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A CONTINUOUS CONDITIONAL GRADIENT METHOD*

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Abstract: In this paper we study the continuous conditional gradient method to solve convex minimization problems in Hilbert space. First, sufficient conditions for convergence are provided and the convergence rate is found for a minimization problem with a strong convex function. Then, the regularized method is considered for a minimization problem with inaccurate initial data. Regularization is based on the continuous conditional gradient method in conjunction with the penalty function method. The sufficient conditions for the convergence of the regularized method are presented, the regularizing operator is constructed, and a stopping rule for the continuous process is proposed.

Keywords: Continuous methods, conditional gradient, regularization.

1. INTRODUCTION

Observe the following minimization problem:

$$J(u) \to \inf, u \in U \tag{1.1}$$

where U is a closed bounded convex set in a real Hilbert space H, function J is continuously Fréchet differentiable and convex on U. For solving problem (1.1) we will observe the continuous conditional gradient method described by differential inclusion

$$\begin{aligned} &u'(t) + \beta(t)u(t) \in \beta(t)(\varepsilon(t) - Arg\min\{\langle J'(u(t)), z - u(t)\rangle : z \in U\}), \\ &t \geq 0, u(0) = u_0 \in U. \end{aligned} \tag{1.2}$$

where u_0 is the given initial point, $t \geq 0$, $\varepsilon(t) \geq 0$, $\beta(t) \geq 0$. Here $u'(t) = \dfrac{du}{dt}$, $J'(u)$ is the gradient of J at the point u, and

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$$\in -Arg\min\{f(u): u \in U\} = \{w \in U : f(w) \le \inf_{u \in U} f(u) + \in\} \tag{1.3}$$

The method (1.2), (1.3) is a continuous version of the well-known iterative conditional gradient method [10, p. 291]:

$$u_{k+1} - (1-\beta_k)u_k \in \beta_k(\varepsilon_k - Arg\min\{\langle J'(u_k), u - u_k\rangle : u \in U\})$$

with $\varepsilon_k \ge 0, k \ge 0, \lim_{k\to\infty} \varepsilon_k = 0, u_0 \in U$ is the given initial point.

We will assume the global existence of a solution to differential inclusion (1.2). Some results related to the existence of a solution can be obtained from the general results of the viability theory [4].

The continuous methods are of some interest for research because they give a large choice of numerical integration methods to solve the corresponding differential equations, and in that way they provide new iterative processes. In [3] one such approach to minimization was presented coupling approximation with the descent method. Continuous methods based on the gradient projection method, proximal point method, Newton method and linearization method were studied in [2], [7], [8] and [12]. All these results are based on studying the trajectories of the corresponding differential equations that describe the continuous method. Let us remark that a general approach to the investigation of the stabilization of trajectories of nonlinear evolution equations in Banach space was given in [1].

2. THE CONDITIONS FOR CONVERGENCE AND CONVERGENCE RATE

In this section we study the convergence of trajectories $u(t), t \ge 0$ of the system (1.2), (1.3). We will begin with the following lemma.

Lemma 2.1. *Let $U \subseteq H$ be a convex closed set in a real Hilbert space; then any trajectory $u(t), t \ge 0$ of the system (1.2), (1.3) is in the set U, i.e. $u(t) \in U, t \ge 0$.*

Proof: The solution to differential equation (1.2), (1.3) is given by

$$u(t) = \frac{u_0}{h(t)} + \frac{1}{h(t)} \int_0^t \beta(s)h(s)w(s)ds, t \ge 0$$

where

$$w(t) \in \varepsilon(t) - Arg\min\{\langle J'(u(t)), z - u(t)\rangle : z \in U\}, t \ge 0,$$

$$h(t) = \exp(\int_0^t \beta(s)ds), t \ge 0, \tag{2.1}$$

It is well known [11, p. 177, Corollary 1], that every closed convex set U in a Hilbert space H can be represented in the following way

$$U = \cap_{\alpha \in I} H_\alpha, H_\alpha = \{x \in H : \langle c_\alpha, x \rangle \le d_\alpha\}, \alpha \in I$$

with $c_\alpha \in H, d_\alpha \in R, \alpha \in I$; I is the set of index α. Scalar products of the elements $u(t)$ and c_α give

$$\langle c_\alpha, u(t) \rangle = \frac{1}{h(t)} \langle c_\alpha, u_0 \rangle + \frac{1}{h(t)} \int_0^t \beta(s) h(s) \langle c_\alpha, w(s) \rangle ds, t \ge 0, \alpha \in I.$$

Since $u_0 \in U, w(s) \in U, s \ge 0, h'(s) = \beta(s)h(s)$, we conclude that $\langle c_\alpha, u(t) \rangle \le d_\alpha$, for all $t \ge 0, \alpha \in I$. Therefore $u(t) \in U$ for every $t \ge 0$. This completes the proof of Lemma 2.1.

In the following theorem we establish some sufficient conditions for the convergence of the method (1.2), (1.3).

Theorem 2.1. *Let the following conditions be satisfied:*

a) U is a closed bounded convex set in a Hilbert space H, function J is continuously differentiable and convex on U;

b) the parameters $\beta(t), \varepsilon(t)$ of method (1.2), (1.3) satisfy

$$\begin{aligned} &\varepsilon(t), \beta(t) \in C[0, +\infty), \varepsilon(t) \ge 0, \beta(t) > 0, t \ge 0, \\ &\lim_{t \to \infty} \varepsilon(t) = 0, \int_0^{+\infty} \beta(s) ds = +\infty. \end{aligned} \tag{2.2}$$

Then, any trajectory of the system (1.2), (1.3) has the properties

$$\lim_{t \to \infty} J(u(t)) = J_*, \lim_{t \to \infty} \rho(u(t), U_*) = 0,$$

where

$$J_* = \inf_{u \in U} J(u), U_* = \{u \in U : J(u) = J_*\}, \rho(u, U_*) = \inf_{u \in U_*} \| u - v \|.$$

Proof: The given assumptions imply that $J_* > -\infty$ and $U_* \neq \emptyset$. Let us consider the function

$$a(t) = J(u(t)) - J_*. \tag{2.3}$$

According to Lemma 2.1, $u(t) \in U$, so $a(t) \ge 0, t \ge 0$. Differentiating and using the relations (1.2), (1.3) lead to inequality

$$a'(t) = \langle J'(u(t), u'(t)) \rangle \le \beta(t) \langle J'(u(t)), u_* - u(t) \rangle + \beta(t)\varepsilon(t), t \ge 0, \tag{2.4}$$

where $u_* \in U_*$. Taking into account the convexity of function J and (2.3), we get

$$\langle J'(u(t)), u_* - u(t)\rangle \le J(u_*) - J(u(t)) = -a(t), t \ge 0.$$

Combining this inequality with (2.4), we obtain

$$a'(t) + \beta(t)a(t) \le \beta(t)\varepsilon(t), t \ge 0, a(0) = a_0, \tag{2.5}$$

where $a(0) = J(u_0) - J_*$.

Note that function h defined in (2.1) satisfies the equation $h'(t) = \beta(t)h(t)$. Now, multiplying the inequality (2.5) by $h(t)$, and integrating on the segment $[0, t]$ we are led to the inequality

$$a(t) \le \frac{a_0}{h(t)} + \frac{1}{h(t)}\int_0^t \beta(s)h(s)\varepsilon(s)ds, t \ge 0. \tag{2.6}$$

From (2.1) and (2.2), it is obvious that $\lim_{t\to\infty} h(t) = \infty$. If $\int_0^\infty \beta(s)h(s)\varepsilon(s)ds < +\infty$, then $\lim_{t\to\infty} a(t) = 0$. Otherwise, according to Lemma 2.1, the limit value of the second term on the right-hand side in (2.6) can be obtained using the L' Hospital rule. So, we obtain

$$\limsup_{t\to\infty} a(t) \le \lim_{t\to\infty} \frac{\beta(t)\varepsilon(t)h(t)}{\beta(t)h(t)} = \lim_{t\to\infty} \varepsilon(t) = 0.$$

This and (2.3) give $\lim_{t\to\infty} J(u(t)) = J_*$. Suppose $\limsup_{t\to\infty} \rho(u(t), U_*) = a$. Since U is a weak compact set in H, there exists a point $u_\infty \in U$ and a sequence $\{u(t_i), i \in N\} \subseteq \{u(t) : t \ge 0\} \subseteq U$ such that

$$\limsup_{t\to\infty} \rho(u(t), U_*) = \lim_{i\to\infty} \rho(u(t_i), U_*), \lim_{i\to\infty} \| (u(t_i) - u_\infty \| = 0.$$

The functions J and $\rho = \rho(u, U_*)$ are weak lower semicontinuous, therefore

$$\lim_{i\to\infty} J(u(t_i)) = J(u_\infty), \lim_{i\to\infty} \rho(u(t_i), U_*) = \rho(u_\infty, U_*) = a.$$

Since $\{u(t_i), i \ge 1\} \subseteq \{u(t), t \ge 0\}$ and $\lim_{t\to\infty} J(u(t)) = J_*$, it is easy to see that $J(u_\infty) = J_*$, i.e. $u_\infty \in U_*$. Consequently, $a=0$, i.e. $\limsup_{t\to\infty} \rho(u(t), U_*) = 0$. This and $\rho(u, U_*) \ge 0$ provide $\lim_{t\to\infty} \rho(u(t), U_*) = 0$. This proves Theorem 2.1.

Further, for a strongly convex function J we will derive an estimate of the convergence rate of the method (1.2), (1.3).

Definition 2.1. *A function* $J: U \to R$ *is strongly convex on a convex set* U *if there exists* $k > 0$ *such that*

$$J(\alpha x+(1-\alpha)y) \le \alpha J(x)+(1-\alpha)J(y)-k\alpha(1-\alpha)\| x-y \|^2,$$

for all $x, y \in U$, $0 \le \alpha \le 1$.

Corollary 2.1. *Let* $J: U \to R$ *be a strongly convex function on a convex set U and let all the other conditions of Theorem 2.1 be satisfied. Then*

$$\| u(t)-u_* \|^2 \le \frac{1}{kh(t)}[J(u_0)-J_*+\int_0^t \varepsilon(s)\beta(s)h(s)ds], t \ge 0. \quad (2.7)$$

where u_* *is the unique solution to the problem (1.1), and h is the function defined in (2.1).*

Proof: As it is known [10, p. 5, Theorem 8], under the given assumption the problem (1.1) has the unique solution u_*, and

$$k \| u-u_* \|^2 \le J(u)-J(u_*), u \in U$$

The estimate (2.7) follows from this inequality, (2.3) and (2.6). This ends the proof of Corollary 2.1.

3. REGULARIZATION

This section deals with problem (1.1) where objective function J and set U are known only approximately.

Consider the following problem:

$$J(u) \to \inf, u \in U \quad (3.1)$$

$$U=\{u \in U_0 : g_i(u) \le 0, i=1,\ldots,m, g_j(u)=0, j=m+1,\ldots,s\}, \quad (3.2)$$

where U_0 is the given closed bounded convex set of a real Hilbert space H, the functions J, g_i are defined and Fréchet differentiable on U_0. Suppose that

$$J_* = \inf_{u \in U} J(u) > -\infty, U_* = \{u \in U : J(u) = J(u_*)\} \ne 0. \quad (3.3)$$

It is well known [9], [11] that, generally speaking, problem (3.1), (3.2) is unstable with respect to perturbations of the initial data J and $g_i, i=1,\ldots,s$. Hence some methods of regularization must be applied. Below we describe and study a regularization based on the continuous conditional gradient method (1.2), (1.3), combined with a penalty function method which incorporates the constraints into the objective function. We will use the simplest penalty function

$$P(u)=\sum_{i=1}^{m}|\max\{g_i(u);0\}|^p+\sum_{i=m+1}^{s}|g_i(u)|^p,\ u\in U_0,\ p>1.$$

Under the given assumptions, the function P is Fréchet differentiable on U_0. Suppose that instead of the exact gradients $J'(u), P'(u)$, only their approximations $J'_u(u,t), P'_u(u,t)$ are known. Note that $J'(u)=\dfrac{dJ(u)}{du}, P'(u)=\dfrac{dP(u)}{du}$. Consider the following method:

$$u'(t)+\beta(t)u(t)\in\beta(t)(\varepsilon(t)-Arg\min\{\langle\widetilde{T}'_u(u(t),t),z-u(t)\rangle:z\in U_0\}) \tag{3.4}$$
$$u(0)=u_0\in U_0,$$

where $\varepsilon(t)\geq 0, \lim_{t\to\infty}\varepsilon(t)=0,$ and the function

$$\widetilde{T}'_u(u,t)=J'_u(u,t)+A(t)P'_u(u,t)+\alpha(t)u,\ u\in U_0,\ t\geq 0, \tag{3.5}$$

is an approximation to gradient

$$T'_u(u,t)=J'(u)+A(t)P'(u)+\alpha(t)u,\ u\in U_0,\ t\geq 0, \tag{3.6}$$

of the Tikhonov function

$$T(u,t)=J(u)+A(t)P(u)+\frac{1}{2}\alpha(t)\|u\|^2. \tag{3.7}$$

The functions $\alpha(t), A(t), \beta(t), \varepsilon(t)$ are parameters of the method (3.4), (3.5).

Note that the regularized continuous methods of the gradient type have been studied in [6],[7],[8], while the regularized iterative conditional gradient method was considered in [13].

We will need the following lemma to prove the main result of this section.

Lemma 3.1. *Suppose the functions* $\alpha,\beta\in C^1[0,+\infty)$ *are positive and*

$$\lim_{t\to\infty}\left[\frac{|\alpha'(t)|}{\alpha(t)\beta(t)}+\frac{|\beta'(t)|}{\beta^2(t)}\right]=0. \tag{3.8}$$

Then

$$\lim_{t\to\infty}h(t)=\lim_{t\to\infty}\beta(t)h(t)=\lim_{t\to\infty}\alpha(t)h(t)=\lim_{t\to\infty}\beta(t)\alpha(t)h(t)=+\infty. \tag{3.9}$$

Proof: Let $n\geq 3, k\geq 2$ be integers. From (3.8) it follows that there exists $t_0\geq 0$ such that

$$-\frac{\beta'(t)}{\beta^2(t)} \le \frac{1}{n}, \frac{\alpha'(t)}{\alpha(t)\beta(t)} \ge \frac{k-n}{n}, t \ge t_0. \tag{3.10}$$

Integrating the first inequality of (3.10), we have

$$\beta(t) \ge \frac{n}{t-t_0+C_0}, t \ge t_0, \; C_0 = \frac{n}{\beta(t_0)}. \tag{3.11}$$

Using (2.1) and (3.11) one can find

$$h(t) \ge \exp\left(\int_{t_0}^{t} \beta(s)ds\right) \ge \exp\left[n \ln \frac{t-t_0+C_0}{C_0}\right] = \left(\frac{t-t_0+C_0}{C_0}\right)^n, t \ge t_0. \tag{3.12}$$

This gives the first relation in (3.9). Combining (3.11) and the second inequality in (3.10), we get

$$\frac{\alpha'(t)}{\alpha(t)} \ge (k-n)\frac{1}{t-t_0+C_0}, t \ge t_0$$

After integration and simple calculation, it is easy to find

$$\alpha(t) \ge \alpha(t_0)\left(\frac{t-t_0+C_0}{C_0}\right)^{k-n}, t \ge t_0. \tag{3.13}$$

Multiplying $h(t)$ with $\beta(t), \alpha(t)$ and $\alpha(t)\beta(t)$, and using (3.11)-(3.13), we obtain

$$\beta(t)h(t) \ge \frac{n}{C_0^n} \cdot (t-t_0+C_0)^{n-1}, t \ge t_0,$$

$$\alpha(t)h(t) \ge \frac{\alpha(t_0)}{C_0^k} \cdot (t-t_0+C_0)^k, t \ge t_0,$$

$$\beta(t)\alpha(t)h(t) \ge \frac{n\alpha(t_0)}{C_0^k} \cdot (t-t_0+C_0)^{k-1}, t \ge t_0.$$

This implies the remaining relations in (3.9). The proof of Lemma 3.1 is completed.

Definition 3.1. *A point $u_* \in U_*$ is called the normal solution to problem (3.1), (3.2) if* $\| u_* \| = \inf\{\| u \| : u \in U_*\}$.

Theorem 3.1. *Assume that*

a) U_0 is a closed bounded convex set in a real Hilbert space H, the functions $J(u), g_i(u), i = 1,\dots,s$ are convex and continuously Fréchet differentiable on U_0, the functions $g_i(u), i = 1,\dots,s$ are bounded on U_0; Lagrange function

$$L(u,\lambda)=J(u)+\sum_{i=1}^{s}\lambda_i g_i(u),\ u\in U_0,\ \lambda\in\Lambda_0,$$

$$\Lambda_0=\{\lambda=(\lambda_1,...,\lambda_s)\in E^s:\lambda_1\geq 0,...,\lambda_m\geq 0\}$$

of the problem (3.1), (3.2) has a saddle point (z_*,λ^*), *that is*

$$L(z_*,\lambda)\leq L(z_*,\lambda^*)\leq L(z,\lambda^*),\ z\in U_0,\ \lambda\in\Lambda_0;$$

b) the approximations $J'_u(u,t), P'_u(u,t)$ *to gradients* $J'(u), P'(u)$ *are such that*

$$\max\{\|J'_u(u,t)-J'(u)\|,\|P'_u(u,t)-P'(u)\|\}\leq\delta(t),\ u\in U_0,\ t\geq 0;$$

c) the parameters $\alpha(t),\beta(t),\delta(t),\varepsilon(t),A(t)$ *of the method (3.4), (3.5) satisfy*

$$\alpha(t),A(t),\beta(t)\in C^1[0,+\infty),\ \delta(t),\varepsilon(t)\in C[0,+\infty),$$
$$\alpha(t)>0,\ \beta(t)>0,\ A(t)\geq 0,\ \delta(t)\geq 0,\ \varepsilon(t)\geq 0,\ t\geq 0, \tag{3.15}$$

$\alpha(t)$ *convex,* $A(t)$ *concave and* $\alpha'(t)\leq 0,\ A'(t)\geq 0$ *on* $[0,+\infty)$

$$\lim_{t\to\infty}[\varepsilon(t)+\delta(t)+\frac{1}{A(t)}]=0,\ \lim_{t\to\infty}\frac{\varepsilon(t)+A(t)\delta(t)}{\alpha(t)}=0, \tag{3.16}$$

$$\lim_{t\to\infty}\alpha(t)A^{\frac{1}{p-1}}(t)=0,\ \lim_{t\to\infty}\frac{|\alpha'(t)|+A'(t)}{\alpha(t)\beta(t)}=0,\ \lim_{t\to\infty}\frac{|\beta'(t)|}{\beta^2(t)}=0. \tag{3.17}$$

Then any solution $u(t), t\geq t_0$, *of the system (3.4), (3.5) converges as* $t\to\infty$ *to the normal solution* u_* *of (3.1), (3.2). Moreover, this convergence is uniform with respect to the choice of the approximations* $J'_u(u,t), P'_u(u,t)$.

Proof: First, note that under the given assumptions, conditions (3.3) are satisfied. It can be proved in the same way as in Lemma 2.1, that $u(t)\in U_0$ for $t\geq 0$. Since U_0 and $g_i, i=1,...,s$ are bounded, we can introduce

$$C=\max\{\sup_{u\in U_0}\|u\|,\ \sup_{u\in U_0}P(u)\}<+\infty. \tag{3.18}$$

Let

$$v(t)\in U_0,\ v(t)=\arg\min\{T(u,t):u\in U_0\},\ t\geq 0 \tag{3.19}$$

where $T(u,t)$ is the Tikhonov function defined in (3.7). Since function $u\to T(u,t)$ is strongly convex for every $t\geq 0$, the points $v(t)$, $t\geq 0$ are well defined [11, p. 232, Theorem 2], and

$$\lim_{t\to\infty}\|v(t)-u_*\|=0. \tag{3.20}$$

Further, we consider the following function:

$$a(t,\tau)=T(u(t),t)-T(v(\tau),\tau),\ t,\tau\ge 0. \tag{3.21}$$

Differentiating with respect to t and using (3.4), (3.7), (3.18) leads to the inequality

$$a_t'(t,\tau)=\langle T_u'(u(t),t),u'(t)\rangle+T_t'(u(t),t)\le$$
$$\beta(t)\langle T_u'(u(t),t),w(t)-u(t)\rangle+C(A'(t)+|\alpha'(t)|),\ t\ge 0,\tau\ge 0$$

where

$$w(t)\in\varepsilon(t)-Arg\min\{\langle\widetilde{T}_u'(u(t),t),z-u(t)\rangle:z\in U\},\ t\ge 0. \tag{3.22}$$

Adding and substracting $\langle\widetilde{T}_u'(u(t),t),w(t)-u(t)\rangle$ from the right-hand side of the last inequality, and using the relations (3.5), (3.14), (3.18), (3.19) and (3.22) we can obtain

$$a_t'(t,\tau)\le\beta(t)\|T_u'(u(t),t)-\widetilde{T}_u'(u(t),t)\|\cdot\|w(t)-u(t)\|+$$
$$\beta(t)\langle\widetilde{T}_u'(u(t),t),v(\tau)-u(t)\rangle+C(A'(t)+|\alpha'(t)|)\le$$
$$2C\beta(t)(1+A(t))\delta(t)+\beta(t)\varepsilon(t)+C(A'(t)+|\alpha'(t)|)+$$
$$\beta(t)\langle\widetilde{T}_u'(u(t),t),v(\tau)-u(t)\rangle,\qquad t\ge 0,\tau\ge 0.$$

Now, adding and subtracting $\langle\widetilde{T}_u'(u(t),t),v(\tau)-u(t)\rangle$ and taking into account (3.5), (3.6), (3.14), (3.18), we have

$$a_t'(t,\tau)\le 4C\beta(t)(1+A(t))\delta(t)+\beta(t)\varepsilon(t)+C(A'(t)+|\alpha'(t)|)+$$
$$\beta(t)\langle T_u'(u(t),t),v(\tau)-u(t)\rangle.\qquad t\ge 0,\tau\ge 0.$$

The relation (3.16) implies that there exists $t_0\ge 0$ such that $A(t)\ge 1$, for $t\ge t_0$. This, (3.21) and convexity of the function $u\to T(u,t)$ give

$$a_t'(t,\tau)\le C_1[\beta(t)(A(t)\delta(t)+\varepsilon(t))+A'(t)+|\alpha'(t)|]+$$
$$\beta(t)[T(v(\tau),t)-T(v(\tau),\tau)]-\beta(t)a(t,\tau),\qquad t,\tau\ge 0. \tag{3.23}$$

where $C_1=\max\{1,8C\}$.

From (3.7) and (3.18), it follows that

$$|T(v(\tau),t)-T(v(\tau),\tau)|\le|A(t)-A(\tau)|P(v(\tau))+|\alpha(\tau)-\alpha(t)|\frac{\|v(\tau)\|^2}{2}\le$$
$$\max\{C,\frac{C^2}{2}\}[|A(t)-A(\tau)|+|\alpha(t)-\alpha(\tau)|],\qquad t\ge 0,\tau\ge 0.$$

Combining this estimate with (3.23), we obtain the inequality

$$\alpha_t'(t,\tau)+\beta(t)a(t,\tau)\le C_2(f(t)+g(t,\tau)),\quad t\ge t_0,\tau\ge 0, \tag{3.24}$$

where $C_2=\max\{1,\frac{C^2}{2},C_1\}$,

$$f(t)=\beta(t)(A(t)\delta(t)+\varepsilon(t))+A'(t)+|\alpha'(t)|,\quad t\ge t_0 \tag{3.25}$$

$$g(t,\tau)=\beta(t)(|A(t)-A(\tau)|+|\alpha(t)-\alpha(\tau)|,\quad t\ge t_0,\tau\ge 0 \tag{3.26}$$

Multiplying (3.24) by $h(t)$ defined in (2.1), and integrating with respect to t over $[t_0,\tau]$ for $\tau\ge t_0$, we get

$$a(\tau,\tau)\le\frac{a(t_0,\tau)}{h(\tau)}+\frac{C_2}{h(\tau)}\int_0^\tau[f(s)+g(s,\tau)]h(s)ds,\quad \tau\ge t_0. \tag{3.27}$$

Since J and P are continuous, taking into account $A(\tau)\ge 1,\tau\ge t_0$, (3.18) and (3.20), (3.21), it is not difficult to show that

$$a(t_0,\tau)\le|T(u(t_0),t_0)|+C_3(A(\tau)+\alpha(\tau)),\quad \tau\ge t_0, \tag{3.28}$$

where $C_3==\max\{\frac{C^2}{2},C+\sup_{\tau\ge 0}|J(v(\tau))|\}$. Further, the strong convexity of the function $u\to T(u,t)$ and (3.21) with $t=\tau$, provide

$$a(\tau,\tau)\ge\frac{\alpha(\tau)}{2}\|u(\tau)-v(\tau)\|^2,\quad \tau\ge 0. \tag{3.29}$$

Finally, from (3.15), (3.26), taking into account the properties of $\alpha'(t)$ and $A'(t)$ we have

$$g(s,\tau)\le G(s,\tau)=\beta(s)(|A'(s)|+|\alpha'(s)|)(\tau-s),\quad s\in[0,\tau],\tau\ge 0. \tag{3.30}$$

Combining (3.28) - (3.30) with (3.27), we derive

$$\|u(\tau)-v(\tau)\|^2\le C_4\frac{|T(u(t_0,t_0))|+A(\tau)+\alpha(\tau)}{\alpha(\tau)h(\tau)}+$$
$$\frac{C_4}{h(\tau)\alpha(\tau)}\int_0^\tau(f(s)+G(s,\tau))h(s)ds,\quad \tau\ge t_0, \tag{3.31}$$

where $C_4=2\max\{1,C_2,C_3\}$. According to Lemma 3.1: $\lim_{\tau\to\infty}\alpha(\tau)h(\tau)=\infty$, $\lim_{\tau\to\infty}\alpha(\tau)\beta(\tau)h(\tau)=\infty$. Using L'Hospital's rule and the conditions (3.17), by simple calculations, we obtain

$$\lim_{\tau\to\infty}\frac{|T(u(t_0,t_0))|+A(\tau)+\alpha(\tau)}{\alpha(\tau)h(\tau)}=$$
$$\lim_{\tau\to\infty}\frac{A(\tau)}{\alpha(\tau)h(\tau)}=\lim_{\tau\to\infty}\frac{A'(\tau)}{(\alpha'(\tau)+\alpha(\tau)\beta(\tau))h(\tau)}= \tag{3.32}$$
$$\lim_{\tau\to\infty}\frac{A'(\tau)}{\alpha(\tau)\beta(\tau)}\cdot\lim_{\tau\to\infty}\frac{1}{h(\tau)}\cdot\lim_{\tau\to\infty}\frac{1}{1+\dfrac{\alpha'(\tau)}{\alpha(\tau)\beta(\tau)}}=0.$$

If $\lim_{\tau\to\infty}\int_0^\tau (f(s)+G(s,\tau))h(s)ds<+\infty$ then the second term on the right-hand side of (3.31) tends to 0 as $\tau\to\infty$. Otherwise, we can use L'Hospital's rule again. This together with $G(\tau,\tau)=0$, the conditions (3.16), (3.17) and the relations (3.25), (3.30) give

$$\lim_{\tau\to\infty}\frac{\int_0^\tau (f(s)+G(s,\tau))h(s)ds}{\alpha(\tau)h(\tau)}=$$
$$\lim_{\tau\to\infty}\frac{f(\tau)h(\tau)+\int_0^\tau G'_\tau(s,\tau)h(s)ds}{\alpha(\tau)\beta(\tau)h(\tau)}\cdot\lim_{\tau\to\infty}\frac{1}{1+\dfrac{\alpha'(\tau)}{\alpha(\tau)\beta(\tau)}}=\lim_{\tau\to\infty}\frac{\int_0^\tau G'_\tau(s,\tau)h(s)ds}{\alpha(\tau)\beta(\tau)h(\tau)}.$$

The last inequality in (3.10) shows that we can use L'Hospital's rule to calculate this limit. Hence

$$\lim_{\tau\to\infty}\frac{\int_0^\tau (f(s)+G(s,\tau))h(s)ds}{\alpha(\tau)h(\tau)}=$$
$$.\lim_{\tau\to\infty}\frac{G'_\tau(\tau,\tau)}{(\alpha(\tau)\beta(\tau))'+\alpha(\tau)\beta^2(\tau)}=\lim_{\tau\to\infty}\frac{G'_\tau(\tau,\tau)}{\alpha(\tau)\beta^2(\tau)}\cdot\frac{1}{\lim_{\tau\to\infty}(1+\dfrac{(\alpha(\tau)\beta(\tau))'}{\alpha(\tau)\beta^2(\tau)})}.$$

This equality and (3.17), imply

$$\lim_{\tau\to\infty}\frac{\int_0^\tau (f(s)+G(s,\tau))h(s)ds}{\alpha(\tau)h(\tau)}=0. \tag{3.33}$$

Relations (3.31) - (3.33) lead us to $\lim_{\tau\to\infty}\|u(\tau)-v(\tau)\|=0$. This and (3.20) give

$$\lim_{\tau\to\infty}\|u(\tau)-u_*\|=0. \tag{3.34}$$

The convergence in (3.20) does not depend on the choice of $J'_u(u,t), P'_u(u,t)$ [11, p. 232, Theorem 2]. From (3.27) it is obvious that the convergence $\lim_{\tau\to\infty} \| u(\tau)-v(\tau)\| = 0$ also does not depend on the choice of $J'_u(u,t), P'_u(u,t)$. Therefore, the convergence (3.34) is uniform with respect to the choice of approximations $J'_u(u,t), P'_u(u,t)$. This completes the proof.

4. STOPPING RULE

In Theorem 3.1 it was assumed that the value of gradients $J'(u), P'(u)$ at any fixed point $u \in U_0$ can be computed with any prescribed accuracy $\delta(t)$, $\lim_{\tau\to\infty}\delta(t)=0$ in the sense of (3.14). However, in practice the initial data are usually given with an error which remains less than some fixed positive number δ. In particular, for problem (3.1), (3.2), the following condition is more realistic than (3.14):

$$\max\{\| J'_\delta - J'(u)\|, \| P'_\delta - P'(u)\| \le \delta, u \in U_0, \tag{4.1}$$

where $\delta \ge 0$ is known. Then, we can attempt to solve problem (3.1), (3.2) by the method

$$\begin{aligned} &u'(t)+\beta(t)u(t) \in \beta(t)(\varepsilon(t) - Arg\min\{\langle \widetilde{T}'_\delta(u,t), z-u(t)\rangle : z \in U_0\}), \\ &t>0, u(0) \in U_0, \end{aligned} \tag{4.2}$$

obtained from (3.4), (3.5)' by replacing $\widetilde{T}'_u(u,t) = J'_u(u,t)+P'_u(u,t)+\alpha(t)u$ with $\widetilde{T}'_\delta(u,t) = J'_\delta(u)+P'_\delta(u)+\alpha(t)u$. However, it is easily seen that the conditions (3.16) for the compatibility of the parameters $\alpha(t), \beta(t), \varepsilon(t), A(t)$ with error parameter $\delta(t)=\delta>0, t\ge 0$, will be obviously destroyed, and so process (4.2) can diverge and its use for large t can become absurd. The question arises, to which reasonable moment $t_\delta = t(\delta)$ should the process (4.2) be continued so that the resulting point $u_\delta = u(t_\delta)$ can be taken as an approximation to u_*, corresponding to the given error level $\delta \ge 0$. It turns out that this question, of practical importance, can be answered on the basis of Theorem 3.1. For that purpose, we can fix any initial point $u_0 \in U_0$ and the functions $\delta(t), \alpha(t), \beta(t), \varepsilon(t), A(t)$ satisfying conditions c) of Theorem 3.1. It can be assumed that $\delta(0) > \delta$. Since, it is not assumed here that conditions (3.14) are satisfied, it must be emphasized that the function $\delta(t)$ now is a parameter of the process (4.2) in no way associated with conditions (3.14) or (4.1). For each fixed $\delta, 0<\delta<\delta(0)$, we will continue the process (4.2) up to the moment $t_\delta = t(\delta)$ for which

$$t_\delta = \sup\{t \ge 0 : \delta(s) \ge \delta, s \in [0,t]\}. \tag{4.3}$$

Since $\lim_{t\to\infty}\delta(t)=0, \delta(0)\ge\delta$, such a moment t_δ will certainly be found. The rule (4.3) for terminating the process (4.2) is called the stopping rule and it can be justified on the basis of the following theorem.

Theorem 4.1. *Let the conditions of Theorem 3.1, apart from (3.14), be satisfied; the approximations* J'_δ, P'_δ *to gradients* J', P' *satisfy the condition (4.1). Let* $u(t), 0 \le t \le t_\delta$ *be obtained by the method (4.2), where the moment* t_δ *is determined in accordance with stopping rule (4.3). Then*

$$\lim_{\delta \to 0} \| u(t_\delta - u_*) \| = 0 .$$

The proof of this theorem is exactly the same as the proof of similar Theorem 2 from [8].

It follows from Theorem 4.1 that the operator R_δ, which puts each set $(J'_\delta, P'_\delta, \delta)$ of (4.1) in correspondence with the point $u_\delta = u(t_\delta)$ determined by the method (4.2), (4.3), is a regularizing operator [6]. An interesting, but as yet uninvestigated problem that arises is how to choose the functions $A(t), \alpha(t), \beta(t), \varepsilon(t), \delta(t)$ which will give an operator R_δ that is optimal in some sense.

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