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ON A GRAPH TRANSFORMATION THAT PRESERVES THE STABILITY NUMBER*

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Abstract: We derive from Boolean methods a transformation which, when applicable, builds from a given graph a new graph with the same stability number and with the number of vertices decreased by one. We next describe classes of graphs for which such a transformation leads to a polynomial algorithm for computing the stability number.

Keywords: Boolean methods, stability number, polynomial algorithms.

1. INTRODUCTION

In the present paper all graphs will be assumed simple (no loops and no multiple edges are allowed). A set S of vertices in a graph $G = (V, E)$ is stable if no two vertices in S are linked by an edge. The maximum size of a stable set in graph G is denoted by $\alpha(G)$ and is called the stability number of G. For a weighted graph G, the maximum weight of a stable set in G is denoted $\alpha_W(G)$.

Given a positive integer k, finding whether an arbitrary graph contains a stable set with at least k vertices is NP-complete [6]. However, there are special classes of graphs for which $\alpha(G)$ can be computed in polynomial time [e.g. 1, 3, 4, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 20].

In some cases, Boolean methods can suggest graph theoretical procedures. Ebenegger, Hammer and de Werra [5] have described the relationship between the maximization of a pseudo-Boolean function and the determination of a stable set

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* The Boolean transformation studied in this paper as well as the definition of a magnet are due to Peter L. Hammer. I would like to thank him for having encouraged me to demonstrate the potentiality of such a transformation.

having maximum weight in a graph. This relation is summarized in Section 2. In the same paper, Ebenegger et al. consider the computation of the stability number $\alpha(G)$ of a graph $G=(V,E)$ (unweighted case) and describe the transformation of the corresponding pseudo-Boolean function which amounts to constructing another graph G' with $\alpha(G')=\alpha(G)-1$. By repeatedly applying this construction, one may compute $\alpha(G)$ (in at most $\alpha(G)\leq|V|$ steps). Unfortunately, the number of vertices is generally increasing when the transformation is applied. However, specialized versions of this construction have provided polynomial algorithms for some classes of graphs [7, 9, 10, 12]. Recently, a different Boolean transformation has been studied in [14].

In order to compute the stability number $\alpha(G)$ of a graph $G=(V,E)$, we study in Section 3 a simplification of the corresponding pseudo-Boolean function; the transformation, when applicable, amounts to constructing another graph $G'=(V',E')$ with $|V'|=|V|-1$ and $\alpha(G')=\alpha(G)$. It is based on the Boolean equality $xy+x\overline{y}=x$.

In Section 4, we describe classes of graphs for which the stability number can be computed in polynomial time by using the above transformation.

A graph $G=(V,E)$ is bipartite if the vertex set V can be partitioned into two sets V_1 and V_2 such that each edge of E has one endpoint in V_1 and the other in V_2; we shall denote such a graph by $G=((V_1,V_2),E)$. A bipartite graph $G=((V_1,V_2),E)$ is complete if each vertex in V_1 is adjacent to all vertices of V_2.

A chordless cycle, or chain, on k vertices is denoted $C_k(v_1,\ldots,v_k)$, or $P_k(v_1,\ldots,v_k)$ (or C_k and P_k for short).

For a vertex x in graph G, we denote by $N_G(x)$ the set of vertices which are adjacent to x in G. Graph G is called H-free if none of its induced subgraphs are isomorphic to H.

For two sets A and B, $A\setminus B$ denotes the set of elements which are in A but not in B. The weight of a set of vertices is the total weight of its elements.

The graph theoretical terms not defined here are borrowed from [2] while for pseudo-Boolean definitions, the reader is referred to [11].

2. PSEUDO-BOOLEAN FUNCTIONS AND CONFLICT GRAPHS

It is known that a pseudo-Boolean function f can always be written in a polynomial form, i.e.,

$$f(x_1,\ldots,x_n)=K+\sum_{i=1}^{p}w_iT_i$$

$$\text{where } T_i=\prod_{j\in A_i}x_j\prod_{k\in B_i}\overline{x}_k \quad \text{with } A_i,B_i\subseteq\{1,\ldots,n\} \quad \text{and } A_i\cap B_i=\emptyset.$$

If all $w_i (1 \leq i \leq p)$ are strictly positive and $K = 0$, we say that f is a posiform. To a posiform f we associate a weighted conflict graph $G = (V, E)$ defined as follows:

$V = \{1, \ldots, p\}$ and each vertex i has a weight w_i,

$E = \{[i, j] \mid \exists\, k \in ((A_i \cap B_j) \cup (A_j \cap B_i))\}$.

Hence, two vertices i and j of G are linked by an edge if x_k appears in T_i (or T_j) while $\bar{x}_k$ appears in T_j (or T_i). It is clear from the definition of G that the maximum value of f is equal to the maximum weight $\alpha_w(G)$ of a stable set in G.

Conversely, for each graph G with positive weights w_u associated with each vertex u of G, there exist posiforms f such that G is the conflict graph of f [5]. Indeed, consider an arbitrary covering of the edge set of G by complete bipartite partial subgraphs $G_i((V_{i_1}, V_{i_2}), E_i)$ of G, $i = 1, \ldots, q$. Notice that $G_1, \ldots, G_q$ are partial, but not necessarily induced subgraphs of G. Then set:

$$f = \sum_{u \in V} w_u T_u$$

where with , .

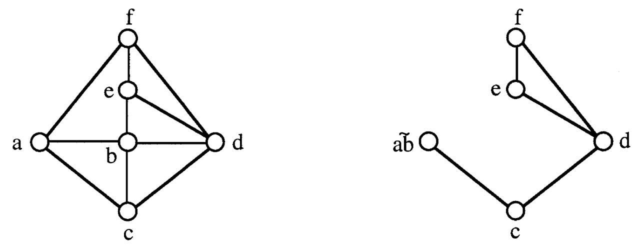
Let and be two terms of the posiform f such that appears in and appears in . Then and . Hence, u is adjacent to v in , showing that G is the conflict graph associated with f.

3. MAGNETS IN GRAPHS

A magnet in a graph is defined as a pair (a, b) of adjacent vertices with the same weight and such that each vertex in is adjacent to each vertex in . In other words, the two endpoints of an edge induce a magnet in a graph G if and only if this edge is not the middle edge of any in G.

Given a magnet (a, b) in a graph G, we consider a new graph, denoted , and obtained from G by replacing vertices a and b by a new vertex having the same weight as a and b, and linked to every common neighbor of a and b in G. This transformation is illustrated in Fig. 1.

The following theorem states that the maximum weight of a stable set in G is not modified by transformation .



a graph G the graph T(G,(a,b))

Figure 1.

Theorem 1. Let be a magnet in a weighted graph . Then .

Proof: We shall give two proofs of this theorem, a Boolean and a graph theoretical one.

Boolean proof:
The edges incident to a or b can be covered by the two following complete bipartite partial subgraphs and of G:

with and ,

with and .

Consider now any covering of the edges in by complete bipartite partial subgraphs . The graphs cover all the edges of E and the associated posiform satisfies and . Hence, . It follows that f has the same maximum value as the posiform . This means that the conflict graph associated with g satisfies and .

But is obtained from G by removing vertices a and b, and by adding a new vertex of weight linked to every vertex v such that appears in . Since appears in if and only if , it follows that .

Graph theoretical proof:
Denote and consider any stable set S in G. If a or b belongs to S then is stable in and the weight of S is equal to the weight of . Otherwise S is stable in . This proves that .

In order to show that , consider any stable set in and define , and :

– if does not belong to then is stable in G;

– if belongs to while then is stable in G and .

– if belongs to while , then both and are empty.

Hence, is stable in G and . ❑

In a graph G, we say that some vertex a dominates some vertex b if . It is well known that if a graph G contains two adjacent vertices a and b such that a dominates b, then the graph obtained from G by removing vertex a has a stability number equal to . This is in fact a corollary of Theorem 1. Indeed, in this case, all weights are equal to one (unweighted case) and . Hence, (a, b) is a magnet in G and the neighbors of in are exactly those of b in G. Therefore, is the graph obtained from G by removing vertex a.

Let a and b be two adjacent vertices in a graph G. If a dominates b, we say that is a d-magnet in G. Notice that if (a, b) is a d-magnet in a graph G, then is always an induced subgraph of G. This is not necessarily the case for all magnets in G. However, we can state the following property.

Property 1. Let (a, b) be a magnet in a graph G, let H be an induced subgraph of , and assume G is H-free.

Then is a vertex in H, and there are at least two adjacent vertices c and d in H that are not adjacent to .

Proof: Let denote the vertex set of H. If does not belong to , then induces an H in G, a contradiction. So assume that belongs to . Define and consider the subgraph of G induced by . If a dominates b, or b dominates a in , then or induces a graph isomorphic to H in G, a contradiction. Hence contains an induced , which means that c and d are adjacent vertices of that are not adjacent to . ❑

Definition 1. A graph H is a demagnetization (or d-demagnetization) of a graph G if

– H does not contain any magnet (or d-magnet), and
– there exists a sequence of graphs such that for a magnet (or d-magnet) in .

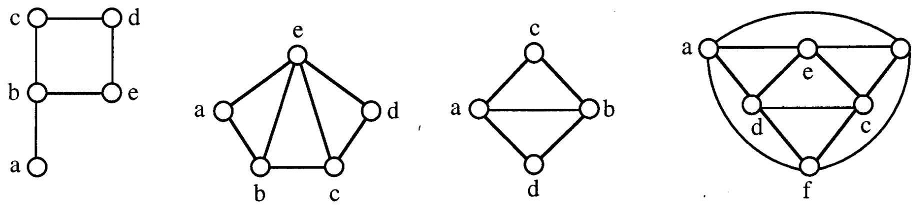
It follows from Theorem 1 that if H is a demagnetization of G then . From now on, we will only consider the unweighted case. In the next section, we study classes of graphs G such that the stability number can be computed in polynomial time by finding the maximum stable set in a demagnetization H of G.

4. TWO POLYNOMIALLY SOLVABLE CASES

Given a in a graph G, we distinguish among three kinds of vertices; and are the extreme vertices of are the interior ones, and all other vertices in G are said to be exterior to .

A vertex v in a graph G is called special if each induced in G contains v as extreme vertex, and each induced in contains v as interior vertex. In particular, if neither G, nor contains an induced , then each vertex in G is special. Section 4.1 considers -free graphs that contain a special vertex; we show that a demagnetization H of such graphs G can be obtained so that the edge set of H is empty (hence, equals the number of vertices in H).

A flag is a graph obtained from a by adding a vertex adjacent to exactly one vertex of the . A gem is the graph obtained from a by adding a vertex adjacent to all four vertices of the . A diamond is a complete graph on four vertices minus one edge. A is the graph obtained from a complete graph on six vertices by removing a perfect matching in it. All these graphs are represented in Fig. 2. Let G be a -free, flag-free, gem-free and -free graph, let H be any d-demagnetization of G, and let L be any demagnetization of H; we prove in Section 4.2 that each connected component of L is either an isolated vertex, or else a .



flag (a,b,c,d,e) gem (a,b,c,d,e) diamond (a,b,c,d) $\overline{3K_2}$(a,b,c,d,e,f)

Figure 2.

4.1. On C_5-free graphs that contain a special vertex

Lemma 1. Let a be a special vertex in a -free graph G, and let b be a vertex adjacent to a that minimizes . Then (a, b) is a magnet in G.

Proof: Argue by contradiction: assume is not a magnet in G. Then G contains an induced . Since is smaller than or equal to , there exists a vertex e adjacent to c but not to a and b in G. Vertex e cannot be adjacent to d, else G contains an induced . Hence, a is not an extreme vertex of in G, a contradiction. ❑

Such a magnet is called an s-magnet. We now prove that if is an s-magnet in a -free graph G, then is also -free and contains as a special vertex.

Lemma 2. Let be an s-magnet in a -free graph G. Then is -free.

Proof: Argue by contradiction: assume contains an induced .

We know by Property 1 that belongs to ; we may assume . Now a must be adjacent to exactly one vertex among and , else G contains an induced or ; we may assume a is adjacent to . It follows that a is not an interior vertex of in , a contradiction. ❑

Lemma 3. Let be an s-magnet in a -free graph G. Then is a special vertex in .

Proof: Argue by contradiction: assume first is not an extreme vertex of an induced in . Then must belong to , else a is an exterior vertex of in G. But cannot be equal to , else G contains an induced or in which a is not an extreme vertex; we may assume . It follows that either a is not an extreme vertex of an induced or in G, or else a is not an interior vertex of an induced or in , a contradiction.

Assume now is not an interior vertex of an induced in . Then must belong to , else a is exterior to in ; we may assume . It follows that contains an induced or in which a is not an interior vertex, a contradiction. ❑

A demagnetization of a graph is called an s-demagnetization if only special magnets are used when applying transformation . The following theorem is a direct consequence of Lemma 2 and Lemma 3.

Theorem 2. Let G be a -free graph that contains a special vertex, and let H be an s-demagnetization of G. Then H has an empty edge set.

Notice that if a special vertex in a graph G is isolated, then both G and are -free (since, otherwise, a would be an exterior vertex of any in G or). The following algorithm can therefore be used for computing the stability number of a -free graph that contains a special vertex.

Input. A - free graph G that contains a special vertex.
Output. The stability number of G.

1. Determine a special vertex a in G. Set .
2. While the edge set of H is not empty do
   If a is an isolated vertex, then set a equal to any nonisolated vertex in H.
   Determine a vertex b adjacent to a that minimizes
   Set .
3. Set equal to the number of vertices in H.

Finding a special vertex a (if any) in a graph can be performed in time. Determining a vertex b adjacent to a that minimizes takes time. Given a magnet in H, the construction of takes time. Since the main loop is performed times, it follows that the above algorithm runs in time.

A graph G is called Meyniel [18] if each odd cycle in G with at least five vertices contains at least two chords. It is easy to observe that a graph G is -free, -free and -free if and only if both G and are Meyniel. Notice that each vertex in such a graph is special; hence, the first step of the above algorithm can be simplified. It therefore follows that if both G and are Meyniel, then the stability number of G can be computed in time by means of the above algorithm.

4.2. On P_5-free, flag-free, gem-free and -free graphs

Lemma 4. Let G be a -free, flag-free, gem-free and -free graph that contains no d-magnet. Then G is diamond-free.

Proof: Argue by contradiction: assume G contains an induced diamond . Since b does not dominate a, there is a vertex e in G adjacent to a but not to b. Vertex e cannot be adjacent to exactly one vertex among c and d else induces a gem in G.

We first show that e cannot be adjacent to both c and d. If this is the case, then there exists a vertex f adjacent to e but not to a in G (else a dominates e). Now f is adjacent to b else G contains an induced flag , gem or gem . Also, f is neither adjacent to c, nor to d, else G contains an induced gem , gem or . Since a does not dominate c, there exists a vertex g adjacent to c but not to a. Vertex g is not adjacent to b, else G contains an induced gem , gem or . It follows that G contains an induced gem , flag , flag or flag , a contradiction.

So e is adjacent neither to a, nor to b. Up to this point, we have proved that any vertex that is adjacent to exactly one vertex among a and b is adjacent neither to c, nor to d. Now, since a does not dominate e, there exists a vertex f adjacent to e but not to a. Vertex f cannot be adjacent to c or d else f would not be adjacent to b (by the above observation) and G would contain an induced flag , flag or flag .

Since a does not dominate c, there exists a vertex g adjacent to c but not to a, and this implies that g is not adjacent to b (by the above observation). So, g is adjacent to e and f else G contains an induced , , flag or flag . Finally, G contains an induced flag or flag , a contradiction. ❑

Corollary 1. Let G be a -free, flag-free, gem-free and -free graph. Let H be a d-demagnetization of G. Then G is -free, flag-free and diamond-free.

Proof: H is an induced subgraph of G. Hence, this corollary directly follows from Lemma 4 and from the fact that a diamond is an induced subgraph of a gem and of a . ❑

Lemma 5. Let be a magnet in a -free, flag-free and diamond-free graph G. Then is also -free, flag-free and diamond-free.

Proof: The fact that is diamond-free follows from Property 1. Notice also that if two vertices c and d are adjacent to in then c is adjacent to d, else G contains an induced diamond . It now remains to prove that H is -free and flag-free. We argue by contradiction and assume first that H contains an induced . According to Property 1 and the above observation, can only be equal to x or u. We may assume . Now, neither a, nor b is adjacent to z in G, else G contains an induced diamond or diamond . So G contains an induced , , fiag or flag , a contradiction.

So, we assume now H contains an induced flag . Again, according to Property 1 and to the above observation, can only be equal to x. Now, neither a nor b can be adjacent to z or u, else or induces a diamond in G. It follows that G contains an induced flag or flag , a contradiction. ❑

Lemma 6. Let G be a -free, flag-free and diamond-free connected graph containing an induced . Then G is .

Proof: Argue by contradiction: let be an induced subgraph of and suppose . Since G is connected, we may assume that there is a vertex a in adjacent to . Vertex a cannot be adjacent to both and else G contains an induced diamond . We may assume that a is not adjacent to . Now a is adjacent to else G contains an induced or flag . Hence, G contains an induced diamond or flag , a contradiction. ❑

Theorem 3. Let G be a -free, flag-free, gem-free and -free graph, let H be a d-demagnetization of G and let L be a demagnetization of H. Then each connected component of L is either

(i) an isolated vertex, or

(ii) a .

Proof: We know from Corollary 1 that H is -free, flag-free and diamond-free. Hence, it follows from Lemma 5 that L is also -free, flag-free and diamond-free.

Consider any connected component of L. We may assume that has a nonempty edge set, else is an isolated vertex and nothing has to be proved. So let a and b be two adjacent vertices in . Since is not a magnet in there is an induced in . Now, since is not a magnet in , there is an induced in (vertex e is possibly equal to b). It follows that f is not adjacent to b in , else contains an induced flag , flag , or diamond . Also, f is adjacent to d in else contains an induced . It follows that contains an induced , which means that , by Lemma 6. ❑

It follows from the above theorem that the stability number of -free, flag-free, gem-free and -free graphs can be computed in polynomial time by means of the following algorithm.

Input. A -free, flag-free, gem-free and -free graph.
Output. The stability number of G.

1. Set .
   While H contains a d-magnet do
   Choose any d-magnet in H and set .

2. Set .
   While L contains a magnet do
   Choose any magnet in L and set .

3. Let n and c be the number of vertices and the number of induced in L, respectively. Set equal to .

Finding a magnet or a d-magnet (if any) in a graph can be performed in time. Since at most magnets are determined in Steps 1 and 2, the above algorithm runs in time.

Let G be an arbitrary graph, and let L be the graph resulting from the application on G of Steps 1 and 2 of the above algorithm. If each connected component of L is either an isolated vertex, or a , then Step 3 can be applied on L in order to compute the stability number of G. According to Theorem 3, such a situation necessarily occurs if G is -free, flag-free, gem-free and -free. It can, however, also occur for other kinds of graphs. For example, if G is a flag, then L contains exactly three isolated vertices, which means that the stability number of a flag is three. There is therefore no need to design a recognition algorithm for -free, flag-free, gem-free and -free graphs. It is more interesting to apply Steps 1 and 2 of the above algorithm to any given graph G, and to check whether the reduced graph L has the desired structure.

5. CONCLUDING REMARKS

One of the aims of this paper was to prove that Boolean methods can suggest graph theoretical procedures. We have studied a simplification on posiforms which, when applicable, amounts to reducing the size of the corresponding conflict graph while preserving its stability number. We have described in section 4 classes of graphs G for which such a transformation leads to a polynomial algorithm for the computation of .

Let be a class of graphs for which the stability number can be determined in polynomial time. Future research in the use of transformation would be to characterize those graphs G that admit a demagnetization H with . We could for example choose as being the class of claw-free graphs [19, 20]. Notice that given any magnet in a claw-free graph G, the graph is also claw-free, by Property 1.

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