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FUZZY CONTROL BASED ON FUZZY RELATION EQUATIONS

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Abstract: The paper discusses fuzzy controllers based on fuzzy relation equations in the case when fuzzy controller inputs are exact. The Godelian implication and minimum function as *t*-norm are considered. The fuzzy relation equation with sup-t composition results in plausible control and the adjoint equation leads to simple fuzzy control (min-max inference method). The computation of fuzzy relations is not necessary. The relation between fuzzy plausible control and simple fuzzy control is discussed. Control algorithms are compared on a simulation example.

Keywords: Fuzzy control theory, fuzzy operators, decision making.

1. INTRODUCTION

Fuzzy logic control is one of the expanding application fields of fuzzy set theory [6]. Recent applications of fuzzy logic control have spread over various areas of automatic control, particularly in process control [3, 7-9, 11]. A fuzzy logic controller (FLC) allows a simple and more human approach to control design due to its ability to determine outputs for a given set of inputs without using a conventional, mathematical model. FLC follows the general strategy of control worked out by a human being. Including a set of control rules and membership functions, the fuzzy controller converts linguistic variables into numeric values required in most applications. Altering control rules or membership functions provides fuzzy controllers with adaptive capabilities that are very important for industry.

A typical FLC is composed of three basic parts: an input signal fuzzification where continuous input signals are transformed into linguistic variables, a fuzzy engine that handles rule inference, and a defuzzification part that ensures exact and physically interpretable values for control variables.

The design of FLC may include: the definition of input and output variables, selection of the data manipulation method, membership function design and the rule (control) base design. The most frequently used data manipulation method is the "min-max-gravity" method (simple fuzzy control). This Mamdani-type controller assumes min-max inference operators and center of gravity defuzzification [6, 8, 11]. However, any t-norm and t-conorm as inference operator can be used. Some properties of FLC using different inference operators can be found in [2, 4].

Simple control is a reasoning procedure based on modus ponens $(A \wedge (A \Rightarrow B)) \Rightarrow B$ tautology [8, 11]. Modus ponens tautology reads:

- *Implication:* if A then B
- *Premise:* A is true
- *Conclusion:* B is true

where A and B are fuzzy statements or propositions.

Approximate reasoning based on other tautologies, such as modus tolens, syllogism or generalized modus ponens, which gives $(A \Rightarrow (A \Rightarrow B)) \Rightarrow B$ [11], have also been suggested [5, 10]. Here we are concerned with plausible control. Fuzzy control is plausible [8] if it fulfil features given with F1-F4. Plausible control reads:

with *implication* if A then B

F1 -	*Premise*	A is true	*Conclusion*	B is true
F2 -	*Premise*	A is not true	*Conclusion*	B is unknown
F3 -	*Premise*	A is more fuzzy	*Conclusion*	B is more fuzzy
F4 -	*Premise*	A is less fuzzy	*Conclusion*	B is B

where feature F1 describes modus ponens tautology.

Inference methods can also be obtained by utilizing fuzzy relation equations [6] with different implication functions. Fuzzy sets and fuzzy relations, calculated for simple control, satisfy neither the fuzzy relational equation with sup-t composition nor the adjoint equation. Therefore, simple control is not "mathematically correct". However, the solutions of fuzzy relation equations are not unique, because the φ-operator presented in [6] and the t-norm [2, 8, 11] are not unique. It is shown in [8] which combinations of different implication functions (φ-operator) and t-norms give plausible control. In this paper the Godelian implication for φ-operator and minimum function (intersection) for t-norm are considered.

Despite being "incorrect", simple control is applied rather than plausible control. Fuzzy control algorithms based on fuzzy relation equations require the calculation of fuzzy relations. Fuzzy relations often need more memory requirements as disposed. Besides, all the necessary calculations for simple control are trivial, demanding minimum time, whish is of great importance, especially for real time applications.

This paper discusses the case when fuzzy controller inputs are not fuzzy. This is the case when measured variables and set points are not fuzzy, or when they are fuzzy, they are defuzzified before succeeding to the fuzzy controller. From the practical aspect, this assumption is not restrictive. The assumption of defuzzified controller inputs leads to many simplifications in fuzzy control based on fuzzy relation equations. These simplifications concern the fuzzy relation for there is no more need to calculate the fuzzy relation. All mathematical computations are similar to fuzzy simple control, providing the enhanced application of plausible control.

2. FUZZY CONTROLLER AND CONTROL BASE

A typical closed loop system with a fuzzy controller is shown in Fig. 1. Controlled variables (inputs to the fuzzy controller) and set points can be first fuzzified and then used. However, that phase is skipped here, causing exact data to succeed to control rules.

The main source of knowledge to construct the set of control rules comes from the control protocol of the human operator. The protocol consists of a set of conditional "if-then" statements, where the first part (if) contains a condition and the second part (then) deals with an action (control) that is to be taken. It conveys the human strategy, expressing which control is to be applied when a certain state of the process being controlled is matched. A set of n control rules (control base) is given with (1)

- IF X_{11} AND X_{1j} AND X_{1m} THEN U_1
- IF X_{i1} AND X_{ij} AND X_{im} THEN U_i
- IF X_{n1} AND X_{nj} AND X_{nm} THEN U_n (1)

Condition X_{ij} is expressed by its membership function $\mu(X_{ij}(x))$, where x belongs to the space of X_{ij}. The space $[X_{ij}]$ is the same for all $i = 1,\dots,n$. In the case when any X_{ij} does not exist in the control base, it is assumed that the membership function equals one over the entire space.

Control variables U_i that are to be applied when certain conditions are satisfied are expressed by membership functions $\mu(U_i(u))$, where u belongs to the space of U_i. All U_i are defined over the same space, here denoted as $[U]$.

Statements like *else* or *or* are easy to incorporate in the control base [1]. For reasons of simplification here we observe the control base in the following form:

- IF X_i THEN U_i; $i = 1,\dots,n$ (2)

where membership functions $\mu(X_i(x))$ are defined over the same space, here denoted with $[X]$. The generalization from (2) to (1) can be easily made.

The defuzzification phase ensures exact and physically interpretable values for control variables. There are several defuzzification methods and here the center of gravity procedure is used [6].

3. SIMPLE FUZZY CONTROL

Supposing the control base is given with (2) and using fuzzy relation notation, simple fuzzy control is given with [6]

- $R_i = X_i * U_i$; $i = 1,\dots,n$
- $R = \cup R_i$
- $U = X \odot R$ (3)

where $*$ denotes the Cartesian product operator and $\odot$ denotes the sup-min composition. Fuzzy relations R_i are defined over product space $[X,U]$ and are calculated as [1,6,11]

$$R_i(x,u) = X_i(x) * U_i(u) = \min\{X_i(x), U_i(u)\} \text{ for all } x \in [X] \text{ and } u \in [U] \quad (4)$$

where $X_i(x)$ and $U_i(u)$ are expressed by their membership functions. In (3) the maximum (union) function is denoted with $\cup$ [11]. The controller's input (condition-value obtained from the system) is denoted with X. Controller input $X(x)$ is defined over the space $[X]$. In the case when the control base is given with (1) X_i and X, in (3), are fuzzy relations.

The calculated fuzzy control is denoted by $U(u)$, which is defined over the space $[U]$. Defuzzification is later applied to obtain the exact control value. Relating to sup-min composition [1, 6], fuzzy control is

$$U(u) = X(x) \odot R(x,u) = \sup_{x\in[X]}\{\min\{X(x), R(x,u)\}\} \quad (5)$$

More generally, operator $\odot$ describes the sup-t composition, and here the minimum function for the t-norm is observed [6, 8, 11]. Subject to (3)

$$U(u) = \sup_{x\in[X]}\{\min\{X(x), \cup_i R_i(x,u)\}\} \quad (6)$$

The sup-min composition is distributed with respect to union [1, 6]

$$U(u) = \sup_{x\in[X]}\{\cup_i \min\{X(x), R_i(x,u)\}\} \quad (7)$$

Subject to (3)

$$U(u) = \sup_{x\in[X]}\{\cup_i \min\{X(x), X_i(x) * U_i(u)\}\} \tag{8}$$

Subject to associatively this can be rewritten as

$$U(u) = \cup_i \sup_{x\in[X]}\{\min\{X(x), X_i(x)\}\} * U_i(u) = \cup_i \Lambda_i * U_i(u) \tag{9}$$

where Λ_i is a scalar value, called the possibility of X with respect to X_i [6], defined by

$$\Lambda_i = \Pi(X / X_i) = \sup_{x\in[X]}\{\min\{X(x), X_i(x)\}\} \tag{10}$$

In simple fuzzy control, Λ_i is a scalar value even in the case when the controller's input is fuzzy. However, here is a particular case, when $X(x)$ is nonfuzzy, which means

$$X(x) = \begin{cases} 1; & x = x_0 \\ 0; & \text{otherwise} \end{cases} \tag{11}$$

In that case, Λ_i is calculated as

$$\Lambda_i = \sup\{\min\{1, X_i(x_0)\}, \min\{0, X_i(x)\}\} = X_i(x_0) \tag{12}$$

4. FUZZY CONTROL BASED ON A FUZZY RELATION EQUATION WITH sup-*t* COMPOSITION

Using fuzzy relation notation and control base (2) this type of fuzzy control is given with [6]

- $R_i = X_i \,\varphi\, U_i\,; \quad i = 1,\ldots,n$
- $R = \cap R_i$
- $U = X \odot R$ (13)

The minimum (intersection) function is denoted with $\cap$ [11], and operator $\odot$ describes the sup-t (here sup-min) composition. Operator φ represents the implication function [6]. In this paper the Godelian implication, defined in [1, 6, 8] is concerned:

$$(X \,\varphi\, Y)(x,y) = X(x) \,\varphi\, Y(y) = \begin{cases} 1; & \mu(X(x)) \le \mu(Y(y)) \\ \mu(X(x)); & \mu(X(x)) > \mu(Y(y)) \end{cases} \tag{14}$$

From (5) and (13) fuzzy control is

$$U(u) = \sup_{x\in[X]}\{\min\{X(x), \cap_i R_i(x,u)\}\} \tag{15}$$

Sup-min composition is not distributed with respect to intersection [1]

$$X \odot (Y \cap Z) <= (X \odot Y) \cap (X \odot Z) \tag{16}$$

Therefore, obtaining fuzzy control $U(u)$, in (13), demands the calculation of fuzzy relation $R(x,u)$. However, in the particular case when the controller's input $X(x)$ is exact (11), equation (15) can be rewritten as:

$$U(u) = \sup\{\min\{1, \cap_i R_i(x_0,u)\}, \min\{0, \cap_i R_i(x,u)\}\} = \cap_i R_i(x_0,u) \tag{17}$$

Subject to (13) and (14), equation (17) leads to

$$U(u) = \cap_i X_i(x_0)\, \varphi\, U_i(u) = \cap_i \Lambda_i\, \varphi\, U_i(u) \tag{18}$$

where Λ_i is a scalar value given by (12). The computations of plausible fuzzy control (18) and simple fuzzy control (9) are very similar, enhancing the possible application of plausible control.

5. FUZZY CONTROL BASED ON AN ADJOINT FUZZY RELATION EQUATION

This fuzzy control, concerning (2), is given with [6]

- $R_i = X_i * U_i$; $i = 1,\ldots,n$
- $R = \cup R_i$
- $U = X \varphi R$ (19)

where operators $*$ and $\cup$ are already defined. Operator φ in (19) denotes the Godelian implication between fuzzy set and fuzzy relation [6]

$$U(u) = \inf_{x\in[X]}\{X(x)\, \varphi\, R(x,u)\} \tag{20}$$

In the particular case when $X(x)$ is nonfuzzy (11), equation (20) leads to:

$$U(u) = \inf_{x\in[X]}\{(1\, \varphi\, R(x_0,u)), (0\, \varphi\, R(x,u))\} \tag{21}$$

Concerning (14) and (19), fuzzy control is given with

$$U(u) = \cup_i R_i(x_0,u) = \cup_i X_i(x_0) * U_i(u) = \cup_i \Lambda_i * U_i(u) \tag{22}$$

Obviously, fuzzy control based on the adjoint fuzzy relation equation, in the case when the controller's input is nonfuzzy, gives the same control algorithm as simple fuzzy control.

6. SIMULATION EXAMPLE

The control problem is to hold an object with mass m on the top of a "hill", Fig. 2. The observed object is influenced by force F:

$$\vec{F} = \overrightarrow{Fm} + \overrightarrow{Fd} + \overrightarrow{Ff} \tag{23}$$

where Fm denotes "movement" force, Fd denotes random disturbance force and Ff denotes fuzzy control force, and

$$Fm = Fg\frac{G'(p)}{\sqrt{1+(G'(p))^2}} = mg\frac{G'(p)}{\sqrt{1+(G'(p))^2}} \tag{24}$$

where $G'(p)$ denotes the derivation of ground shape. Two ground shapes (functions) were considered

- $G1(p) = e^{-p^2}$
- $G2(p) = -p^2$ (25)

The object's position (p) and velocity (v) are calculated in discrete time (t) domain as

- $\overrightarrow{p(t+1)} = \overrightarrow{p(t)} + \overrightarrow{v(t)}\Delta t$
- $\overrightarrow{v(t+1)} = \overrightarrow{v(t)} + \overrightarrow{F(t)} / m\Delta t - C_f \overrightarrow{v(t)}$ (26)

where friction constant is denoted with C_f and Δt represents sample time. Concrete values were

- $m = 10\ Kg$
- $g = 9.81\ m/s^2$
- $\Delta t = 0.01\ s$
- $C_f = 0.04$

The control base is given with Table 1. Membership functions for position and force are shown in Fig. 3, and velocity membership functions are shown in Fig. 4.

Experiments with simple fuzzy control and plausible control were done under the same circumstances - the same set of control rules, the same membership functions and the same disturbance force. Simulation was done for 3000 sampling intervals and the sum of the squares of the applied disturbance force was

$$\sum_{t=1}^{3000} F_d{}^2(t) = 2.176 \cdot 10^6\ [N^2]$$

For both ground functions, two initial points were considered $p(0) = 0$ and $p(0) = 1$ with exact controller input. Position error (PE) and fuzzy power (FP) were as results concerned.

- $\text{PE} = \sum_{t=1}^{t=3000} p^2(t)\ [m^2]$
- $\text{FP} = \sum_{t=1}^{t=3000} {F_f}^2(t)\ [10^6 N^2]$ (27)

The obtained results with simple fuzzy control are shown in Table 2 and with plausible control in Table 3.

7. CONCLUSION

It is shown that in the case when fuzzy controllers' inputs are exact, simple fuzzy control and fuzzy control based on the sup-t relation equation result in similar control algorithms. Both algorithms do not require calculation and memory storage of the fuzzy relation. Here was minimum function as t-norm and Godelian implication function observed, that gives plausible control. The application possibility of plausible control is, therefore, enhanced.

Fuzzy control based on the adjoint relation equation results in the same control algorithm as simple fuzzy control when controllers' inputs are nonfuzzy.

The simulation example shows that the quality of system behavior with a fuzzy plausible controller can be superior compared to system behavior with a simple fuzzy controller.

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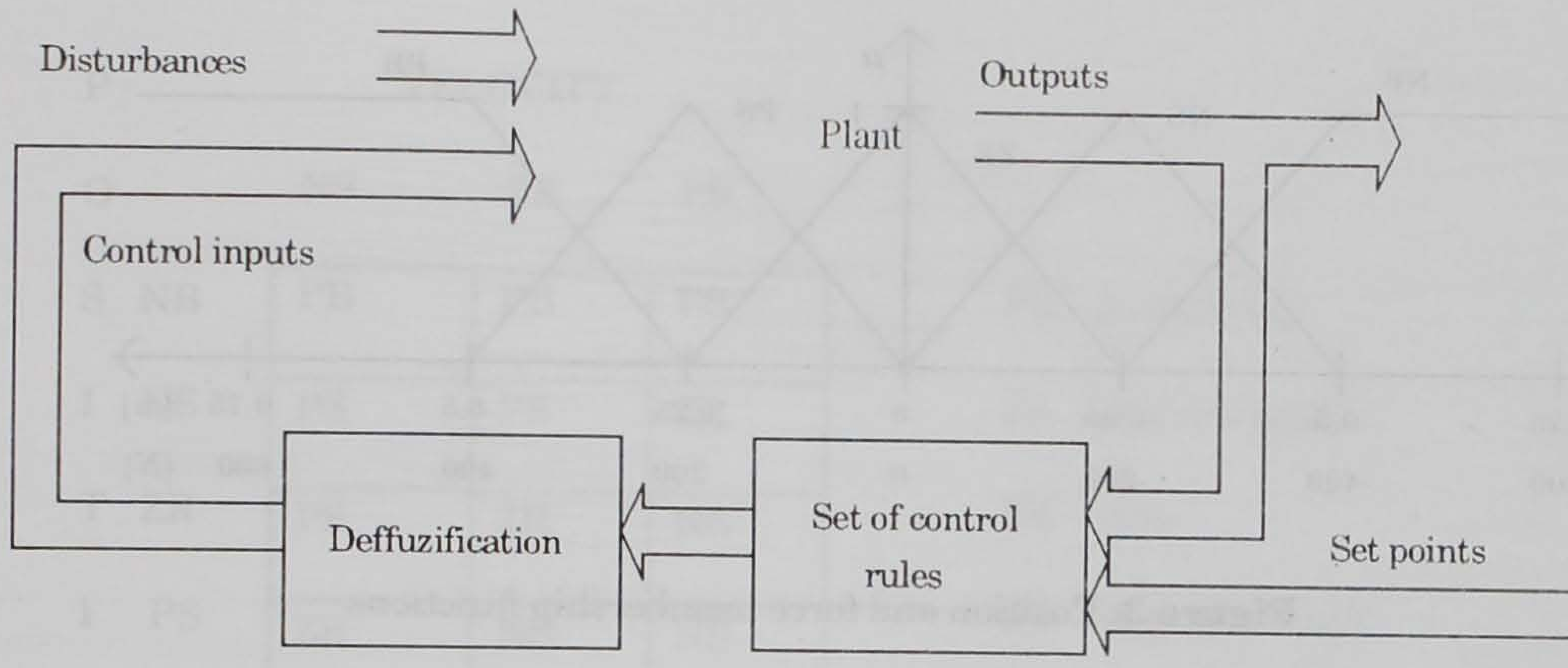

Figure 1: Closed loop system with fuzzy controller

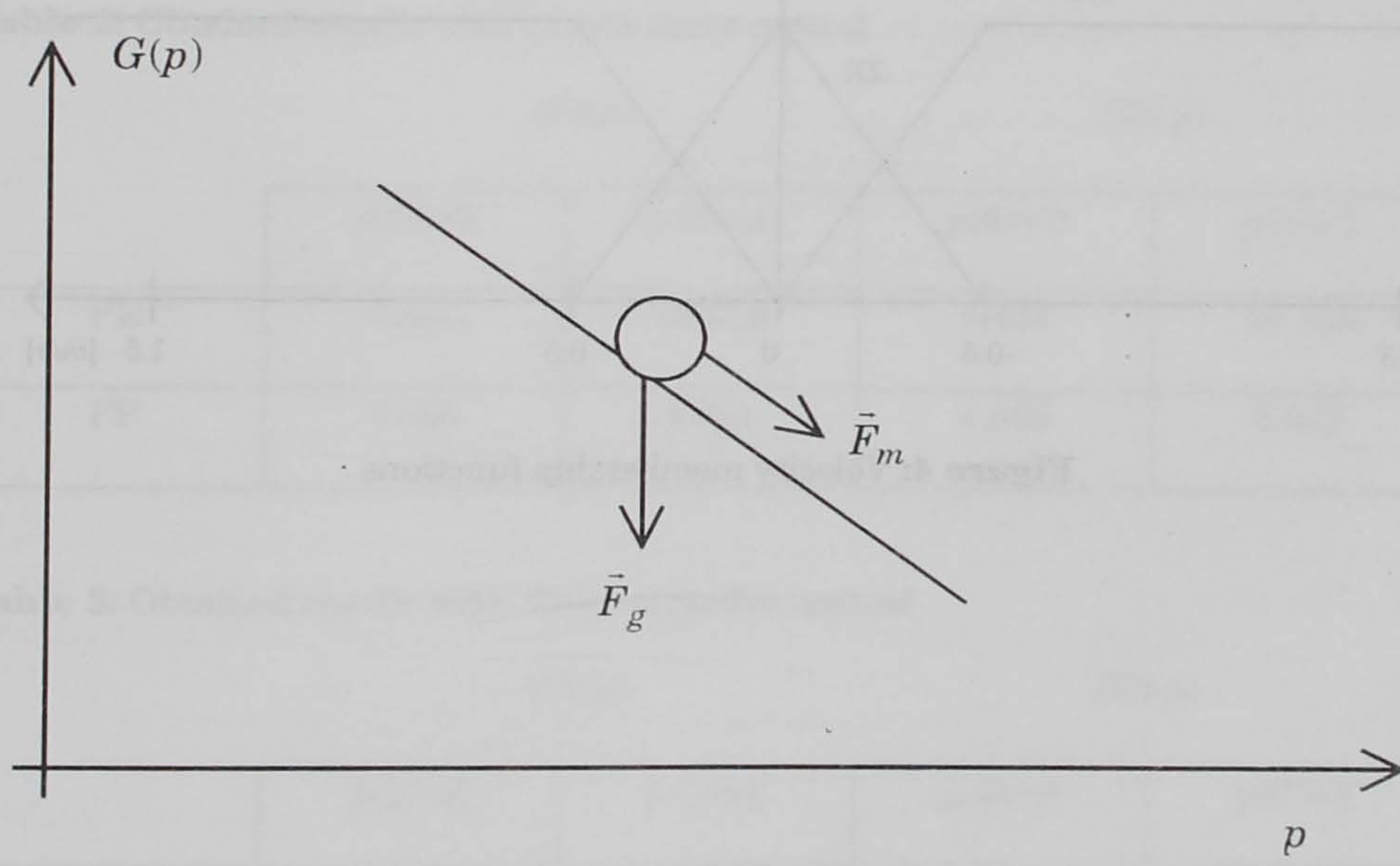

Figure 2: Simulation example

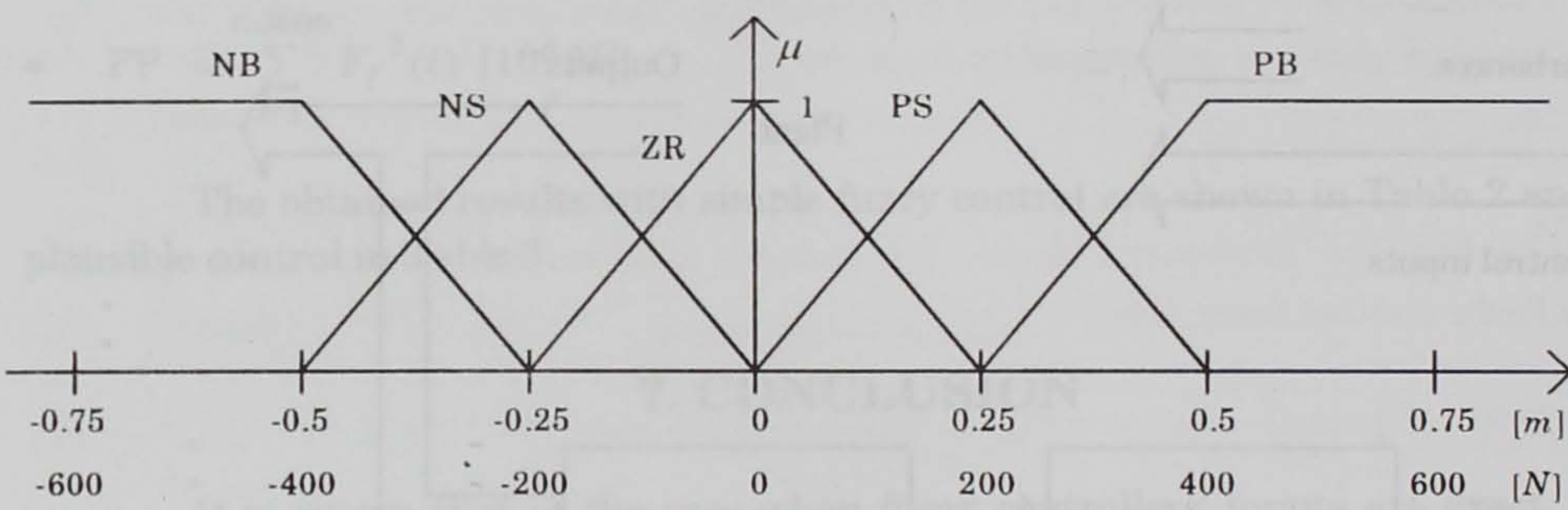

Figure 3: Position and force membership functions

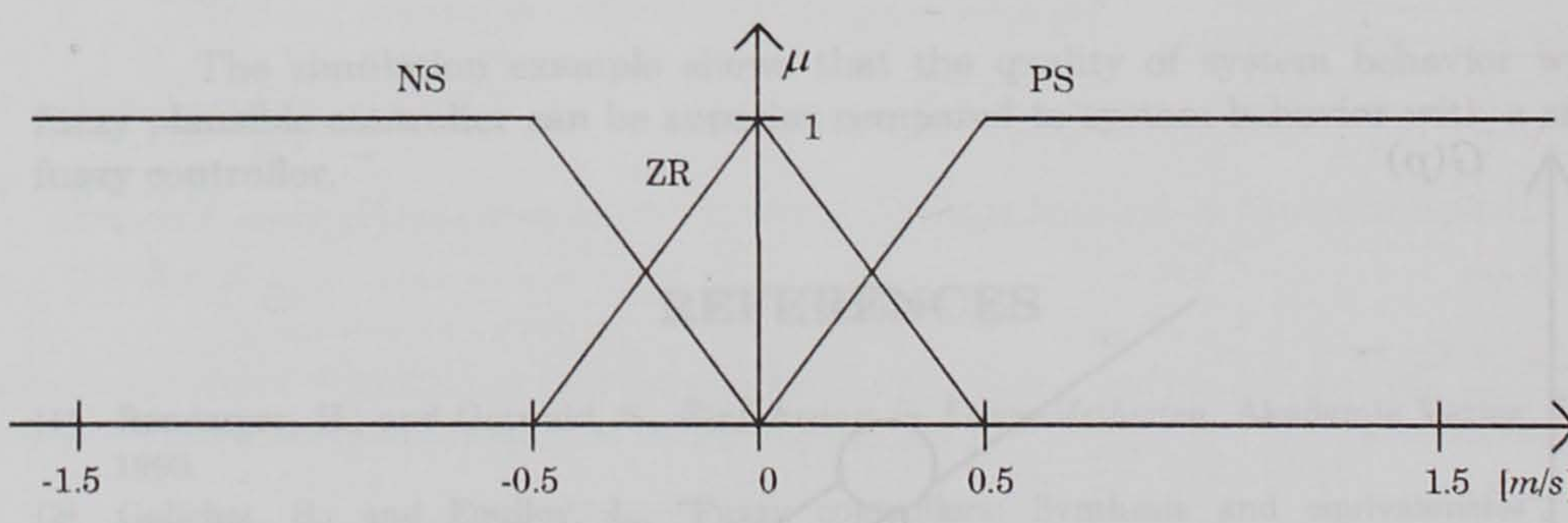

Figure 4: Velocity membership functions

Table 1: Fuzzy control force

POSITION \ VELOCITY	NS	ZR	PS
NB	PB	PB	PS
NS	PS	PS	ZR
ZR	PS	ZR	NS
PS	ZR	NS	NS
PB	NS	NB	NB

PB - positive big

PS - positive small

ZR - zero

NS - negative small

NB - negative big

Table 2: Obtained results with simple fuzzy control

	$G1(p)$		$G2(p)$	
	$p(0)=0$	$p(0)=1$	$p(0)=0$	$p(0)=1$
PE	0.535	20.018	0.434	19.306
FP	4.589	5.921	4.656	6.022

Table 3: Obtained results with fuzzy plausible control

	$G1(p)$		$G2(p)$	
	$p(0)=0$	$p(0)=1$	$p(0)=0$	$p(0)=1$
PE	0.172	19.857	0.162	19.047
FP	6.727	7.949	6.803	7.992