EQUIVALENTS OF THE VARIATIONAL PRINCIPLE IN
FUZZY AND PROBABILISTIC METRIC SPACES

Mila STOJAKOVIĆ
Department of Mathematics,
Faculty of Engineering, University of Novi Sad,
21000 Novi Sad, Yugoslavia

Ljiljana GAJIĆ
Institute of Mathematics,
Faculty of Science, University of Novi Sad,
21000 Novi Sad, Yugoslavia

Abstract: Some equivalents of Ekeland's variational principle in fuzzy and probabilistic metric space of the Menger type are proved.

Keywords: Variational principle, fuzzy metric space, Menger space, fixed point.

1. INTRODUCTION

The variational principle, given first Ekeland [2], was proved for metric spaces. After that, this problem received a great deal of attention. Numerous powerful applications in various fields of mathematics were given.

Fuzzy metric space was introduced by Kaleva and Seikkala [3]. The variational principle and its equivalents in this kind of space have been considered in many recent papers, some of them are [1], [7].

The organization of this paper is the following.

Section 2 contains necessary definitions and notions. Section 3 is devoted to the form of the variational principle and its equivalents. Some theorems of the fixed point type for single and multi valued mappings are given.

In Section 4 we give some results for probabilistic metric space of the Menger type, using theorems from previous sections.
2. PRELIMINARIES

The notion of a fuzzy metric space was introduced by Kaleva and Seikkala in [3].

Throughout the paper let $R = (-\infty, \infty)$, $R^+ = [0, \infty)$. Let $F$ denote the set of all fuzzy numbers, that is, the set of all fuzzy sets $u : R \to [0,1]$ such that for every $\alpha \in (0,1]$ the set
\[ u_\alpha = \{ x \in R : u(x) \geq \alpha \} \]
is compact and convex. If for all $\alpha \in (0,1]$, $u_\alpha \subseteq R^+ \cup \{x\}^*$, then $u$ belongs to the set of non-negative fuzzy numbers $F^+$.

It is obvious that, if $u \in F^+$, then $u_\alpha = [a_\alpha, b_\alpha]$, $a_\alpha, b_\alpha \in R^+ \cup \{x\}^*$ for all $\alpha \in (0,1]$.

Let $X$ be a nonempty set, $d : X \times X \to F^+$, $L$ and $R$ are symmetric mappings from $[0,1]^2 \to [0,1]$ nondecreasing in both arguments such that $L(0,0) = 0$, $R(1,1) = 1$. We shall denote
\[ |d(x,y)|_\alpha = |\lambda_\alpha(x,y), \rho_\alpha(x,y)| \]
and by $I_{[\alpha]}$, the indicator function of $\alpha$.

The quadruple $(X, d, L, R)$ is a fuzzy metric space and $d$ a fuzzy metric iff
1. $d(x,y) = I_{[0]} \iff x = y$, 
2. $d(x,y) = d(y,x)$ for all $x, y \in X$
3. for all $x, y, z \in X$
\[ d(x,y)(\varepsilon + \delta) \geq L(d(x,z)(\varepsilon), d(z,y)(\delta)) \]
whenever $\varepsilon \leq \lambda_1(x,z), \delta \leq \lambda_1(z,y)$ and $\varepsilon + \delta \leq \lambda_1(x,y)$
\[ d(x,y)(\varepsilon + \delta) \leq R(d(x,z)(\varepsilon), d(z,y)(\delta)) \]
whenever $\varepsilon \geq \lambda_1(x,z), \delta \geq \lambda_1(z,y)$ and $\varepsilon + \delta \geq \lambda_1(x,y)$

If $\lim_{\alpha \to 0} R(\alpha, \alpha) = 0$, then the family $U = \{ U(\varepsilon, \alpha) : \varepsilon > 0, \alpha \in (0,1] \}$ of sets $U(\varepsilon, \alpha) = \{(x,y) \in X \times X : \rho_\alpha(x,y) < \varepsilon \}$ forms the basis for a Hausdorff uniformity on $X \times X$.
The sets
\[ N_x(\varepsilon; \alpha) = \{ y \in X : \rho_{\alpha}(x, y) < \varepsilon \} \]
form the basis for a Hausdorff topology on \( X \) and this topology is metrizable.

The subset \( A \subset X \) is fuzzy bounded if there exists a \( u \in F^+ \), \( \lim_{\alpha \to \infty} u(a) = 0 \)
such that \( \lambda_{x}(x, y) \leq a_{\alpha} \) and \( \rho_{\alpha}(x, y) \leq b_{\alpha} \) for all \( x, y \in A \). \( \alpha \in (0, 1] \), where \( u_{\alpha} = [a_{\alpha}, b_{\alpha}] \).

The diameter \( d(A) \) of fuzzy bounded set \( A \subset X \) is
\[ d(A) = \sup_{x, y \in A} d(x, y) \]
It is obvious that
\[ (d(A))_{\alpha} = [\sup_{x, y \in A} \lambda_{x}(x, y), \sup_{x, y \in A} \rho_{\alpha}(x, y)] = [\bar{d}_{\alpha}(A), \overline{d}_{\alpha}(A)], \alpha \in (0, 1] \].

3. THE VARIATIONAL PRINCIPLE AND ITS EQUIVALENTS IN FUZZY METRIC SPACES

Let \((X, d, L, R)\) be a fuzzy metric space such that \( \lim_{\alpha \to 0} R(a, a) = 0 \),\( \lim_{\alpha \to \infty} d(x, y)(a) = 0 \) for all \( x, y \in X \) and let \( \phi : X \times X \to (-\infty, \infty] \) be a function which is

lower semicontinuous in the second argument, \( \phi(x, x) = 0 \) for all \( x \in X \), \( \phi(x, x) \leq \phi(x, z) + \phi(z, y) \) for all \( x, y, z \in X \),

there exists \( x \in X \) such that \( \inf_{x \in X} \phi(x, x) > -\infty \). \( \phi(x, x) \leq \phi(x, z) + \phi(z, y) \) for all \( x, y, z \in X \),

The relation \( \leq \) is introduced by the equivalence
\[ x \leq y \iff \forall \alpha \in (0, 1], \rho_{\alpha}(x, y) + \phi(x, y) \leq 0 . \]

Lemma 1. If the function \( \phi : X \times X \to (-\infty, \infty] \) satisfies (2), then the relation \( \leq \) is reflexive, antisymmetric and transitive (relation \( \leq \) is an order in \( X \)).

Proof: Since for all \( \alpha \in (0, 1] \) \( \rho_{\alpha}(x, x) = 0 = -\phi(x, x) \), we get that \( x \leq x \) for all \( x \in X \).
Further, is \( x \leq y \) and \( y \leq x \), then for every \( \alpha \in (0, 1] \)

\[ \rho_{\alpha}(x, y) \leq -\phi(x, y) \] and \[ \rho_{\alpha}(y, x) \leq -\phi(y, x) \].

The equality \( \rho_{\alpha}(x, y) = \rho_{\alpha}(y, x) \) implies that for all \( \alpha \in (0, 1] \)
To prove the transitivity we assume that \( x \leq y \) and \( y \leq z \). This means that 
\[
d(x,y)(a) = 0 \quad \text{for all } a > -\phi(x,y) > 0 \quad \text{and} \quad d(y,z)(b) = 0 \quad \text{for all } b > -\phi(y,z) > 0.
\]
On the other hand, for every \( \varepsilon > -\phi(x,z) \geq -\phi(x,y) - \phi(y,z) \), there exist \( a > -\phi(x,y) > 0 \) and \( b > -\phi(y,z) > 0 \) such that 
\[
d(x,z)(c) \leq R(d(x,y)(a), d(y,z)(b)) = R(0,0) = 0,
\]
that is, for all \( \alpha \in (0,1] \)
\[
\rho_\alpha(x,z) \leq -\phi(x,z) \iff x \leq z.
\]
For every \( x \in X \) we define the set \( S(x) = \{ y \in X : x \leq y \} \), where the relation \( \leq \) is introduced by (4). Let \( x_0 \in X \) be such that
\[
(a) \quad \text{any nondecreasing Cauchy sequence in } S(x_0) \text{ has an upper bound in } X \text{ and}
\]
\[
(b) \quad \text{for any } x \in S(x_0) \text{ and } \varepsilon > 0, \text{ there exists } y \in S(x_0) \text{ such that } \overline{d}_\alpha(S(y)) < \varepsilon \text{ for every } \alpha \in (0,1].
\]
In the next seven theorems, let \((X,d,L,R)\) be a fuzzy metric space (not necessarily complete, \( \phi : X \times X \to (-\infty, \infty) \) be such that (2) is satisfied and the relation \( \leq \) defined by (4) be an order in \( X \) satisfying (a) and (b).

**Theorem 1.** If \((*)\) is satisfied, then there exists \( x^* \in S(x_0) \) such that for all \( x \in X \setminus \{ x^* \} \)
\[
\rho_\alpha(x^*,x) + \phi(x^*,x) > 0 \text{ for some } \alpha \in (0,1].
\]  
(5)

**Proof:** Using assumption (b), we shall form a Cauchy sequence \( \{x_n\}_{n=N}^{\infty} \). Since \( x_0 \in S(x_0) \), for \( \varepsilon = 1 \) there exists \( x_1 \in S(x_0) \) such that \( \overline{d}_\alpha(S(x_1)) < 1 \). If \( \varepsilon = \frac{1}{2} \), then there exists \( x_2 \in S(x_1) \) such that \( \overline{d}_\alpha(S(x_2)) < \frac{1}{2} \). Continuing this process, for \( \varepsilon = \frac{1}{n} \) there exists \( x_n \in S(x_{n-1}) \) such that \( \overline{d}_\alpha(S(x_n)) < \frac{1}{n} \) for all \( \alpha \in (0,1] \). The sequence \( \{x_n\}_{n=N}^{\infty} \) is nondecreasing \( x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \) and it is Cauchy sequence \( \overline{d}_\alpha(x_n,x_m) < \min \left\{ \frac{1}{n}, \frac{1}{m} \right\} \) for all \( \alpha \in (0,1] \).

By (a) we get that there exists an upper bound \( x^* \in X \). Since \( x_n \leq x^* \), this means that \( x^* \in S(x_n) \) and \( x^* \in \bigcap_{n=N}^{\infty} S(x_n) \). But \( \overline{d}_\alpha(S(x_n)) \to 0 \), which implies that \( x^* = \lim_{n \to \infty} x_n \). In order to prove (5) we assume that there exists \( x \in X \setminus \{ x^* \} \) such that
\[ d_\alpha (x^*, x) + \phi(x^*, x) \leq 0 \quad \text{for all} \quad \alpha \in (0, 1], \]

that is, \( x^* \preceq x \). Then \( x \) together with \( x^* \) belongs to \( S(x_0) \) for all \( n \in \mathbb{N} \) and \( \rho_\alpha (x, x^*) \leq \bar{d}_\alpha (S(x_n)) \leq \frac{1}{n} \). Putting \( n \to \infty \), we get that \( \rho_\alpha (x, x^*) = 0 \) for all \( \alpha \in (0, 1] \) which means that \( x = x^* \). Since we chose \( x \) from \( X \setminus \{x^*\} \), we have that the assumption \( x^* \preceq x \) is not correct and hence (5) is true.

**Theorem 2.** Let (\( \ast \)) be satisfied. If \( A \subset X \) has the property that for every \( x \in S(x_0) \setminus A \) there exists \( y \in S(x_0) \setminus \{x^*\} \) such that \( x \preceq y \), then there exists \( x^* \in S(x_0) \cap A \).

**Proof:** From Theorem 1 we know that there exists \( x^* \in S(x_0) \) such that \( x \preceq x^* \) for all \( x \in X \setminus \{x^*\} \). It is obvious that \( x^* \in A \), i.e. \( x^* \in S(x_0) \cap A \).

**Theorem 3.** Let (\( \ast \)) be satisfied. If for every \( x \in S(x_0) \) with \( \inf_y \phi(x, y) < 0 \) there exists \( y \in X \setminus \{x\} \) such that \( x \preceq y \), then there exists \( x^* \in S(x_0) \) such that \( \phi(x^*, y) \geq 0 \) for all \( y \in X \).

**Proof:** If \( A = \{x \in X : \inf_y \phi(x, y) \geq 0\} \), then the assumptions from the theorem could be formulated by: for every \( x \in S(x_0) \setminus A \) there exists \( y \in X \setminus \{x\} \) such that \( x \preceq y \). Now we can apply Theorem 2 which means that there exists \( x^* \in S(x_0) \cap A \).

**Theorem 4.** If the conditions (\( \ast \)) are satisfied and if \( f : X \to X \) is a function satisfying \( x \preceq f(x) \) for all \( x \in X \), then \( f \) has a fixed point \( x^* \in S(x_0) \).

**Proof:** By Theorem 1, there exists \( x^* \in S(x_0) \) such that \( x^* \preceq x \) for every \( x \in X \setminus \{x^*\} \). If we suppose that \( f(x^*) \neq x^* \), then for some \( \alpha \in (0, 1] \)

\[ \rho_\alpha (x^*, f(x^*)) + \phi(x^*, f(x^*)) > 0, \]

that is, \( x^* \not\preceq f(x^*) \). This contradicts the assumption of the theorem that \( x \preceq f(x) \) for all \( x \in X \).

**Theorem 5.** Let the conditions (\( \ast \)) be satisfied. If \( F : X \to 2^X \setminus \{\emptyset\} \) is a multivalued mapping such that for every \( x \in X \) and every \( y \in F(x) \) \( x \preceq y \), then there exists \( x^* \in S(x_0) \) such that \( F(x^*) = \{x^*\} \).
**Proof:** If \( f : S(x_0) \to X \) is a selection of \( F \), we can apply Theorem 4 to \( f \), which means that there exists \( x^* \in S(x_0) \) such that \( f(x^*) = x^* \). If \( F(x) \neq \{x\} \) for all \( x \in S(x_0) \), then either \( x \in F(x) \) or \( x \not\in F(x) \). The selection formed by \( f(x) = y \in F(x) \setminus \{x\} \) has no fixed point, which is a contradiction. This means that there exists \( x^* \in S(x_0) \) such that \( F(x^*) = \{x^*\} \).

**Theorem 6.** Let the conditions (*) be satisfied. If \( F : X \to 2^X \setminus \{\emptyset\} \) is a multivalued mapping such that for every \( x \in S(x_0) \) \( F(x) \) there exists \( y \in X \setminus \{x\} \) for which \( x \leq y \), then there exists \( x^* \in S(x_0) \) such that \( F(x^*) = \{x^*\} \).

**Proof:** Invoking Theorem 1, \( x^* \in S(x_0) \) is an element for which \( x^* \not\leq x \) for all \( x \in X \setminus \{x^*\} \). The supposition that \( x^* \in F(x^*) \) means that \( x^* \in S(x_0) \setminus F(x^*) \). Then there exists \( y \in X \setminus \{x^*\} \) such that \( x^* \leq y \) which contradicts \( x^* \not\leq x \) for all \( x \in X \setminus \{x^*\} \).

**Theorem 7.** The statements of Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5 and Theorem 6 are equivalent.

**Proof:** So far we have proved the implications Theorem 1 \( \Rightarrow \) Theorem 2, Theorem 2 \( \Rightarrow \) Theorem 3, Theorem 1 \( \Rightarrow \) Theorem 4, Theorem 4 \( \Rightarrow \) Theorem 5 and Theorem 1 \( \Rightarrow \) Theorem 6.

The implications Theorem 6 \( \Rightarrow \) Theorem 4 and Theorem 5 \( \Rightarrow \) Theorem 4 are obvious, since the single-valued mapping is a special case of multivalued mapping.

It only remains to prove that Theorem 3 \( \Rightarrow \) Theorem 1 and Theorem 4 \( \Rightarrow \) Theorem 1.

To prove that Theorem 3 \( \Rightarrow \) Theorem 1, we shall assume that \( x^* \in S(x_0) \) from Theorem 3 (\( \phi(x^*,x) \geq 0 \) for all \( x \in X \)) is such that (5) does not hold, i.e.

\[
\rho_\alpha (x^*,x) + \phi(x^*,x) \leq 0 \text{ for all } \alpha \in (0,1) \text{ and all } x \in X \setminus \{x^*\}.
\]

(6)

It is obvious that if \( \phi(x^*,x) \geq 0 \) for all \( x \in X \) and \( \rho_\alpha (x^*,x) \geq 0 \), then together with (6) we get \( \rho_\alpha (x^*,x) = 0 \) for all \( \alpha \in (0,1) \). But we chose \( x \) from \( X \setminus \{x^*\} \), that is, \( \rho_\alpha (x^*,x) > 0 \) for some \( \alpha \in (0,1) \). This is a contradiction.

To complete the proof, one needs to show that Theorem 4 \( \Rightarrow \) Theorem 1. If Theorem 1 does not hold, then for every \( x \in S(x_0) \) there exists \( y \in X \setminus \{x\} \) such that \( x \leq y \). We shall form \( f : S(x_0) \to X \) by \( f(x) = y \).

If Theorem 4 holds, then there \( x^* \in S(x_0) \) such that \( f(x^*) = x^* \) which contradicts the supposition that \( y \in X \setminus \{x\} \).
Lemma 2. If \((X, d, L, R)\) is a fuzzy metric space and the function \(\phi : X \times X \to (-\infty, \infty)\) satisfies (1), (2) and (3), then for any \(x \in S(x)\) and \(\varepsilon > 0\), there exists \(y \in S(x)\) such that \(d_\alpha(S(y)) < \varepsilon\) for every \(\alpha \in (0.1]\).

Proof: Let \(x \in S(x)\). Then

\[
\inf_{z \in S(x)} \phi(x, z) \geq \inf_{z \in S(x)} |\phi(x, z) - \phi(x, x)| \geq \inf_{z \in S(x)} \phi(x, z) - \phi(x, x) > -\varepsilon,
\]

that is, there exists \(\alpha \in R: \inf_{z \in S(x)} \phi(x, z) = \alpha\). We shall choose \(y \in S(x)\) such that \(\phi(x, y) \leq \alpha + \varepsilon/2\).

Similarly as in the previous case, we get

\[
b = \inf_{z \in S(y)} \phi(y, z) \geq \inf_{z \in S(x)} \phi(x, z) - \phi(x, y) \geq -\varepsilon/2.
\]

Finally, to finish the proof, it remains to show that \(\overline{d}_\alpha(S(y)) < \varepsilon\).

If \(z_1, z_2 \in S(y)\), then for every \(\alpha \in (0.1]\)

\[
\rho_\alpha(y, z_1) \leq \phi(y, z_1) \leq \frac{\varepsilon}{2} \iff d(y, z_1)(\frac{\varepsilon}{2}) < \alpha,
\]

\[
\rho_\alpha(y, z_2) \leq \phi(y, z_2) \leq \frac{\varepsilon}{2} \iff d(y, z_2)(\frac{\varepsilon}{2}) < \alpha.
\]

Since \(\lim_{\alpha \to 0} R(\alpha, \alpha) = 0\) for every \(\alpha \in (0.1]\) there exists \(\beta \in (0.1]\) such that \(R(\beta, \beta) < \alpha\). Then we get

\[
d(z_1, z_2)(\varepsilon) \leq R(d(y, z_1)(\frac{\varepsilon}{2}), d(y, z_2)(\frac{\varepsilon}{2})) \leq R(\beta, \beta) < \alpha
\]

which means that \(\rho_\beta(z_1, z_2) < \varepsilon\) for every \(\alpha \in (0.1]\). Hence, \(\overline{d}_\beta(S(y)) = \sup_{z_1, z_2 \in S(y)} \rho_\beta(z_1, z_2) \leq \varepsilon\) for all \(\alpha \in (0.1]\).

Lemma 3. \([5]\) If \((X, d, L, R)\) is a complete fuzzy metric space and the function \(\phi : X \times X \to (-\infty, \infty)\) satisfies (1) and (2), then every nondecreasing Cauchy sequence has an upper bound in \(X\).

Theorem 8. If \((X, d, L, R)\) is a complete fuzzy metric space and the function \(\phi : X \times X \to (-\infty, \infty)\) satisfies (1), (2) and (3), then the next six statements are equivalent:
there exists $x^* \in S(\bar{x})$ such that for all $y \in X \setminus \{x^*\}$

$$p_a(x^*.x) + \phi(x^*.x) > 0 \text{ for some } a \in (0,1),$$

ii. if $A \subset X$ has the property that for every $x^* \in S(\bar{x}) \cap A$ there exists $y \in X \setminus \{x^*\}$ such that $x \leq y$, then there exists $x^* \in S(\bar{x}) \cap A$,

iii. if for every $x \in S(x)$ with $\inf_{y \in X} \phi(x, y) < 0$ there exists $y \in X \setminus \{x\}$ such that $x \leq y$, then there exists $x^* \in S(\bar{x})$ such that $\phi(x^*, y) \geq 0$ for all $y \in X$,

iv. if $f : X \to X$ is a function satisfying $x \leq f(x)$ for all $x \in X$, then $f$ has a fixed point $x^* \in S(\bar{x})$,

v. if $F : X \to 2^X \setminus \{0\}$ is a multivalued mapping such that for every $x \in X$ and every $y \in F(x)$, $x \leq y$, then there exists $x^* \in S(\bar{x})$ such that $F(x^*) = \{x^*\}$,

vi. if $F : X \to 2^X \setminus \{0\}$ is a multivalued mapping such that for every $x \in S(\bar{x})$ $F(x)$ there exists some $y \in X \setminus \{x\}$ for which $x \leq y$, then there exists $x^* \in S(\bar{x})$ with $x^* \notin F(x^*)$.

**Proof:** Using Theorem 7, Lemma 2 and Lemma 3, we get Theorem 8.

4. THE VARIATIONAL PRINCIPLE AND ITS EQUIVALENTS IN PROBABILISTIC METRIC SPACES

The function $F : \mathbb{R} \to [0,1]$ ($\mathbb{R}$ denotes the set of reals) which is left continuous, nondecreasing with $\sup_x \mathbb{R} F(x) = 1$, is a distribution function. Let $D$ be the set of all distribution functions.

The triplet $(X, F, t)$ where $X$ is any set, $F : X \times X \to D$ is such that

$$F_{x,y} = F_{y,x} \quad \text{for all } x, y \in X,$$

$$F_{x,y}(v) = 1 \quad \text{for all } v > 0 \iff x = y,$$

$$F_{x,y}(0) = 0 \quad \text{for all } x, y \in X,$$

$$F_{x,y}(v + u) \geq t(F_{x,z}(v), F_{z,y}(u)) \quad \text{for all } x, y, z \in X \text{ and } u, v \in \mathbb{R},$$

and $t : [0,1] \times [0,1] \to [0,1]$ is commutative, nondecreasing, associative and $t(a, 1) = a$ for all $a \in [0,1]$, is a Menger space.
In [3] it was proved by Kaleva and Seikkala that every Menger space \((X,F,t)\) is a fuzzy metric space \((X,d,L,R)\) where \(u_{x,y} = \sup \{v : F_{x,y}(v) = 0\}\) and

\[
d(x,y)(u) = \begin{cases} 0, & u < u_{x,y} \\ 1 - F_{x,y}(u), & u \geq u_{x,y} \end{cases}
\]

\[
R(a,b) = 1 - t(1 - a, 1 - b), \quad a, b \in [0,1]
\]

\(L = 0\)

If \(\phi : X \times X \to (-\infty, \infty)\) is a function satisfying (1), (2) and (3), then we define the relation \(\leq\) by

\[
x \leq y \iff F_{x,y}(u) \geq H(u + \phi(x,y)) \text{ for all } u > 0
\]

where \(H(u) = \begin{cases} 0, & u \leq 0 \\ 1, & u > 0 \end{cases}\). For every \(x \in X\) by \(S(x)\) we denote set \(S(x) = \{y \in X : x \leq y\}\).

**Lemma 4.** The relation \(\leq\) defined by (7) is an order in \(X\).

**Theorem 9.** Let \((X,F,t)\) be a complete Menger space such that \(\lim_{n \to \infty} t(a,a) = 1\) and \(\phi : X \times X \to (-\infty, \infty)\) be a function satisfying (1), (2) and (3). Then the statements (ii)-(vi) (from Theorem 8) and

(i) there exists \(x^* \in S(x)\) such that for all \(x \in X \setminus \{x^*\}\)

\[F_{x,y}(u) < H(u + \phi(x,y)) \text{ for some } u \in R^+\]

are equivalent.

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