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A PRODUCTION INVENTORY MODEL FOR VARIABLE DEMAND AND PRODUCTION

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Abstract: This study presents a production inventory model for a variable market in which the production rate at any instant depends on the demand and the inventory level. Three market demand rates are addressed. The proposed model can assist managers in accurately determining the production type under the variform market demand. A simple algorithm to compute the optimal production scheduling period is developed. The algorithm is illustrated by considering a numerical example of this class. Finally, we also discuss the sensitivity of these solutions to changes in underlying parameter values.

Keywords: Production inventory, variable demand, optimal production scheduling period.

1. INTRODUCTION

Many mathematical models have been developed for controlling inventory. However, the formulation of production inventory policies for controlling the production rate has received relatively little attention. Goswami and Chaudhuri [6] developed order-level inventory models for deteriorating items in which the finite production rate is proportional to the time dependent demand rate. Balkhi and Benkherouf [1] considered a production lot size inventory model in which arbitrary production and demand rate depends on the time function. Furthermore, Bhunia and Maiti [2] assumed that the production rate is a variable. They also presented inventory models in which the production rate depends on either on-hand inventory or demand. In practice, demand and inventory level may influence production. A situation in which the demand decreases (or increases) may cause the manufacturers to decrease (or increase) their production as well. Also, the production rate may either increase or decrease with the inventory level. Thus, the effect of the production rate on inventory systems warrants further study. In this study, we further extend the above to formulate an inventory model by simultaneously considering the variable demand and production rate.

The standard EOQ model assumes a constant and known demand rate over an infinite planning horizon. Mak [9] proposed a production lot size inventory model with a uniform demand rate over a fixed time horizon. However, most items experience a variable demand rate. Hence, the EOQ model must obviously be modified. Many studies have extended the EOQ model in order to accommodate time varying demand patterns. Goswami and Chaudhuri [5, 6] assumed a linear trend in demand. Hariga [7], Bose et al. [3] and Hong et al. [8] considered an inventory model with time-proportional demand. Wee [13] developed a deterministic lot-size inventory model for deteriorating items with shortage and a declining market. Wagner and Whitin [12] discussed the discrete version of the above problem and developed a dynamic programming algorithm for the determination of EOQ. Buchanan [4] presented two myopic heuristics for more general demand functions using a dynamic programming formulation. In addition, Mandal and Phaudar [10] and Urban [11] discussed the inventory model with an inventory-level-dependent demand rate.

This study attempts to develop a production inventory model by assuming that the finite production rate is proportional to both the demand rate and the inventory level, when the demand rate follows a time function. The mathematical formula of the expected cost function is derived. Then the optimal production scheduling period and maximum inventory level can be easily solved by using the Intermediate Value Theorem method and the theory of majorization. Three situations of the market demand patterns, i.e., growth, maturity and decline, are considered and a numerical example is used to illustrate the proposed model. Finally, the sensitivity of these solutions to changes in underlying parameter values as well as the advantages of the proposed model are addressed in the conclusions.

2. ASSUMPTIONS AND NOTATION

The mathematical model of the production inventory problem considered in this paper is developed on the basis of the following assumptions and notations. Additional notation will be introduced later when needed.

1. A single item is considered over a prescribed period of T units of time, where $T = t_1 + t_2$; t_1 and t_2 are the durations of the production period and after the production period, respectively.
2. At time t $(0 \le t \le T)$, the demand rate is $D(t)$.
3. At time t $(0 \le t \le T)$, the on-hand inventory is $I(t)$.
4. Production rate, $P(t)$, at any instant depends on both the demand and the inventory level. That is at time t $(0 \le t \le t_1)$, $P(t) = a + bD(t) - cI(t)$, $a > 0$, $0 \le b < 1$, and $0 \le c < 1$.

5. Shortages are not allowed.
6. The inventory system involves only one stocking point; I_m represents the maximum inventory level.
7. The relevant costs are the inventory holding cost C_i per unit per time unit and the setup cost C_s per new cycle, which are all known and constant during period T.

3. THE MATHEMATICAL MODEL

By the above assumptions and notations, the inventory level starts at time $t=0$ and reaches I_m maximum level after t_1 time units have elapsed. Then the production is stopped, the stock level declines continuously and the inventory level becomes zero at time t_1+t_2 $(=T)$. Our purpose is to find the optimal values of t_1, T and I_m that minimize the average cost K over the time horizon $[0,T]$.

The expression for the differential equations governing the stock status during period $[0,T]$ can be written as

$$\frac{dI(t)}{dt} = P(t) - D(t) = a + (b-1)D(t) - cI(t), \qquad 0 \le t \le t_1, \tag{1}$$

and

$$\frac{dI(t)}{dt} = -D(t), \qquad t_1 \le t \le t_1 + t_2. \tag{2}$$

Using the boundary condition: $I(t)=0$ for $t=0,T$, and after adjusting for the constant of integration, Eqs. (1) and (2) are clearly equivalent to the following equations

$$I(t) = \frac{a}{c}[1-e^{-ct}] + (b-1)e^{-ct}\int_0^t D(t)e^{ct}\,dt = \frac{a}{c}[1-e^{-ct}] + (b-1)f(t), \quad 0 \le t \le t_1, \tag{3}$$

and

$$I(t) = \int_t^{t_1+t_2} D(t)\,dt = g(t), \qquad t_1 \le t \le t_1 + t_2. \tag{4}$$

Again $I(t_1) = I_m$; thus, t_1 and t_2 are related by the equation

$$I_m = \frac{a}{c}[1-e^{-ct_1}] + (b-1)f(t_1) = g(t_2). \tag{5}$$

We derive the number of inventory for the period $[0, T]$ as

$$\int_0^{t_1} I(t)dt + \int_{t_1}^{t_1+t_2} I(t)dt = \int_0^{t_1} \left\{ \frac{a}{c}[1-e^{-ct}] + (b-1)f(t) \right\} dt + \int_{t_1}^{t_1+t_2} g(t)dt \,. \tag{6}$$

Hence, the total average cost of the inventory system is

$$K = \text{setup cost} + \text{holding cost}$$
$$= \frac{C_s}{T} + \frac{C_i}{T} \left\{ \frac{at_1 - 1}{c} + \frac{ae^{ct_1}}{c^2} + (b-1)F(t_1) + G(t_2) \right\}. \tag{7}$$

4. SOLUTION PROCEDURE

The above cost function K is a function of two variables t_1 and t_2. However, they are not independent and are related by Eq. (5). The problem is to determine the optimal value of t_1 that minimizes the total average cost K. We take the first and second derivatives of K with respect to t_1 as follows:

$$\frac{dK}{dt_1} = -\frac{C_s}{T^2}[1 + \frac{dt_2}{dt_1}] + \frac{C_i}{T} \left\{ \frac{a}{c}[1-e^{ct_1}] + (b-1)f(t_1) + g(t_2)\frac{dt_2}{dt_1} \right\}, \tag{8}$$

and

$$\frac{d^2K}{dt_1^2} > 0 \,.$$

Let $h(t_1) = \frac{dK}{dt_1}$, then h increases with respect to t_1, and t_1^* is the optimal value if and only if $h(t_1^*) = 0$. Since K is convex with respect to t_1, the Newton-Raphson method can be used to find the optimal value of t_1. However, it may not be easy for a practitioner with limited mathematical knowledge to understand the Newton-Raphson method. In this section, we shall present a simple algorithm to compute the optimal value of t_1. Before describing the algorithm, we need the following theorem.

Intermediate Value Theorem. *Let h be a continuous function on $[L, U]$, and let $h(L)h(U) < 0$. Then, there exits a number $d \in [L, U]$ such that $h(d) = 0$.*

Since $h(t)$ is strictly increasing, the following algorithm is based on the above theorem and the uniqueness of the root of Eq. (8). Now, we are in a position to outline the algorithm to determine the optimal value t_1^*. Note that $h(0) < 0$ and $h(t_U) > 0$. Hence, $0 < t_1^* < t_1$.

Step 1. Let $\varepsilon > 0$.

Step 2. Let $t_L = 0$ and $t_U = t_1$.

Step 3. Let $t = \dfrac{t_L + t_U}{2}$.

Step 4. If $|h(t)| < \varepsilon$, go to Step 6. Otherwise, go to Step 5.

Step 5. If $h(t) > 0$, set $t_U = t$. If $h(t) < 0$, set $t_L = t$. Then, go to Step 3.

Step 6. $t_1^* = t$ and exit the optimal value.

We obtain the optimal value of t_1 by using the Intermediate Value Theorem method. The optimal values of T, I_m and the minimum total average cost K can be obtained from Eqs. (5) and (7) respectively.

5. SOME CASES

The optimal solution procedure developed in the previous section is now illustrated with three market situations of growth, maturity and declining demand trend models. In the first model, the demand function is $D(t) = \alpha + \beta t$, where $\alpha, \beta > 0$. The second model presents a constant demand function $D(t) = \alpha$. Finally, the third model has the demand function $D(t) = \alpha e^{-\lambda t}$, where λ is a constant governing the decreasing rate.

5.1. Increasing demand pattern (growth) $D(t) = \alpha + \beta t$

Arguing as in Eqs. (5) and (7) above, the basic condition for this model becomes

$$t_2 = \frac{-(\alpha + \beta t_1) + \sqrt{(\alpha + \beta t_1)^2 + 2\beta I(t_1)}}{\beta}, \tag{9}$$

where $I(t_1) = \left[\dfrac{a + (b-1)\alpha}{c} - \dfrac{(b-1)\beta}{c^2}\right](1 - e^{-ct_1}) + \dfrac{(b-1)\beta t_1}{c}$,

and

$$K = \frac{C_s}{T} + \frac{C_i}{T}\left\{\left[\frac{a + (b-1)\alpha}{c} - \frac{(b-1)\beta}{c^2}\right]\left(t_1 + \frac{e^{-ct_1} - 1}{c}\right) + \frac{(b-1)\beta t_1^2}{2c} + \right.$$
$$\left. + \frac{\alpha t_2^2}{2} + \frac{\beta t_1 t_2^2}{2} + \frac{\beta t_2^3}{3}\right\}, \tag{10}$$

where $c \to 0$. We then obtain the same model as that given by Bhunia and Maiti's [2] second model, that is the model reduces to an inventory system where the production rate depends on demand.

$$\lim_{c\to 0} K = \frac{C_s}{T} + \frac{C_i}{T}\left\{\frac{[a+(b-1)\alpha]t_1^2}{2} + \frac{(b-1)\beta t_1^3}{6} + \frac{\alpha t_2^2}{2} + \frac{\beta t_1 t_2^2}{2} + \frac{\beta t_2^3}{3}\right\}. \quad (11)$$

If we assume $b = 0$, the model changes to an inventory system where the production rate depends on the on-hand inventory; the model is the same as Bhunia and Maiti's [2] first model.

$$K = \frac{C_s}{T} + \frac{C_i}{T}\left\{\left[\frac{a-\alpha}{c} + \frac{\beta}{c^2}\right]\left(t_1 + \frac{e^{-ct_1} - 1}{c}\right) - \frac{\beta t_1^2}{2c} + \frac{\alpha t_2^2}{2} + \frac{\beta t_1 t_2^2}{2} + \frac{\beta t_2^3}{3}\right\}. \quad (12)$$

5.2. Uniform demand pattern (maturity) $D(t) = \alpha$

In this case, Eqs. (5) and (7) reduce to

$$t_2 = \frac{a+(b-1)\alpha}{c\alpha}(1 - e^{-ct_1}). \quad (13)$$

and

$$K = \frac{C_s}{T} + \frac{C_i}{T}\left\{\frac{a+(b-1)\alpha}{c}\left(t_1 + \frac{e^{-ct_1} - 1}{c}\right) + \frac{\alpha t_2^2}{2}\right\}. \quad (14)$$

5.3. Decreasing demand pattern (decline) $D(t) = \alpha e^{-\lambda t}$

Substituting the expression of this type of demand into Eqs. (5) and (7), we can obtain

$$t_2 = -\frac{1}{\lambda}\ln\left[1 - \frac{\lambda a}{c\alpha}(1 - e^{-ct_1}) + \frac{(b-1)\lambda}{\lambda - c}(e^{-\lambda t_1} - e^{-ct_1})\right], \quad (15)$$

and

$$K = \frac{C_s}{T} + \frac{C_i}{T}\left\{\frac{a}{c}\left(t_1 + \frac{e^{-ct_1} - 1}{c}\right) - \frac{(b-1)\alpha}{\lambda - c}\left[\frac{e^{-ct_1} - 1}{c} - \frac{e^{-\lambda t_1} - 1}{\lambda}\right] + \frac{\alpha}{\lambda}\left(\frac{1 - e^{-\lambda t_2}}{\lambda} - t_2 e^{-\lambda t_2}\right)\right\}. \quad (16)$$

In each of the three cases, if the increasing demand model's parameters b, c and β, the uniform demand model's parameters b and, c as well as the decreasing demand model's parameters b, c and λ are zero, the model reduces to an EOQ model. In this situation, the cost function becomes

$$K = \frac{C_s}{T} + \frac{C_i}{T}\left\{\frac{(a-\alpha)t_1^2}{2} + \frac{\alpha t_2^2}{2}\right\}. \tag{17}$$

6. A NUMERICAL EXAMPLE

To illustrate the results so far, we use the following example, which is adapted from the example of Bhunia and Maiti [2]. For this model, let $a = 200$ units/month, $b = 0.3$, $c = 0.3$, $C_s = \$100$ for each new cycle, $C_i = \$1$/unit/month, $\alpha = 100$ units, $\beta = 20$, $\lambda = 0.3$. The optimum values of t_1 and T, along with minimum total average cost per month and optimum values of I_m, are calculated for the model. Next, the values are compared using different situations, as shown in Table 1. This table reveals the following:

1. When market demand goes up, the optimal production scheduling period is 0.9734 month and the total average cost is \$110.32;
2. When market demand is mature, the optimal production scheduling period is 0.9175 month and the total average cost is \$104.27;
3. When market demand declines, the optimal production scheduling period is 0.6897 month and the total average cost is \$101.41; and
4. When market demand declines, the optimal production scheduling period t_1, cycle time T, the maximum inventory level I_m and the total average cost are relatively small compared to the above two market demand situations.

Table 1: Results of the numerical example

Demand	t_1	T	I_m	K
$D(t) = \alpha + \beta t$	0.9734	1.7862	103.71	110.32
$D(t) = \alpha$	0.9175	1.9604	104.28	104.27
$D(t) = \alpha e^{-\lambda t}$	0.6897	1.6756	85.35	101.41

7. SENSITIVITY ANALYSIS

With the above numerical example, the optimal values of t_1, T, I_m, and the total average inventory cost K for the fixed set $\phi = \{\alpha, \beta, a, b, c, \lambda\}$ of parametric values are denoted by t_1^0, T^0, I_m^0 and K^0, respectively. Therefore, $t_1^0 = 0.9734\,(0.6897)$, $T^0 = 1.7862\,(1.6756)$, $I_m^0 = 103.71\,(85.35)$, and $K^0 = 110.32\,(101.41)$. Now, when only one of the parameters in the set of parametric values changes by a fixed proportion and all other parameters remain unchanged, let t_1^*, T^*, I_m^* and K^* denote the corresponding optimal values, respectively. Then we calculate the following sensitivity measures for 30% changes on either side of the parameters.

S. P. P. = Sensitivity of the optimum production scheduling period

$$= \left(\frac{t_1^*}{t_1^0} - 1\right) \times 100;$$

S. P. T. = Sensitivity of the optimum production cycle time

$$= \left(\frac{T^*}{T^0} - 1\right) \times 100;$$

S. M. I. = Sensitivity of the maximum inventory level

$$= \left(\frac{I_m^*}{I_m^0} - 1\right) \times 100; \text{ and}$$

S. T. C. = Sensitivity of the optimum total average cost

$$= \left(\frac{K^*}{K^0} - 1\right) \times 100.$$

Table 2 summarizes these results. The increase in the parameter is indicated by the + sign and the decrease by the - sign attached to it. Based on the sensitivity analysis, we can infer the following:

1. The optimal production scheduling period t_1 is insensitive to changes in parameters λ and β, slightly sensitive to changes in b and c and highly sensitive to changes in α and a;
2. The optimal production cycle time T is insensitive to changes in parameters β and c, moderately sensitive to changes in b, λ and α and highly sensitive to changes in a;

3. The maximum inventory level I_m is insensitive to changes in parameters λ, c and β, slightly sensitive to changes in b and α and highly sensitive to changes in a;
4. The optimal total average cost K is insensitive to changes in parameter β, slightly sensitive to changes in b, α, λ and c and highly sensitive to changes in a; and
5. Increasing or decreasing parameter α decreases the minimum total average cost.

Table 2: Sensitivity analysis

Parameters	t_1^*	T^*	I_m^*	K^*	S.P.P.	S.P.T.	S.M.I.	S.T.C.
α+: 130	1.2542	1.8992	104.18	108.34	28.85	6.33	0.5	-1.8
α−: 70	0.7419	1.7611	96.85	107.24	-23.78	-1.41	-6.6	-2.8
β+: 26	0.9986	1.7634	103.95	111.55	2.59	-1.28	0.2	1.1
β−: 14	0.9517	1.8192	103.58	108.85	-2.23	1.85	-0.1	-1.3
a+: 260	0.6715	1.5898	112.60	121.03	-31.01	-11	8.57	9.7
a−: 140	2.0978	2.6650	83.74	86.96	115.51	49.2	-19.3	-21.2
b+: 0.39	0.9044	1.7391	105.53	112.50	-7.09	-2.64	1.8	2
b−: 0.21	1.0570	1.8448	101.64	107.85	8.59	3.28	-2	-2.2
c+: 0.39	1.0201	1.8222	103.00	109.44	4.8	2.02	-0.7	-2.6
c−: 0.21	0.9323	1.7549	104.37	111.14	-4.22	-1.75	0.63	0.7
λ+:0.39	0.6818	1.7233	85.59	99.29	-1.15	2.85	0	-2
λ−: 0.21	0.6986	1.6364	85.12	103.27	1.29	-2.34	0	2

8. CONCLUSIONS

This study considers a production inventory system with the assumption that the production rate at any instant depends on demand and the inventory level. Three demand patterns, i.e., growth, maturity and decline, arĕ addressed. The developed model can be easily solved by using the Intermediate Value Theorem method. The analysis and the results provided in this study can be useful in helping management to plan production and control inventory. A numerical example demonstrates that applying the proposed model can obtain the minimum total average cost subject to different market demands. The sensitivity of the solution to changes in the values of different parameters has been discussed. According to these results, the proposed model is highly sensitive with respect to parameters α and a, slightly sensitive to parameters b, λ and c, and insensitive to parameter β. A future study should incorporate more realistic assumptions into the proposed model, such as relaxing the terminal condition of zero inventory at the end of the production cycle.

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