Yugoslav Journal of Operations Research
9 (1999), Number 2, 157-168

SEMIDEFINITE RELAXATIONS OF THE TRAVELING SALESMAN PROBLEM

Dragoš CVETKOVIĆ,
University of Belgrade, Faculty of Electrical Engineering
Bulevar Revolucije 73, 11000 Belgrade, Yugoslavia

Mirjana ČANGALOVIĆ, Vera KOVAČEVIĆ-VUJČIĆ,
University of Belgrade, Faculty of Organizational Sciences
Jove Ilića 154, 11040 Belgrade, Yugoslavia

Abstract: We apply semidefinite programming to the symmetric traveling salesman problem (TSP). The TSP is modeled as a problem of discrete semidefinite programming. In order to estimate the optimal objective value, a class of semidefinite relaxations of the TSP model is defined. Experimental results with randomly generated examples show that the proposed relaxation gives considerably better bounds than 2-matching and 1-tree relaxations.

Keywords: Semidefinite programming, combinatorial optimization, traveling salesman problem.

1. INTRODUCTION

Semidefinite programming (SDP) has wide applications in different classes of optimization problems (see e.g. [12]). Especially, there is growing interest in the use of SDP in combinatorial optimization, where suitable semidefinite relaxations have been developed for a number of NP-hard problems. Some examples are the recently introduced semidefinite relaxation for the max-cut problem, the well-known semidefinite relaxation for the stable set problem, semidefinite relaxations for graph colouring problems, etc. (see [7], [11] for a survey). The purpose of this paper is to investigate the applications of SDP to the traveling salesman problem. Some preliminary results in this direction have been obtained in [3].

The symmetric traveling salesman problem (TSP) is one of the best known NP-hard combinatorial optimization problems. There is extensive literature on both the theoretical and practical aspects of the TSP. The theory includes algorithms and heuristics for solving the problem with the emphasis on complexity questions. A nice collection of papers summarizing the most important theoretical results related to the TSP can be found in [10] (see also [4], [9]). In this paper we develop a discrete semidefinite programming model of the TSP which is based on the notion of the Laplacian of a graph. A class of semidefinite relaxations for this model is defined and its efficiency is examined.

The paper is organized as follows: In Section 2 we briefly review the results of Fiedler [6] related to the algebraic connectivity of graphs and extensions to edge-weighted graphs. We show that algebraic connectivity can be characterized in terms of the positive semidefiniteness of a suitably chosen linear transformation of the Laplacian. Section 3 presents a discrete semidefinite programming TSP model and a class of its semidefinite relaxations which give lower bounds to the objective function value.

Section 4 summarizes experimental results with randomly generated examples which show that the proposed semidefinite relaxation gives considerably better bounds than 2-matching and 1-tree relaxations.

2. LAPLACIAN OF GRAPHS AND ALGEBRAIC CONNECTIVITY

Let $G=(V,E)$ be an undirected simple graph, where $V=\{1,\dots,n\}$ is the set of vertices and E is the set of edges. The Laplacian $L(G)$ of graph G is defined as $L(G)=D(G)-A(G)$, where $D(G)$ is the diagonal matrix with vertex degrees on the diagonal and $A(G)$ is the adjacency matrix of G. One may also describe $L(G)$ by means of its quadratic form

$$(L(G)x,x)=\sum_{\substack{\{i,j\}\in E\\ i<j}}(x_i-x_j)^2$$

where $x=(x_1,\dots,x_n)\in R^n$. The matrix $L(G)$ is symmetric and positive semidefinite. If $\lambda_1\le\dots\le\lambda_n$ are eigenvalues of $L(G)$ then $\lambda_1=0$ with the corresponding eigenvector $e=(1,\dots,1)$. All other eigenvalues have eigenvectors which belong to the set

$$S = \left\{ x = (x_1, \ldots, x_n) \in R^n \,\middle|\, \sum_{i=1}^{n} x_i = 0 \right\}.$$

The second smallest eigenvalue λ_2 of $L(G)$, according to M. Fiedler [6], is called the algebraic connectivity of G and denoted by $a(G)$. In [6] the following theorem is proved:

Theorem 1. *The algebraic connectivity* $a(G)$ *satisfies the following properties:*

(*i*) $a(G) \geq 0$. $a(G) > 0$ *if and only if* G *is connected.*

(*ii*) *If* $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ *and* $E_1 \subseteq E_2$ *then* $a(G_1) \leq a(G_2)$. ♦

The next theorem shows that the level of the algebraic connectivity of G can be characterized in terms of the positive semidefiniteness of a suitably chosen matrix:

Theorem 2. *Let* $L(G)$ *be the Laplacian of graph* G, *and let* α *and* β *be real parameters such that* $\beta > 0$, $n\alpha - \beta \geq 0$. *Then* $a(G) \geq \beta$ *if and only if the matrix* $X = L(G) + \alpha J - \beta I$ *is positive semidefinite, where* J *is the* $n \times n$ *matrix with all entries equal to one and* I *is the unit matrix of order* n.

Proof: Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L(G)$ and let $x^1 = e$ and $x^i \in S, i = 2, \ldots, n$ be the corresponding eigenvectors which form a basis for R^n. It is well-known that J has two eigenvalues: 0, with multiplicity $n-1$ and S as its eigenspace, and n with e as its eigenvector. Therefore

$$Xe = (L + \alpha J - \beta I)e = (\alpha n - \beta)e$$

$$Xx^i = (L + \alpha J - \beta I)x^i = (\lambda_i - \beta)x^i, \quad i = 2, \ldots, n$$

which means that $\alpha n - \beta$ and $\lambda_i - \beta, i = 2, \ldots, n$ are eigenvalues of X with eigenvectors $e, x^2, \ldots, x^n$, respectively.

If X is positive semidefinite then $\lambda_2 - \beta \geq 0$, i.e. $a(G) \geq \beta$. Suppose that $a(G) = \lambda_2 \geq \beta$. If $n\alpha \leq \lambda_2$ then the smallest eigenvalue of X is equal to $n\alpha - \beta$. As $n\alpha - \beta \geq 0$ it follows that X is positive semidefinite. In the case when $\lambda_2 < n\alpha$ the smallest eigenvalue of X is $\lambda_2 - \beta$ and from the assumption $\lambda_2 - \beta \geq 0$ it follows that X is positive semidefinite. ♦

The concept of the Laplacian and algebraic connectivity can be extended to graphs with positively weighted edges. A C-edge-weighted graph $G_C = (V,E,C)$ is defined by a graph $G = (V,E)$ and a symmetric nonnegative weight matrix C such that $c_{ij} > 0$ if and only if $\{i,j\} \in E$. Now the Laplacian $L(G_C)$ is defined as $L(G_C) = \mathrm{diag}(r_1,\dots,r_n) - C$, where r_i is the sum of the i-th row of C. Another way to describe $L(G_C)$ is the following:

$$(L(G_C)x,x) = \sum_{\substack{\{i,j\}\in E\\ i<j}} c_{ij}(x_i - x_j)^2$$

The Laplacian $L(G_C)$ has characteristics similar to those of $L(G)$. Namely it is symmetric, positive semidefinite with the smallest eigenvalue $\lambda_1 = 0$ and the corresponding eigenvector e. As before, the algebraic connectivity $a(G_C)$ is the second smallest eigenvalue of $L(G_C)$, which enjoys similar properties to those in Theorems 1 and 2.

Theorem 3 [6]. *The generalized algebraic connectivity $a(G_C)$ has the following properties: $a(G_C) \geq 0$, $a(G_C) > 0$ if and only if G_C is connected.* ♦

Theorem 4. *Let $L(G_C)$ be the Laplacian of weighted graph G_C and let α and β be real parameters such that $\beta > 0$, $n\alpha - \beta \geq 0$. Then $a(G_C) \geq \beta$ if and only if $L(G_C) + \alpha J - \beta I$ is positive semidefinite.*

Proof: If $\lambda_1 \leq \dots \leq \lambda_n$ are eigenvalues and $x^1,\dots,x^n$ are the corresponding eigenvectors of $L(G_C)$ we again have $\lambda_1 = 0$, $x^1 = e$, $x^2,\dots,x^n \in S$ and the proof follows the same lines as in Theorem 2. ♦

Further properties of the Laplacian matrices of graphs can be found in [8].

3. SEMIDEFINITE RELAXATIONS OF TSP

In this section we shall assume that $G = (V,E)$ is a complete undirected graph, where, as before, $V = \{1,\dots,n\}$ is the set of vertices and E is the set of edges. To each edge $\{i,j\} \in E$ a distance (cost) d_{ij} is associated such that the distance matrix $D = [d_{ij}]_{n \times n}$ is symmetric and $d_{ii} = 0$, $i = 1,\dots,n$.

Now the symmetric traveling salesman problem (TSP) can be formulated as the problem of finding the Hamiltonian circuit of G with minimal cost.

It is easy to see that a spanning subgraph H of G is a Hamiltonian circuit if and only if its Laplacian $L(H) = [l_{ij}]_{n \times n}$ satisfies the following conditions:

$$l_{ii} = 2, \ i = 1,\dots,n \tag{1}$$

$$\lambda_2 > 0 \tag{2}$$

where $\lambda_2 = a(H)$ is the second smallest eigenvalue of $L(H)$. Namely, if $L(H)$ satisfies (1) then H is a 2-matching, i.e. it is either a Hamiltonian circuit or a collection of at least two disjoint subcircuits meeting all of the vertices. According to (i) of Theorem 1, condition (2) guarantees that H is connected, which implies that H is a Hamiltonian circuit.

It is well-known in the theory of graph spectra (see [5]) that the Laplacian of a circuit with n vertices has the spectrum

$$2 - 2\cos(2\pi j / n), \ j = 1,\dots,n$$

The second smallest eigenvalue is obtained for $j = 1$ and $j = n-1$, i.e. $\lambda_2 = \lambda_3 = 2 - 2\cos(2\pi/n)$. In the sequel this value will be denoted by h_n, i.e. $h_n = 2 - 2\cos(2\pi/n)$.

The next theorem gives the basis for the discrete semidefinite programming model of the TSP.

Theorem 5. *Let H be a spanning subgraph of G such that $d(i) = 2, i = 1,\dots,n$, where $d(i)$ is the degree of vertex i with respect to H, and let $L(H) = [l_{ij}]_{n \times n}$ be the corresponding Laplacian. Let α and β be real parameters such that $\alpha > h_n / n$, $0 < \beta \le h_n$. Then H is a Hamiltonian circuit if and only if the matrix $X = L(H) + \alpha J - \beta I$ is positive semidefinite, where J is the $n \times n$ matrix with all entries equal to one and I is the unit matrix of order n.*

Proof: The conditions of Theorem 5 guarantee that H is a 2-matching. Also, from the given assumptions it follows that $n\alpha - \beta > 0$ and hence the conditions of Theorem 2 are satisfied. Suppose that H is a Hamiltonian circuit. Then $a(H) = h_n \ge \beta$ and, according to Theorem 2, the matrix X is positive semidefinite. Suppose now that X is positive semidefinite. By Theorem 2 it follows that $a(H) \ge \beta > 0$ and consequently (2) holds, which implies that H is a Hamiltonian circuit. ♦

Remark 1: The conditions of Theorem 5 guarantee that the smallest eigenvalue of X is always equal to $a(H)-\beta$.

It follows from Theorem 5 that a spanning subgraph H of G is a Hamiltonian circuit if and only if its Laplacian $L(H)$ satisfies condition (1) and

$$X = L(H) + \alpha J - \beta I \text{ is positive semidefinite, where } \alpha > h_n / n,\ 0 < \beta \le h_n \qquad (2')$$

Condition (2') forbids 2-matchings with subcircuits since $a(H)=0$ obviously implies that the smallest eigenvalue of X is negative. Starting from (1) and (2') the following discrete semidefinite programming model of the TSP can be defined:

$$\min\ F(X) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(-\frac{1}{2}d_{ij}\right)x_{ij} + \frac{\alpha}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij} \qquad (3)$$

subject to

$$x_{ii} = 2 + \alpha - \beta,\quad i = 1,\ldots,n \qquad (4)$$

$$\sum_{j=1}^{n} x_{ij} = n\alpha - \beta,\quad i = 1,\ldots,n \qquad (5)$$

$$x_{ij} \in \{\alpha - 1, \alpha\},\quad i,j = 1,\ldots,n,\ i < j \qquad (6)$$

$$X \ge 0 \qquad (7)$$

where $X \ge 0$ denotes that the matrix $X = [x_{ij}]_{n \times n}$ is symmetric and positive semidefinite and α and β are chosen according to Theorem 5. Matrix $L = X + \beta I - \alpha J$ represents the Laplacian of a Hamiltonian circuit if and only if X satisfies (4)-(7). Indeed, constraints (4)-(6) provide that L has the form of a Laplacian with diagonal entries equal to 2, while according to (2') condition (7) guarantees that L corresponds to the Hamiltonian circuit. Therefore, if X^* is the optimal solution of problem (3)-(7) then $L^* = X^* + \beta I - \alpha J$ is the Laplacian of the optimal Hamiltonian circuit of G with the objective function value $\sum_{i=1}^{n}\sum_{j=1}^{n}(-\frac{1}{2}d_{ij})l_{ij}^* = F(X^*)$.

A natural semidefinite relaxation of the traveling salesman problem is obtained when discrete conditions (6) are replaced by inequality conditions:

$$\min F(X) \tag{8}$$

subject to

$$x_{ii} = 2 + \alpha - \beta, \quad i = 1,\dots,n \tag{9}$$

$$\sum_{j=1}^{n} x_{ij} = n\alpha - \beta, \quad i = 1,\dots,n \tag{10}$$

$$\alpha - 1 \le x_{ij} \le \alpha, \quad i, j = 1,\dots,n, \; i < j \tag{11}$$

$$X \ge 0 \tag{12}$$

It is easy to see that the relaxation (8)-(12) can be expressed in the form of an SDP problem. Indeed, constraint (9) can be represented as $A_i \circ X = 2 + \alpha - \beta$, where A_i is a symmetric $n \times n$ matrix with 1 at the position (i,i) and all other entries are equal to 0. Similarly, condition (10) is equivalent to $B_i \circ X = 2(n\alpha - \beta)$, where B_i has 2 at the position (i,i) while all the remaining elements of the i-th row and the i-th column are equal to 1, and all the other entries are zero. Finally, condition (11) can be expressed as $2(\alpha - 1) \le C_{ij} \circ X \le 2\alpha$, where C_{ij} has 1 at the positions (i,j) and (j,i) and zero otherwise. Since SDP problem (8)-(12) depends on parameters α and β it represents a class of semidefinite relaxations of TSP. In the sequel, members of this class will be refered to as SDP relaxations.

Let us denote by D and D^0 the feasible set of problem (8)-(12) and its relative interior. For each $X \in D$ the corresponding Laplacian $L = X + \beta I - \alpha J$ can be interpreted as the Laplacian of the weighted graph $G_L = (V, E_L, C_L)$, where $E_L = \{\{i, j\} \in E \mid l_{ij} < 0\}$ and $C_L = 2I - L$. According to Theorem 4, if α and β satisfy the conditions of Theorem 5 then $X \ge 0$ is equivalent to $a(G_L) \ge \beta$. Hence, by Theorem 3 graph G_L is connected. It immediately follows that 2-matchings with disjoint subcircuits cannot correspond to any X in D.

It is easy to see that $D^0 \ne \emptyset$. Indeed, if e.g. $\hat{L} = (2 + \frac{2}{n-1})I - \frac{2}{n-1}J$ then $\hat{X} = \hat{L} + \alpha J - \beta I = (2 + \frac{2}{n-1} - \beta)I + (\alpha - \frac{2}{n-1})J$ has the eigenvalues $2 + \frac{2}{n-1} - \beta$ with the multiplicity $n-1$ and $n\alpha - \beta$ with the multiplicity 1. Since $n\alpha - \beta > 0$ and $2 + \frac{2}{n-1} - \beta \ge 2 + \frac{2}{n-1} - h_n > 0$ for $n \ge 4$, it follows that $\hat{X} \in D^0$, $n \ge 4$.

According to Remark 1 for $\beta < h_n$ matrices X which correspond to the Laplacians of Hamiltonian circuits are in D^0, while for $\beta = h_n$ these matrices belong to $D \setminus D^0$. It is clear that the best relaxation is obtained for $\beta = h_n$. For that reason in the numerical experiments reported in Section 4 parameter β is always chosen to be equal to h_n. Concerning the parameter α, it is always sufficient to choose $\alpha = 1$.

4. NUMERICAL EXPERIMENTS

The semidefinite relaxation proposed in this paper is substantially different from existing TSP relaxations. In this section we will compare it with the well-known 2-matching and 1-tree relaxations (see e.g. [2], [10]). It should first be noted that SDP relaxation (8)-(12) cannot be theoretically compared either with 2-matching or with 1-tree. Indeed, if we consider TSP model (3)-(7) it is easy to see that X which corresponds to the Laplacian of a 2-matching satisfies (4)-(6) but need not satisfy (7). In the case of a 1-tree, the condition (4) is relaxed, (5) and (6) hold trivially, while (7) holds due to the following argument: A 1-tree T represents a spanning subgraph of G which contains only one circuit C_m with m vertices, where $3 \leq m \leq n$. According to *(ii)* of Theorem 1, $a(T) \geq a(C_m)$. As $a(C_m) = h_m \geq h_n$ it follows that $a(T) \geq h_n \geq \beta$.

Preliminary numerical experiments with 55 randomly generated TSP models of dimension $10 \leq n \leq 20$ indicate that SDP relaxation (8)-(12) with $\beta = h_n$ and $\alpha = 1$ gives considerably better lower bounds than 1-tree and 2-matching relaxations. Namely, 1-tree was worse in all 55 cases, while 2-matching was worse in 23 cases and better only in one case. The experiments were performed on a 486 PC using CSDP 2.2 software package developed by B. Borchers [1]. The inequality conditions (11) were handled by adding $n^2 - n$ slack variables which increased the dimensions of the unknown matrix in SDP relaxation to $n^2 \times n^2$. This was the main reason for limiting the dimensions of TSP test examples to 20. It should be pointed out that CSDP 2.2 software package showed very good perfomance and it never needed more than 30 iterations in order to get the solution accurate to 6-8 significant digits.

Table 1 contains details of the numerical experiments. Column 1 indicates the dimension of the problem, column 2 contains the ordinal numbers of randomly generated problems for the given dimension, columns 3, 4 and 5 contain the optimal objective function value of 2-matching, 1-tree and SDP relaxations, respectively. An asterisk denotes that the function value corresponds to the integral Laplacian matrix, i.e. the optimal solution of the TSP is obtained.

Table 1.

Dimension of TSP	Problem No.	2-matching	1-tree	SDP
10	1	1681*	1275	1680.9950*
	2	2778*	2185	2777.9920*
	3	1594	1164	1626.1300
	4	2059*	1924	2058.9950*
	5	2682	2143	2672.4910
11	1	2884*	2605	2884.0000*
	2	2231	1684	2258.4940
	3	1565*	1270	1565.0000*
	4	1222	936	1226.8920
	5	1999	1431	1999.0000
12	1	2962*	1724	2962.0000*
	2	2416*	1824	2416.0000*
	3	1267*	970	1267.0010*
	4	2434*	1664	2434.0000*
	5	1936	1393	1981.7260
13	1	1742*	1330	1742.0000*
	2	2042	1671	2064.4350
	3	1786*	1141	1786.0010*
	4	2616	2000	2650.3250
	5	2458*	1635	2458.0000*
14	1	1503*	1047	1503.0000*
	2	2269*	1839	2269.0000*
	3	1955	1241	1985.5090
	4	2153	1663	2170.5680
	5	2000*	1212	2000.0000*
15	1	1548*	811	1548.0010*
	2	1415*	759	1415.0000*
	3	1813	1532	1813.0000
	4	2438	1726	2455.6730
	5	1749*	1271	1749.0000*
16	1	2579*	1989	2579.0000*
	2	2189*	1201	2189.0000*
	3	2130	1458	2147.9210
	4	1435	850	1447.7110
	5	2595*	1616	2595.0010*

Table 1. (Continued)

Dimension of TSP	Problem No.	2-matching	1-tree	SDP
17	1	1184*	911	1183.9970*
	2	2607*	2001	2606.9930*
	3	1665*	1300	1664.9950*
	4	1564	1064	1568.4940
	5	2182	1413	2192.2420
18	1	2586	2008	2606.5870
	2	2271	1531	2273.2220
	3	1562*	925	1562.0080*
	4	2490*	1733	2490.0000*
	5	1805	1281	1815.9110
19	1	1224*	782	1223.9960*
	2	2040	1269	2039.9930
	3	1418*	1046	1417.9960*
	4	1895	1531	1897.4480
	5	2006	1641	2015.5880
20	1	1937	1295	1953.2990
	2	2410	1440	2410.2790
	3	2584	1744	2585.1740
	4	1754	1369	1758.4360
	5	1812	1029	1817.7610

Although the average behavior of SDP relaxation is better than 1-tree, it is possible to construct TSP instances with a special structure for which the contrary holds. This is illustrated by the following example.

Example 1. Let for $n = 20$ a spanning subgraph W of a complete graph $G = (V, E)$ be given by Figure 1. Note that W does not contain any Hamiltonian circuit. Consider the TSP with distance 1 associated to each edge of W, and all other distances equal to 100. In this case both 2-matching and SDP relaxations give lower bound 20, while 1-tree gives 515, which is the optimal value of the TSP. Such a behavior can be explained as follows.

Due to the special structure of the TSP here the weighted graph corresponding to the optimal solution of the SDP relaxation has the same structure as W. Namely, if X is the optimal solution of (8)-(12), $\tilde{L} = \tilde{X} + h_n I - J$ and $E_{\tilde{L}} = \{\{i, j\} \in E \mid l_{ij} < 0\}$ then $W = (V, E_{\tilde{L}})$. Therefore, although $G_{\tilde{L}}$ is connected it does not contain a Hamiltonian circuit. That is the principal reason for a bad lower estimation of the optimal value of the TSP obtained by SDP and 2-matching relaxations. On the other hand, if node 1 is removed W decomposes to a collection of 6 subtours and hence any 1-tree has to contain the "expensive" edges of G, i.e. edges which are not in W. Therefore, 1-tree relaxation in this case gives a better lower bound. ♦

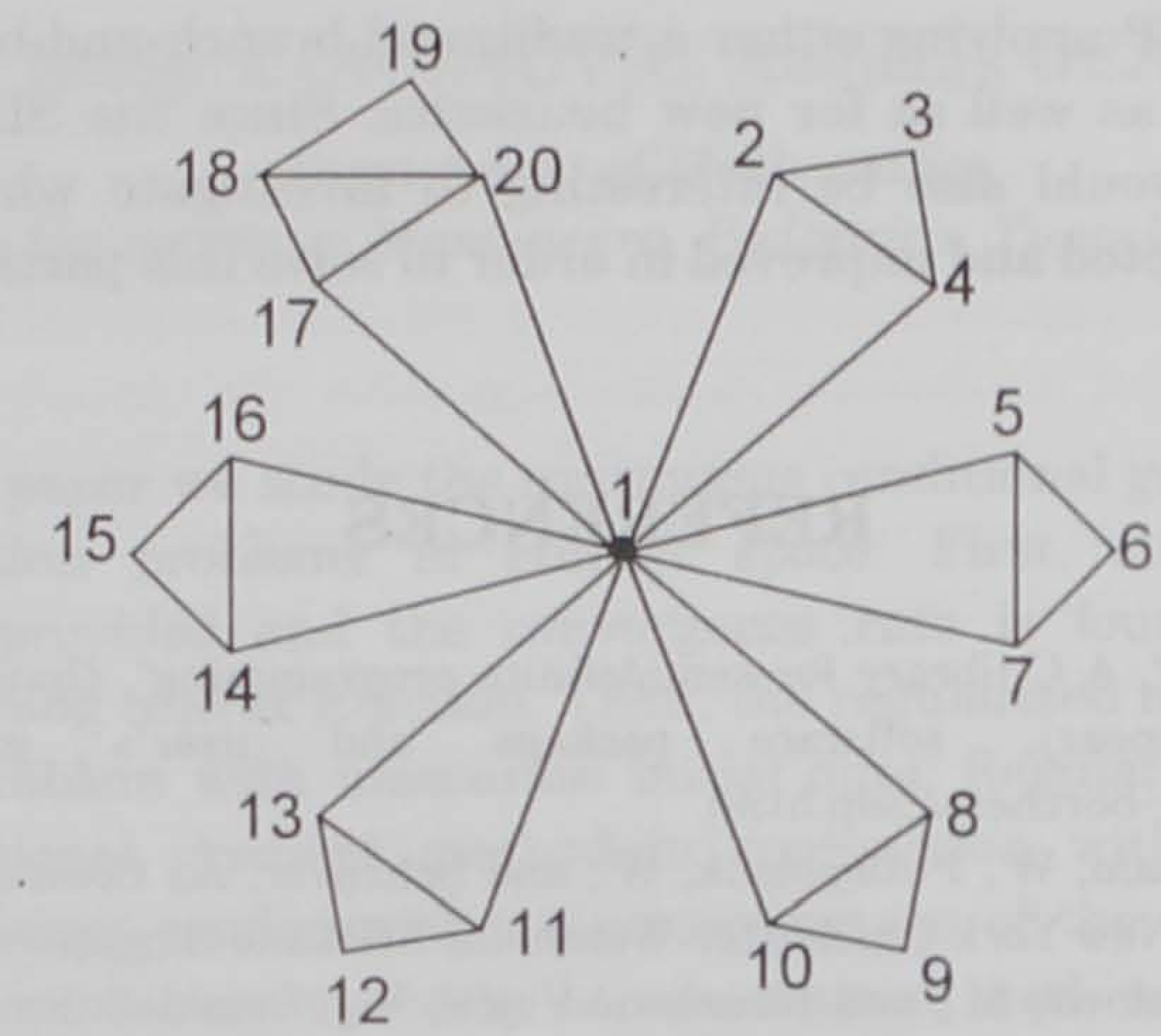

Figure 1.

Example 1 shows that the weighted graph which corresponds to the optimal solution of the SDP relaxation need not contain any Hamiltonian circuit. However, if β in SDP problem (8)-(12) is increased over h_n the edge connectivity of the corresponding weighted graph will also be increased (see [6]), together with the probability that it contains a Hamiltonian circuit. Unfortunatelly, for $\beta > h_n$ matrices which correspond to Hamiltonian circuits are no longer in the feasible set of (8)-(12), which means that the optimal value of SDP relaxation need not be a lower bound for the optimal value of the TSP. Nevertheless, this observation could be the basis for some heuristics for constructing suboptimal Hamiltonian circuits.

5. CONCLUSIONS

In this paper we define a class of semidefinite relaxations of a classical NP-hard combinatorial optimization problem - the symmetric traveling salesman problem. The efficiency of the proposed SDP relaxation is discussed and some preliminary numerical experiments are reported, showing that the SDP relaxation gives better bounds than the well-known 2-matching and 1-tree relaxations.

The proposed relaxation could be used as the basis for developing new exact algorithms for the TSP applying either a traditional branch-and-bound approach or a polyhedral approach, as well as for new heuristics. Since the SDP relaxation has a special structure it would also be interesting to investigate whether existing SDP methods could be adapted and improved in order to solve this particular problem more efficiently.

REFERENCES

[1] Borchers, B., "CSDP, A C library for semidefinite programming", *Optimization Methods and Software* (to appear), software package and user's guide available at: http://www.nmt.edu/~borchers/csdp.html.

[2] Cook, W., Cunningham, W., Pulleyblank, W., and Schrijver, A., *Combinatorial Optimization*, John Wiley & Sons, New York-Chichester-Weinheim-Brisbane-Singapore-Toronto, 1998.

[3] Cvetković, D., Čangalović, M., and Kovačević-Vujčić, V., "Semidefinite programming and the traveling salesman problem", in: R. Petrović , D. Radojević (eds.), *Proceedings of XXV Yugoslav Symposium on Operations Research*, Herceg Novi, Yugoslavia, 1998, 239-242.

[4] Cvetković, D., Dimitrijević, V., and Milosavljević, M., *Variations on the Travelling Salesman Theme*, Libra Produkt, Belgrade, 1996.

[5] Cvetković, D., Doob, M., and Sachs, H., *Spectra of Graphs, 3rd Ed.*, Johann Ambrosius Barth, Heidelberg and Leipzig, 1995.

[6] Fiedler, M., "Laplacian of graphs and algebraic connectivity", in: *Combinatorics and Graph Theory*, Vol. 25, Banach Center Publications, PWN-Polish Scientific Publishers, Warsaw, 1989, 57-70.

[7] Goemans, M., "Semidefinite programming in combinatorial optimization", *Mathematical Programming*, 79 (1997) 143-161.

[8] Merris, R., "Laplacian matrices of graphs: A survey", *Linear Algebra and its Applications*, 197, 198(1994) 143-176.

[9] Laporte, G., "The traveling salesman problem: An overview of exact and approximate algorithms", *European Journal of Operational Research*, 59(1992) 231-247.

[10] Lawler, E.L., Lenstra, J.K., Rinnooy Kan, A.H.G., and Shmoys, D.B., *The Traveling Salesman Problem*, John Wiley & Sons, Chichester-New York-Brisbane-Toronto-Singapore, 1985.

[11] Rendl, F., "Semidefinite programming and combinatorial optimization", Technical Report Woe-19, TU Graz, Austria, December 1997.

[12] Vandenberghe, L., and Boyd, S., "Semidefinite programming", *SIAM Review*, 38(1996) 49-95.