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MAXIMUM ENTROPY AND UTILITY IN A TRANSPORTATION SYSTEM*

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Abstract: The object of the present paper is to study the interrelation and equivalence between methods of maximum entropy and utility in determining the trip distributions in a transportation system.

Keywords: Maximum entropy, utility, trip distribution, transportation system, Bose-Einstein and Fermi-Dirac entropies.

1. INTRODUCTION

In recent years there has been a searching concern with transportation problems in urban and environmental modelling. The subject has become an active field of interest and scientists of different backgrounds are trying to enrich this field by different approaches. A recent and more general approach is based on the concept of entropy [Shannon & Weaver, 1949] and the principle of maximum entropy estimation [Jaynes, 1957]. This approach was developed profoundly by Wilson [1967, 1970, 1979] and others [Webber, 1979].

Another route to the problem is through the concept of the economic model of utility. The interrelation between maximum entropy and maximum utility was pointed

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out by Wilson [1971]. The object of the present paper is to study the equivalence or interrelation between these two principles in the context of trip distribution. The approach is, however, different from that of Wilson [1971] and is more in line with that of Beckmann [1974].

The paper is planned as follows: In Section (2) we considere trip distribution based on the maximum entropy (Shannon) model and derive the same distributions from the maximum-utility model with different cost functions. In Section (3) we have a similar task with the Bose-Einstein and Fermi-Dirac form of entropy.

2. TRIP DISTRIBUTION: THE ENTROPY (SHANNON) APPROACH

Let us consider a model city with a Central Business District (CBD), and a set of locations or sites $\{i, i = 1,2,\ldots,n\}$ and a set of working places or sectors $\{j, j = 1,2,\ldots,m\}$ of the C.B.D.

Let T_{ij} be the number of trips originating from the ith location place to the jth working place (destination). Then the entropy of the trip distribution is given by (analogous to Shannon the entropy):

$$S = -\sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij} \ln T_{ij} \tag{1}$$

Let us assume that the total trips originating from site i and the total number of persons employed in the j-th working place be prescribed:

$$\begin{aligned} \sum_{j=1}^{m} T_{ij} &= A_i, \quad (i = 1,2,\ldots,n) \\ \sum_{i=1}^{n} T_{ij} &= B_j, \quad (j = 1,2,\ldots,m) \end{aligned} \tag{2}$$

and further that the total cost of transportation be fixed:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij} C(r_{ij}) = \hat{C} \tag{3}$$

where r_{ij} is the distance of the jth sector in the CBD from the ith location site and $C(r_{ij})$ is the cost function. Constraints (2) and (3) are insufficient to determine trip distribution T_{ij}. This can, however, be estimated by Jaynes' maximum-entropy principle [Jaynes, 1957].

According to this principle, the least biased distribution will be that which maximizes entropy S given by (1) subject to constraints (2) and (3). The maximization yield [Wilson, 1967] is

$$T_{ij} = a_i b_j e^{-\lambda C(r_{ij})} \tag{4}$$

Where parameters a_i, b_j and λ are to be determined by the equations:

$$\begin{aligned} &\sum_{j=1}^{m} a_i b_j e^{-\lambda C(r_{ij})} = A_i \quad (i = 1,2,\ldots,n) \\ &\sum_{i=1}^{n} a_i b_j e^{-\lambda C(r_{ij})} = B_j \quad (j = 1,2,\ldots,m) \end{aligned} \tag{5}$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j C(r_{ij}) e^{-\lambda C(r_{ij})} = \hat{C} \tag{6}$$

2.1. Trip Distribution: Utility Approach

In the above discussion the distribution was determined as a problem of the theory of information. Since the information (or constraints) available to us are insufficient to uniquely determine the exact distribution, we used the maximum entropy principle of statistical inference. Let us now try to set up an economic model of choice behaviour that underlines trip making. Here we are to distinguish the trips by purposes. In any city work trips are really the result of some basic choices. A person usually accepts a job first or decides to form a business centre and then looks for a suitable place to live. Now, in looking for a place to live he will always try to maximise his utility as far as possible with limited available resources.

A person looking at various potential residences associates a rating or utility index with each. Let the probability density function of utility x for a particular location i be $f_i(x)$, $x \geq a > 0$. Then the probability of something in such a location having utility equal to u or greater (better) than u is given by:

$$P(u) = \int_u^{\infty} f_i(x)\,dx \tag{7}$$

This utility u includes rent, access to schools, hospitals, gas connection, local shopping facilities, electricity board etc., except the distance from CBD. So the net utility of a place is the utility minus the transportation cost $(u - Kr_{ij})$ where r_{ij} is the distance of the ith sector from the jth working place in the CBD and K is the conversion factor of distance to utility.

A person will now try to maximize the net utility. Since the economic power of each person is different, he will accept a place to live with a satisfying level of utility s (say) so that:

$$(u - Kr_{ij}) \geq s$$

or

$$u \geq s + Kr_{ij} \tag{8}$$

Now if A_i is the total number of housing supply in the ith sector at distance r_{ij} from the jth working place, then the probability of a person living in this level is:

$$A_i P(s + Kr_{ij}) \tag{9}$$

If the total number of persons employed in the jth working place in the CBD is B_j, then [Beckmann, 1974]:

$$T_{ij} \sim A_i B_j P(s + Kr_{ij}) \tag{10}$$

Comparing (4) and (10) we observe that:

$$a_i b_j e^{-\lambda C(r_{ij})} \sim A_i B_j P(S + Kr_{ij})$$

or

$$C(r_{ij}) \sim \ln P(S + Kr_{ij}) + \ln(A_i B_j) - \ln(a_i b_j) \tag{11}$$

So, this shows that a cost can always be found so that entropy maximization and utility maximization become equivalent. Let us now illustrate the equivalence with some specific examples by finding the approprite cost functions.

Examples:

(I) First suppose the distribution of utility to be the negative exponential:

$$f_i(x) = e^{-\alpha x}, \quad x \geq 0 \tag{12}$$

Then from (10):

$$T_{ij} \sim A_i B_j \int_{s+Kr_{ij}}^{\infty} e^{-\alpha x} dx \sim \frac{A_i B_j e^{-\alpha(s+Kr_{ij})}}{\alpha}$$

Now from (4) we see that if the cost of travel $C(r_{ij})$ is of the form:

$$C(r_{ij}) \sim S + Kr_{ij} \tag{13}$$

then the two methods lead to the same type of distribution.

(II) If

$$f_i(x) = \frac{e^{-x} x^{m-1}}{\Gamma(m)}, \quad x \geq 0 \tag{14}$$

Then

$$T_{ij} \sim A_i B_j \int_{s+Kr_{ij}}^{\infty} \frac{e^{-x} x^{m-1}}{\Gamma(m)} dx \sim A_i B_j e^{-(S+Kr_{ij})} (A_1 + A_2 r_{ij} + \cdots + A_m r_{ij}^{m-1})$$

which leads to entropy maximization distribution if we take:

$$C(r_{ij}) \sim [(S + Kr_{ij}) - \ln \sum_{\mu=1}^{m-1} A_\mu r^{\mu-1}] \tag{15}$$

(III) If

$$f_i(x) = \frac{a^2}{x}, \quad x \geq a \tag{16}$$

Then

$$T_{ij} \sim A_i B_j \int_{s+Kr_{ij}}^{\infty} \frac{a^2}{x} dx \sim \frac{A_i B_j a^2}{(S + Kr_{ij})^2}$$

which is the generalised gravity model. We observe that if we take the cost function:

$$C(r_{ij}) \sim \ln(S + Kr_{ij}) \tag{17}$$

the maximum-entropy distribution leads to the utility distribution.

3. TRIP DISTRIBUTION: THE ENTROPY (BOSE-EINSTEIN AND FERMI-DIRAC) APPROACH

In this section we shall try to obtain trip distribution based on entropies other than that of Shannon and examine the role of the utility function in generating these types of distributions.

Let T_{ij} be the number of trips from the ith sector (origin) to the jth working place, and the constraints or information available be the same as those of (2) and (3).

Now the problem is to estimate T_{ij} on the basis of information (constraints (2) and (3)). We apply Jaynes' maximum entropy principle with quantum measure of entropy:

$$\hat{S} = -\sum_{i=1}^{n}\sum_{j=1}^{m} T_{ij} \ln T_{ij} + a\sum_{i=1}^{n}\sum_{j=1}^{m}(1+aT_{ij})\ln(1+aT_{ij}) \tag{18}$$

where $a = +1$ for the Bose-Einstein entropy and $a = -1$ for the Fermi-Dirac entropy.

The maximization of entropy $\hat{S}$ subject to constraints (2) and (3) leads to the distribution [Kapur, 1990]:

$$T_{ij} = \frac{1}{a_i b_j e^{\lambda C(r_{ij})} - a} \tag{19}$$

The value $a = 1$ in (19) corresponds to the Fermi-Dirac distribution of trips. In this case at most one trip end is permitted per destination (job). The value $a = -1$ in (19) corresponds to the Base-Einstein distribution of trips. This corresponds to unlimited ends per destination [Fisk, 1985].

We shall discuss the feasibility of distribution (19) in reality later on.

3.1. Trip Distribution: Utility Approach

We have seen in section (2) that:

$$T_{ij} \sim A_i B_j P(s + Kr_{ij}) \tag{20}$$

and

$$P(u) = \int_u^{\infty} f_i(x)dx$$

where $f_i(x)$ is some kind of utility function for the ith location. This may be the potential function of the jth work place or may be some function which depends on the utility of the ith origin (sector-living place) and the attraction of the jth work place. A_i and B_j are some prescribed values related to the ith origin (living place) and jth destination (work place). Again we observe from (10) and (19):

$$A_i B_j P(S + Kr_{ij}) \sim \frac{1}{a_i b_j e^{\lambda C(r_{ij})} - a}$$

or

$$a_i b_j e^{\lambda C(r_{ij})} \sim \frac{1}{A_i B_j P(S + Kr_{ij})} + a$$

or, $$C(r_{ij}) \sim \ln[\frac{a}{a_i b_j} + \frac{1}{A_i a_i B_j b_j P(S+Kr_{ij})}] \tag{21}$$

So, costs can also be found which can make entropy maximization and utility equivalent. The equivalence will also be illustrated with some specific examples.

Examples:

(i) Let us consider utility function:

$$f_i(x) = \frac{ae^x}{(e^x + a)^2}, \quad x > a \tag{22}$$

Then we have:

$$T_{ij} \sim A_i B_j \int_{s+Kr_{ij}}^{\infty} \frac{ae^x}{(e^x + a)^2} dx \sim A_i B_j \frac{a}{e^{s+Kr_{ij}} + a} = \frac{aA_i B_j}{e^{s+Kr_{ij}} + a} \tag{23}$$

We observe that, for $a = \pm 1$, the distribution (23) so obtained resembles the quantum distribution of trips with $C(r_{ij}) \sim S + Kr_{ij}$.

(ii) We have already seen that the utility distribution follows a negative exponential as $f_i(x) = e^{-\alpha x}, x \geq 0$. T_{ij} turns out to be of the form:

$$T_{ij} \sim \frac{A_i B_j e^{-\alpha(s+Kr_{ij})}}{\alpha} \tag{24}$$

which leads to the quantum distribution of trips if:

$$C(r_{ij}) \sim \ln(e^{\alpha(S+Kr_{ij})} \pm 1) \tag{25}$$

4. CONCLUSION

The paper is concerned with two approaches to the problem of decision making in a transportation system. The two approaches are based on the concepts of entropy and utility. In the present paper we have tried to show the interrelation between the above two approaches mathematically in the case of trip distribution. In this sense the paper is of a theoretical type. But it is not purely hypothetical because some of the functions used in this paper have been successfully applied in some other models. For example the exponential utility function has been used in the risk-sharing problem [Kapur, 1990]. The choice of the utility functions $f(x)$ is somewhat of an ad hoc nature. It will depend on different economic problems or situations and its success will also be based on the proper choice of the utility function [Tribus, 1969]. In this paper we have selected a number of utility functions $f(x)$ and have shown how the maximum-utility method which is of great commercial importance can be converted into a well-established statistical decision theory based on the maximum-entropy principle [Wilson, 1970, Jumarie, 1990]. Regarding the applicability of the Bose-Einstein and Fermi-Dirac entropies we state that though the Shannon entropy has a wide range of applicability, Bose-Einstein and Fermi-Dirac entropies have also been applied successfully in the case of work-trip distribution and commodity distribution respectively [Fisk, 1985]. Ours is the first approach to show the equivalence between the maximum utility and maximum-entropy principle based on the Bose-Einstein and Fermi-Dirac entropies.

A large number of papers have been written applying the maximum-entropy technique and maximum utility techniques but our aim is to show that both entropy and utility can be adopted by skillful proponents to explain almost any form of transportation problem by use of either an appropriate entropy or an utility function.

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