{P_z(\delta)}\right]. \tag{26}$$

Since it is not easy to solve Eqs. (25) and (26) for $T$ and $\delta$, an iterative algorithm can be employed to find the optimal $(T, \delta, L)$.

**Algorithm 3**

Step 1. For each $L_i$, $i = 0,1,2,..., n$, perform (i) to (v).

(i) Start with $\delta_{i1} = 0$, and then get $P_z(\delta_{i1}) = 0.5$ and $G(\delta_{i1}) = 0.3989$ by checking the table from Silver and Peterson [11] or Brown [2].

(ii) Substitute $\delta_{i1}$, $G(\delta_{i1})$, and $P_z(\delta_{i1})$ into (26), and use a numerical search method to obtain $T_{i1}$.

(iii) Using $T_{i1}$, determine $G(\delta_{i2})$ from (25).

(iv) Check $G(\delta_{i2})$ from Silver and Peterson [11] or Brown [2] to find $\delta_{i2}$, and then $P_z(\delta_{i2})$.

(v) Repeat (ii) to (iv) until no change occurs in the values of $T_i$ and $\delta_i$.

Step 2. For each $(T_i, \delta_i, L_i)$, compute the corresponding expected total annual cost $EAC_N(T_i, \delta_i, L_i)$, $i = 0,1,2,..., n$,.

Step 3. Find $\min_{i=0,1,2,....,n} EAC_N(T_i, \delta_i, L_i)$. If $EAC_N(T^*, \delta^*, L^*) = \min_{i=0,1,2,....,n} EAC_N(T_i, \delta_i, L_i)$, then $(T^*, \delta^*, L^*)$ is the optimal solution. And hence, the target inventory level is $R^* = D(T^* + L^*) + \delta^* \sigma \sqrt{T^* + L^*}$.

**Example 3.** Consider the same data as in Example 1. Using Algorithm 3, the optimal values for this example are shown in Table 7.

**Table 7:** Summary of the optimal procedure solution for Algorithm 3 ( $T_i$ , $L_i$ in week)

| $\beta$ | $(T^*, R^*, L^*)$ | $EAC_N(T^*, R^*, L^*)$ |
|---|---|---|
| 0.0 | (9.34, 217, 8) | $ 2,719.77 |
| 0.5 | (9.41, 218, 8) | 2,690.34 |
| 0.8 | (9.46, 218, 8) | 2,666.85 |
| 1.0 | (9.48, 218, 8) | 2,654.93 |

By comparing the results in Table 3 and Table 7, it is obvious that the continuous review model is less expensive than the periodic review, and the amount of cost difference decreases as $\beta$ increases.

### 4.2. Distribution free case

In this subsection, since the form of the probability distribution of protection interval demand $X$ is unknown, we cannot determine the exact value of $E(X-R)^+$. Therefore, using the same proposition as presented in the continuous review case, we can obtain the least upper bound of the expected demand shortage at the end of the cycle as follows:

For any $f_X \in \mathbf{F}$, let $\mathbf{F}$ denote the class of p.d.f. $f_X$'s with finite mean $D(T+L)$ and variance $\sigma^2(T+L)$, then

$$E(X-R)^+ \le \frac{1}{2}\left\{\sqrt{\sigma^2(T+L)+[R-D(T+L)]^2}-[R-D(T+L)]\right\}. \tag{27}$$

The upper bound (27) is tight.

Based upon the results of (27) and $R = D(T+L)+\delta\,\sigma\sqrt{T+L}$ , the safety factor $\delta$ can be viewed as a decision variable instead of $R$, and thus model (20) is reduced to

$$\min EAC_U(T,\delta,L) = \frac{A+C(L)}{T}+\frac{hDT}{2}+h\sigma\sqrt{T+L}\left[\delta+\frac{1}{2}(1-\beta)\left(\sqrt{1+\delta^2}-\delta\right)\right]$$

subject to

$$\sigma\sqrt{T+L}\,(\sqrt{1+\delta^2}-\delta) \le 2\alpha D(T+L)\,. \tag{28}$$

The Lagrangian function of this model is given by

$$\begin{aligned} EAC_U(T,\delta,L,\lambda,S) = {} & \frac{A+C(L)}{T}+\frac{hDT}{2}+h\sigma\sqrt{T+L}\left[\delta+\frac{1}{2}(1-\beta)(\sqrt{1+\delta^2}-\delta)\right] \\ & +\lambda\left[\sigma\sqrt{T+L}\,(\sqrt{1+\delta^2}-\delta)+S^2-2\alpha D(T+L)\right], \end{aligned} \tag{29}$$

where $\lambda$ is a Lagrange multiplier and $S^2$ is a nonnegative slack variable.

As mentioned in the above cases, it can be shown that the slack variable $S^2 = 0$. And, for fixed $(T, \delta, \lambda)$, $EAC_U(T, \delta, L, \lambda)$ has a minimum value at the end points of $[L_i, L_{i-1}]$. Further, for fixed $L \in [L_i, L_{i-1}]$, the minimum value of $EAC_U(T, \delta, L, \lambda)$ will occur at the point $(T, \delta, \lambda)$ which satisfies $\partial EAC_U(T, \delta, L, \lambda)/\partial T = 0$, $\partial EAC_U(T, \delta, L, \lambda)/\partial \delta = 0$, and $\partial EAC_U(T, \delta, L, \lambda)/\partial \lambda = 0$. Simplifying these equations leads to

$$\left[\frac{A + C(L)}{T^2} - \frac{hD}{2} + 2\lambda\alpha D\right](T + L)^{1/2} = \frac{h\sigma\delta}{2} + \frac{\sigma}{4}(\sqrt{1+\delta^2} - \delta)[h(1-\beta) + 2\lambda], \qquad (30)$$

$$\lambda = h\left[\frac{\sqrt{1+\delta^2}}{\sqrt{1+\delta^2} - \delta} - \frac{1}{2}(1-\beta)\right], \qquad (31)$$

and

$$\sqrt{1+\delta^2} - \delta = \frac{2\alpha D}{\sigma}\sqrt{T + L}. \qquad (32)$$

Substituting (31) and (32) into (30), we get the review period as

$$T = \sqrt{\frac{2[A + C(L)]}{hD(1 - 2\alpha\beta)}}, \qquad (33)$$

where $\alpha < 1/2$. The deriving process is similar to (22), and hence we omit it.

A similar algorithm procedure as proposed in Section 3.2 can be performed to obtain the optimal solutions.

**Example 4.** Using the same data in Example 1 and applying a similar procedure as in Algorithm 2 yields the optimal values given in Table 8.

**Table 8:** Summary of the optimal procedure solution ( $T_i$ , $L_i$ in week )

| $\beta$ | $(T_*, R_*, L_*)$ | $EAC_U(T_*, R_*, L_*)$ |
|---|---|---|
| 0.0 | (9.72, 263, 8) | $ 3,555.91 |
| 0.5 | (9.80, 263, 8) | 3,522.67 |
| 0.8 | (9.85, 264, 8) | 3,506.20 |
| 1.0 | (9.88, 264, 8) | 3,495.08 |

From Table 9, we can see that the amount of EVAI increases as $\beta$ increases.

**Table 9 :** Calculation of EVAI for the periodic review model

| $\beta$ | $EAC_N(T_*, R_*, L_*)$ | $EAC_N(T^*, R^*, L^*)$ | EVAI |
|---|---|---|---|
| 0.0 | $ 3,496.78 | $ 2,719.77 | $ 777.01 |
| 0.5 | 3,492.95 | 2,690.34 | 802.61 |
| 0.8 | 3,494.29 | 2,666.85 | 827.44 |
| 1.0 | 3,495.08 | 2,654.93 | 840.15 |

**Remark 5:** Analogous to the argument in Remark 4 (continuous review case), from Table 7 and Table 8, it is obvious that, for the periodic review case, increasing the backorder rate $\beta$ results in a decrease in the minimum expected total annual cost, but it results in an increase in the optimal review period. And the optimal target inventory level and lead time as $\beta$ varies is robust.

## 5. CONCLUSION

In this study, we presented a mixture inventory model with backorders and lost sales, where the stockout cost term in the objective function is replaced by a service level constraint. First, we extended Ouyang and Wu's [10] continuous review model by simultaneously optimizing order quantity, reorder point, and lead time. Next, we developed a periodic review inventory model in which review period, target inventory level, and lead time are treated as decision variables. For these two models, we assumed that the lead time/protection interval demand follows a normal distribution, and found the optimal solution. Then, we relaxed this assumption and applied the minimax decision criterion to solve the distribution free case.

In future research on this problem, it would be of interest to consider an inventory model involving the problem of net present value. Another possible extension of this work may be conducted by considering the backorder rate $\beta$ as a decision variable.

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