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AN ANALYTICAL INVENTORY MODEL FOR EXPONENTIALLY DETERIORATING INVENTORY WITH RANDOM LEAD TIME UNDER RANDOM INPUT

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Abstract: In the classical EOQ model, it is tacitly assumed that the lead time is zero and the quantity received matches with the quantity requisitioned and there is no damage or deterioration of units in inventory. However, in practice, we have observed that due to a variety of reasons there is a lead time and the quantity received does not match the quantity ordered. In this paper, an attempt is made to develop an analytical EOQ model when units in inventory are subjected to deterioration, when there is significant lead time and the quantity received does not match the quantity ordered. The effect of various parameters on procurement quantity and average expected total cost of the inventory system is studied with the help of a hypothetical numerical illustration.

Keywords: Deterioration, random lead time, random input.

1. INTRODUCTION

Due to a variety of reasons, viz., machine breakdown, workers' strike, electricity failure, shortages of machines and raw material etc., it is found that the quantity received does not match the quantity requisitioned but may be a random variable depending on the quantity ordered. Silver (1976) has developed an EOQ model when the quantity received is uncertain. Kalro and Gohil (1982) have extended this result to allow shortages. Noori and Keller (1986) have developed a probabilistic model under random input. Shah and Shah (1992a) developed an EOQ model for deteriorating items under random input which was extended by Shah and Shah (1992b) for finite production rates. An order level lot size inventory model for deteriorating items under random supply was developed by Gor and Shah (1994). In practice, it is observed that there is significant lead time between the placement of an order and its realization into the inventory system. Various authors have tried to develop models taking into account fixed or random lead time. Kulscar (1979, 1980) presented an analytical inventory model with stochastic lead time which is more or less similar to that of Liberatore (1979). The most important difference is that, in this model, the distribution of lead time is taken in a finite interval, contrary to that in (1979). The model presented by Kulscar (1980) is the cost optimization version of the simplest member (Prekopa (1963), Ziermann (1972)). Analyzing this model, Kulscar (1980) succeeded in establishing an analytical way that, if some assumptions are weakened, leads to an optimal solution. The result obtained is compared with that in (1979), whose limits can be given to the optimal values of parameters. The structure of the limits is the same as in the model given by Gernscer (1973) which assumes continuous reviewing and stochastic demand. Kulscar's idea has been extended by Shah and Shah (1994) for deteriorating items without shortages.

In this paper, an attempt is made to develop an EOQ model for exponentially deteriorating items with stochastic lead time when the quantity received is uncertain. The effect of various parameters on reorder point, procurement quantity and average expected total cost is shown on a numerical illustration.

2. ASSUMPTIONS AND NOTATIONS

The model is based on the following assumptions:

1. The model deals with the stocking of single items.
2. The demand rate of R units per time unit is known and constant.
3. Shortages are allowed.
4. Lead time is a random variable with probability density function (p.d.f.) $g(l)$, $0 \le l \le \lambda$ and

$$\mu = \int_0^{\lambda} l g(l) dl \tag{1}$$

as the mean lead time where λ is maximum lead time.

5. The replenishment size Q is the decision variable. The order size is Q units per replenishment. However, the actual quantity received Y, is a normal random variable with

$$\begin{aligned} E(Y) &= bQ \\ V(Y) &= \sigma_0^2 + \sigma_1^2 Q^2 \end{aligned} \tag{2}$$

where $b > 0$ is the bias factor and σ_0^2 and σ_1^2 are known constants.

6. Costs taken into account are [in $]:

   C = unit cost

   i = inventory holding charges per unit per time unit

   $C_1 = C * i$ holding cost per unit quantity per time unit

   C_3 = ordering cost per order

   known and constant during the period under consideration.

7. Let Q denote lot size

   S denote reorder point

   $T(Y,l)$ denote cycle time depending on Y, the quantity which is actually received.

8. Units in inventory are subjected to deterioration at a constant rate θ. Deteriorated units cannot be repaired or replaced.

Though $T(Y,l)$ is the decision variable for the validity of the present model it is required that $T(Y,l) \geq \lambda$. As a cycle, we consider the time interval between two successive replacement of orders.

3. MATEMATICAL MODEL

Let $Z(t \mid Y,Q,l)$ denote on hand inventory at time t of a cycle, when lead time is l, $0 \leq l \leq \lambda$, then

$$\frac{R}{\theta}\{e^{\theta(\lambda-t)}-1\}, \qquad 0 \leq t \leq l$$

$$Z(t \mid Y,Q,l) = Ye^{-\theta(t-1)} + \frac{R}{\theta}\{e^{\theta(\lambda-t)}-1\}, \qquad l \leq t \leq T \tag{3}$$

Under the condition that the order placed at the beginning of the cycle time may arrive in the system at any time during $[0,\lambda]$, we find the following:

1) The order level S must be sufficiently large so as to meet the demand and deterioration, even when lead time l has maximum value λ. This suggests that the reorder point must be

$$S = \frac{R}{\theta}\{e^{\theta\lambda}-1\}.$$

2) A cycle should also end with on hand inventory of S units. The lot size q must be sufficiently large so that even if $l=0$ (i.e. even if the ordered quantity arrives at the beginning of the cycle), we should have

$$Z(t\mid Y,Q,l)=S\ ,$$

which implies

$$T(Y,l)=\frac{1}{\theta}\log\left(1+\frac{\theta Y}{R}e^{\theta(l-\lambda)}\right) \tag{4}$$

Now on hand inventory per time unit is

$$I_1(T,l)=\int_0^{T(Y)} Z(t\mid Y,Q,l)dt= \tag{5}$$

$$=\frac{R}{\theta^2}\{e^{\theta\lambda}-e^{\theta(\lambda-T(Y))}-\theta T(Y)\}+\frac{Y}{\theta}\{1-e^{-\theta(T(Y)-l)}\}$$

and the number of units that deteriorate during the cycle for given lead time l is given by

$$D(Y,l)=Y-RT(Y,l) \tag{6}$$

Thus, average expected cycle time is

$$T(Y)=\frac{1}{\theta}\int_0^{\lambda}\log\left(1+\frac{\theta Y}{R}e^{\theta(l-\lambda)}\right)g(l)dl\ , \tag{7}$$

average expected inventory is

$$I_1(Y)=\frac{R}{\theta^2}\int_0^{\lambda}\{e^{\theta\lambda}-e^{\theta(\lambda-T(Y))}-\theta T(Y)\}\,g(l)dl+\frac{Y}{\theta}\int_0^{\lambda}\{1-e^{-\theta(T(Y)-l)}\}\,g(l)dl \tag{8}$$

and average expected units that deteriorate is

$$D(Y)=Y-R\int_0^{\lambda}T(Y,l)g(l)dl \tag{9}$$

Using Eqs. (8) and (9), we have the total cost of the system

$$K_1(Y)=CD(Y)+C_1I_1(Y)+C_3 \tag{10}$$

Assuming, Y to be normally distributed with Eq. (2), we have

$$E(Y^2)=\sigma_0^2+(\sigma_1^2+b^2)Q^2 \tag{11}$$

and $E(Y-bQ)^3=0$, which gives

$$E(Y^3)=3bQ(\sigma_0^2+\sigma_1^2Q^2)+b^3Q^3 \tag{12}$$

As it is not easy to obtain the expectations of various terms involved in Eq. (10), we use a series approximation of the terms involved in $T(Y)$, and hence, $I_1(Y)$ and $D(Y)$, under the assumption that θ is very small. Neglecting terms of order θ^2, we have Eqs. (7), (8), and (9) as

$$T(Y)=\frac{Y}{R}(1+\theta(\lambda-\mu))-\frac{\theta Y^2}{2R^2} \tag{13}$$

$$I_1(Y)=\frac{Y^2}{2R}-\frac{\theta Y^3}{3R^2}+Y(\lambda-\mu)+\frac{\theta Y^2}{R}(\mu-\lambda)+\frac{\theta Y(\lambda^2-\nu)}{2} \tag{14}$$

and

$$D(Y)=\frac{\theta Y^2}{2R}-\theta(\mu-\lambda)Y \tag{15}$$

Then, total expected cost $K(Q)$ per time unit is given by

$$K(Q)=\frac{CE(D(Y))+C_1E(I_1(Y))+C_3}{E(T(Y))} \tag{16}$$

Using Eqs. (2), (11) and (12), the expected value of units that deteriorate $E(D(Y))$, the expected value of inventory in the system $E(I_1(Y))$, and the expected value of cycle time $E(T(Y))$ are as follows:

$$E(D(Y))=\frac{\theta}{2R}\{\sigma_0^2+(\sigma_1^2+b^2)Q^2\}-\theta(\mu-\lambda)bQ \tag{17}$$

$$\begin{aligned}E(I_1(Y))=&\frac{1}{2R}\{\sigma_0^2+(\sigma_1^2+b^2)Q^2\}+\\&+\frac{Q}{R}\{\sigma_0^2+(\sigma_1^2+b^2)Q^2\}(\mu-\lambda)-\\&-\frac{\theta}{3R^2}\{3bQ(\sigma_0^2+\sigma_1^2Q^2)+b^3Q^3\}-bQ\{\lambda-\mu+\frac{\theta}{2}(\lambda^2-\nu)\}\end{aligned} \tag{18}$$

where

$$\nu=\int_0^{\lambda} l^2 g(l)dl \tag{19}$$

and

$$E(T(Y)) = \frac{bQ}{R}X - \frac{\theta}{2R^2}\{\sigma_0^2 + (\sigma_1^2 + b^2)Q^2\} \tag{20}$$

where

$$X = 1 + \theta(\mu - \lambda) \tag{21}$$

$$A = \sigma_1^2 + b^2 \tag{22}$$

hence,

$$E(T(Y))^{-1} = \frac{R}{bQX} + \frac{\theta}{2b^2Q^2X^2}\{\sigma_0^2 + (\sigma_1^2 + b^2)Q^2\} \tag{23}$$

respectively. Substituting values from Eqs. (17), (18) and (23), in Eq. (16), the average expected total cost $K(Q)$ per time unit is given by

$$K(Q) = \frac{E_1}{Q^2} + \frac{E_2}{Q} + E_3Q + E_4Q^2 + E_5 \tag{24}$$

where

$$E_1 = \frac{\theta\sigma_0^2}{2b^2X^2}\left\{\frac{C_1\sigma_0^2}{2R} + C_3\right\} \tag{25}$$

$$E_2 = \frac{1}{bX}\left\{\sigma_0^2\left[\frac{C_1}{2} + \theta\left[\frac{C}{2} + C_1(\lambda - \mu)(1 + 1/2X)\right]\right] + C_3R\right\} \tag{26}$$

$$E_3 = \frac{A}{bX}\left\{\frac{C_1 + C\theta}{2} + C_1\theta(\mu - \lambda)(1 + 1/2X)\right\} \tag{27}$$

$$E_4 = \frac{C_1\theta}{RX}\left\{\frac{A}{4Xb^2}(\sigma_0^2 + b^2) - (\sigma_1^2 + b^2/3)\right\} \tag{28}$$

and

$$E_5 = \frac{R}{X}\left\{(C_1 + C\theta)(\lambda - \mu) + \frac{C_1\theta(\lambda^2 - \nu)}{2}\right\} - \frac{C_1\theta\sigma_0^2}{RX}\left\{1 - \frac{\sigma_1^2}{4b^2X}\right\} + \frac{A}{2b^2X^2}\left\{\frac{C_1\sigma_0^2}{2R} + C_3\theta\right\} \tag{29}$$

with

$$X = 1 + \theta(\mu - \lambda)$$

$$A = \sigma_1^2 + b^2$$

$$\nu = \int_0^{\lambda} l^2 g(l) dl$$

For optimum value of $Q = Q_0$, $dK(Q)/dQ = 0$, i.e. it is the solution of

$$2E_4 Q^4 + E_3 Q^3 - E_2 Q - 2E_1 = 0 \tag{30}$$

where constants E_1, E_2, E_3 and E_4 are defined in Eqs. (25) - (29). The solution of Eq. (30) can be obtained by Newton Raphson's method taking the initial iterate as Ross's (1970) formula

$$Q = Q_0 = \sqrt{\frac{2C_3 R + C_1 \sigma_0^2}{C_1 (b^2 + \sigma_1^2)}}. \tag{31}$$

For average expected cost to be minimum, we have

$$\frac{d^2 K(Q_0)}{dQ_0^2} = \frac{6E_1}{Q_0^4} + \frac{2E_2}{Q_0^3} + 2E_4 > 0 \tag{32}$$

4. PARTICULAR CASES

Let us study the following two cases:

Case I: Keeping variance fixed (equivalently, $\sigma_1 = 0$) and $\sigma_0 > 0$. Then average expected total cost can be computed from Eq. (24) by putting $\sigma_1 = 0$.

Case II: When the variance is directly proportional to the quantity ordered, that is, $\sigma_0 = 0$ and $\sigma_1 > 0$. The average expected total cost can be computed from Eq. (24) by putting $\sigma_0 = 0$.

Consider the probability density function,

$$g(l) = \begin{cases} 1/\lambda, & 0 \le l \le \lambda \\ 0, & \text{otherwise} \end{cases} \tag{33}$$

then,

$$\mu = \lambda / 2 \tag{34}$$

and

$$\nu = \lambda^2 / 3 \tag{35}$$

Using Eqs. (34) and (35), the average expected total cost of an inventory system (Eq. (24)) is given by

$$K(Q) = \frac{\theta\sigma_0^2}{2b^2X^2Q^2}\left\{\frac{C_1\sigma_0^2}{2R} + C_3\right\} +$$

$$+\frac{1}{bXQ}\left\{\sigma_0^2\left[\frac{C_1}{2} + \theta\left[\frac{C}{2} + C_1\lambda / 2(1 + 1/2X)\right]\right] + C_3R\right\} +$$

$$+\frac{A}{bX}Q\left\{\frac{C_1 + C\theta}{2} - C_1\theta\lambda / 2(1 + 1/2X)\right\} +$$

$$+\frac{C_1\theta}{RX}\left\{\frac{A}{4Xb^2}(\sigma_0^2 + b^2) - (\sigma_1^2 + b^2/3)\right\}Q^2 +$$

$$+\frac{R}{X}\left\{(C_1 + C\theta)(\lambda - \mu) + \frac{C_1\theta\lambda^2}{3}\right\} - \frac{C_1\theta\sigma_0^2}{RX}\left\{1 - \frac{\sigma_1^2}{4b^2X}\right\}$$

with

$$X = 1 - \theta\lambda / 2 \tag{53}$$

5. STATISTICAL ANALYSIS BASED ON A HYPOTETHICAL PROBLEM

Consider an inventory system with the following parameters:

Unit cost C = \$ 10.00 per unit.

Inventory holding charges i = \$ 0.10 per year.

Inventory holding cost C_1 = \$ 1.00 per unit per annum.

Demand rate R = 2500 units per year.

Replenishment cost C_3 = \$ 100.00 per order.

5.1. Construction of tables

We construct the following tables:

Table 1: Relation between θ and b

$\lambda = 0.0444 \quad \sigma_1^2 = 1.00 \quad \sigma_0^2 = 5.00$

θ \ b		0.75	0.80	0.85
0.10	S	111.25	111.25	111.25
	Q	400.99	391.58	382.23
	K	36736.16	33960.63	31658.23
0.15	S	111.37	111.37	111.37
	Q	358.80	350.39	342.03
	K	54614.90	50463.37	47020.33
0.20	S	111.49	111.49	111.49
	Q	327.62	319.94	312.31
	K	72593.34	67057.41	62466.80

Table 2: Relation between σ_0^2 and σ_1^2

$\theta = 0.1 \quad b = 0.75 \quad \lambda = 0.0444 \quad S = 111.25$

σ_0^2 \ σ		0.10	0.20	0.30
5.00	Q	616.9280	575.2496	540.8909
	K	16017.66	18333.77	20645.02
10.00	Q	616.9343	575.2555	540.8965
	K	16018.68	18335.03	20646.52
15.00	Q	616.9407	575.2615	540.9021
	K	16019.69	18336.29	20648.03

Table 3: Relation between θ and σ_1^2

$\sigma_0^2 = 0.0 \quad b = 0.75 \quad \lambda = 0.0444$

σ_1^2	θ	0.01	0.02	0.03
0.10	S	111.02	111.05	111.07
	Q	829.09	794.37	763.65
	K	2338.82	3855.12	5371.99
0.15	S	111.02	111.05	111.07
	Q	799.51	766.07	736.47
	K	2479.79	4108.73	5738.33
0.20	S	111.02	111.02	111.07
	Q	772.88	740.56	711.96
	K	2619.74	4361.27	6103.57

Table 4: Relation between θ and σ_0^2

$\sigma_1^2 = 0.0 \quad b = 0.75 \quad \lambda = 0.0444$

σ_0^2	θ	0.01	0.02	0.03
5.00	S	111.02	111.05	111.07
	Q	899.5884	861.7766	828.3442
	K	2056.94	3347.71	4638.85
10.00	S	111.02	111.05	111.07
	Q	899.5834	861.7713	828.3387
	K	2055.74	3346.56	4637.75
15.00	S	111.02	111.02	111.07
	Q	899.5785	861.7661	828.3333
	K	2054.53	3345.40	4636.64

Table 5.1: Relation between σ_0^2 and σ_1^2

$\theta = 0.1 \quad b = 1.0 \quad \lambda = 0.0444 \quad S = 111.02$

σ_0^2	σ_1^2	0.10	0.20	0.30
5.00	Q	643.3813	616.0463	591.9085
	K	2215.87	2375.62	2533.97
10.00	Q	643.3845	616.0497	591.9116
	K	2217.21	2377.08	2535.55
15.00	Q	643.3881	616.0531	591.9150
	K	2218.54	2378.54	2537.14

Table 5.2: Relation between σ_0^2 and σ_1^2

$\theta = 0.02 \quad b = 1.0 \quad \lambda = 0.0666 \quad S = 166.61$

σ_0^2	σ_1^2	0.10	0.20	0.30
5.00	Q	616.5785	590.4233	567.3115
	K	3668.73	3955.39	4240.60
10.00	Q	616.5822	590.4270	567.3150
	K	3670.02	3956.81	4242.15
15.00	Q	616.5860	590.4305	567.3184
	K	3671.30	3958.22	4243.69

Table 5.3: Relation between σ_0^2 and σ_1^2

$\theta = 0.03 \quad b = 1.0 \quad \lambda = 0.0888 \quad S = 222.30$

σ_0^2	σ_1^2	0.10	0.20	0.30
5.00	Q	593.0026	567.8780	545.6652
	K	5132.40	5546.50	5959.11
10.00	Q	593.0065	567.8818	545.6688
	K	5133.64	5547.87	5960.61
15.00	Q	593.0105	567.8855	545.6724
	K	5134.88	5549.24	5962.10

5.2. Interpretations

1) In Table 1, we study the variations in the order level S, the optimum value of Q_0 and total optimum expected cost affected by changes in the values of b and θ. We find that as b increases, $Q = Q_0$ decreases and so does $K(Q_0)$. However, as θ increases, order level S decreases while the optimum value of economic order quantity $Q = Q_0$ and total expected cost $K(Q_0)$ increases. Even analytically, from Eq. (31) it can be seen that $\partial Q / \partial b < 0$, that is Q is a decreasing function of b.

2) From Table 2, we derive the relation between σ_0^2 and σ_1^2 keeping b and θ and lead time constant. It can be observed that as σ_1^2 increases, the optimum order quantity $Q = Q_0$ decreases but $K(Q_0)$ increases, whereas an increase in σ_0^2 results in an increase in both the optimum value of $Q = Q_0$ and total expected cost at optimum Q_0. From Eq. (31) we have Q to be a decreasing function of σ_1, which we observed in the numerical illustration.

3) In Table 3, the effects on model due to changes in θ and σ_1^2 are studied. Here, we find that both Q_0 and $K(Q_0)$ increase as θ increases while an increase in σ_1^2 results in a decrease in the optimum order quantity $Q = Q_0$ and an increase in minimum total expected cost $K(Q_0)$.

4) In Table 4, we study the relation between θ and σ_0^2 taking $\sigma_1^2 = 0.0$ and lead time and bias factor to be constant. As θ increases, procurement quantity decreases while total cost increases, whereas increase in σ_0^2 results in a decrease in purchase units and average expected total cost.

5) In Tables 5.1, 5.2, 5.3, we study the effects of changes in σ_0^2 and σ_1^2 for different values of θ and lead time keeping b constant.

It can be seen that order level increases with an increase in deterioration and lead time. A comparative study of these three Tables suggests that as deterioration increases purchase quantity decreases while total average expected optimum cost of the inventory system increases.

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