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EQUIVALENTS OF THE VARIATIONAL PRINCIPLE IN FUZZY AND PROBABILISTIC METRIC SPACES

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Abstract: Some equivalents of Ekeland's variational principle in fuzzy and probabilistic metric space of the Menger type are proved.

Keywords: Variational principle, fuzzy metric space, Menger space, fixed point.

1. INTRODUCTION

The variational principle, given first Ekeland [2], was proved for metric spaces. After that, this problem received a great deal of attention. Numerous powerful applications in various fields of mathematics were given.

Fuzzy metric space was introduced by Kaleva and Seikkala [3]. The variational principle and its equivalents in this kind of space have been considered in many recent papers, some of them are [1], [7].

The organization of this paper is the following.

Section 2 contains necessary definitions and notions. Section 3 is devoted to the form of the variational principle and its equivalents. Some theorems of the fixed point type for single and multi valued mappings are given.

In Section 4 we give some results for probabilistic metric space of the Menger type, using theorems from previous sections.

2. PRELIMINARIES

The notion of a fuzzy metric space was introduced by Kaleva and Seikkala in [3].

Throughout the paper let $R = (-\infty, \infty)$, $R^+ = [0, \infty)$. Let F denote the set of all fuzzy numbers, that is, the set of all fuzzy sets $u : R \to [0,1]$ such that for every $\alpha \in (0,1]$ the set

$$u_\alpha = \{ x \in R : u(x) \geq \alpha \} \neq 0$$

is compact and convex. If for all $\alpha \in (0,1]$, $u_\alpha \in R^+ \cup \{\infty\}$, then u belongs to the set of non-negative fuzzy numbers F^+.

It is obvious that, if $u \in F^+$, then $u_\alpha = [a_\alpha, b_\alpha]$, $a_\alpha, b_\alpha \in R^+ \cup \{\infty\}$ for all $\alpha \in (0,1]$.

Let X be a nonempty set, $d : X \times X \to F^+$, L and R are symmetric mappings from $[0,1] \times [0,1] \to [0,1]$ nondecreasing in both arguments such that $L(0,0) = 0$, $R(1,1) = 1$. We shall denote

$$[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]$$

and by $I_{\{a\}}$ the indicator function of a.

The quadruple (X, d, L, R) is a fuzzy metric space and d a fuzzy metric iff

1. $d(x,y) = I_{\{0\}} \Leftrightarrow x = y$,
2. $d(x,y) = d(y,x)$ for all $x, y \in X$
3. for all $x, y, z \in X$

$$d(x,y)(\varepsilon + \delta) \geq L(d(x,z)(\varepsilon), d(z,y)(\delta))$$
$$\text{whenever } \varepsilon \leq \lambda_1(x,z), \delta \leq \lambda_1(z,y) \text{ and } \varepsilon + \delta \leq \lambda_1(x,y)$$
$$d(x,y)(\varepsilon + \delta) \leq R(d(x,z)(\varepsilon), d(z,y)(\delta))$$
$$\text{whenever } \varepsilon \geq \lambda_1(x,z), \delta \geq \lambda_1(z,y) \text{ and } \varepsilon + \delta \geq \lambda_1(x,y)$$

If $\lim_{a \to 0} R(a,a) = 0$, then the family

$$U = \{U(\varepsilon, \alpha) : \varepsilon > 0,\ \alpha \in (0,1]\}$$

of sets

$$U(\varepsilon, \alpha) = \{(x,y) \in X \times X : \rho_\alpha(x,y) < \varepsilon\}$$

forms the basis for a Hausdorff uniformity on $X \times X$.

The sets

$$N_x(\varepsilon,\alpha) = \{y \in X : \rho_\alpha(x,y) < \varepsilon\}$$

form the basis for a Hausdorff topology on X and this topology is metrizable.

The subset $A \subset X$ is fuzzy bounded if there exists a $u \in F^+$, $\lim_{a\to\infty} u(a) = 0$ such that $\lambda_\alpha(x,y) \le a_\alpha$ and $\rho_\alpha(x,y) \le b_\alpha$ for all $x,y \in A$, $\alpha \in (0,1]$, where $u_\alpha = [a_\alpha, b_\alpha]$.

The diametar $d(A)$ of fuzzy bounded set $A \subset X$ is $d(A) = \sup_{x,y\in A} d(x,y)$. It is obvious that

$$(d(A))_\alpha = [\sup_{x,y\in A} \lambda_\alpha(x,y),\ \sup_{x,y\in A} \rho_\alpha(x,y)] = [\underline{d}_\alpha(A), \overline{d}_\alpha(A)],\ \alpha \in (0,1].$$

3. THE VARIATIONAL PRINCIPLE AND ITS EQUIVALENTS IN FUZZY METRIC SPACES

Let (X,d,L,R) be a fuzzy metric space such that $\lim_{a\to 0} R(a,a) = 0$, $\lim_{a\to\infty} d(x,y)(a) = 0$ for all $x,y \in X$ and let $\phi : X \times X \to (-\infty,\infty]$ be a function which is

lower semicontinuous in the second argument, (1)

$\phi(x,x) = 0$ for all $x \in X$, (2)

$\phi(x,x) \le \phi(x,z) + \phi(z,y)$ for all $x,y,z \in X$,

there exists $x \in X$ such that $\inf\limits_{x\in X} \phi(x,x) > -\infty$. (3)

The relation $\lesssim$ is introduced by the equivalence

$$x \lesssim y \Leftrightarrow \forall \alpha \in (0,1],\ \rho_\alpha(x,y) + \phi(x,y) \le 0. \tag{4}$$

Lemma 1. *If the function $\phi : X \times X \to (-\infty,\infty]$ satisfies (2), then the relation $\lesssim$ is reflexive, antisymmetric and transitive (relation $\lesssim$ is an order in X).*

Proof: Since for all $\alpha \in (0,1]$ $\rho_\alpha(x,x) = 0 = -\phi(x,x)$, we get that $x \lesssim x$ for all $x \in X$. Further, is $x \lesssim y$ and $y \lesssim x$, then for every $\alpha \in (0,1]$

$$\rho_\alpha(x,y) \le -\phi(x,y) \text{ and } \rho_\alpha(y,x) \le -\phi(y,x).$$

The equality $\rho_\alpha(x,y) = \rho_\alpha(y,x)$ implies that for all $\alpha \in (0,1]$

$$2\rho_\alpha(x,y) \le -\phi(x,y) - \phi(y,x) = -\phi(x,x) = 0 \Rightarrow x = y .$$

To prove the transitivity we assume that $x \precsim y$ and $y \precsim z$. This means that $d(x,y)(a) = 0$ for all $a > -\phi(x,y) > 0$ and $d(y,z)(b) = 0$ for all $b > -\phi(y,z) > 0$. On the other hand, for every $\varepsilon > -\phi(x,z) \ge -\phi(x,y) - \phi(y,z)$, there exist $a > -\phi(x,y) > 0$ and $b > -\phi(y,z) > 0$ such that

$$d(x,z)(\varepsilon) \le R(d(x,y)(a), d(y,z)(b)) = R(0,0) = 0 ,$$

that is, for all $\alpha \in (0,1]$

$$\rho_\alpha(x,z) \le -\phi(x,z) \Leftrightarrow x \precsim z .$$

For every $x \in X$ we define the set $S(x) = \{y \in X : x \precsim y\}$, where the relation $\precsim$ is introduced by (4). Let $x_0 \in X$ be such that

(a) any nondecreasing Cauchy sequence in $S(x_0)$ has an upper bound in X, and

(b) for any $x \in S(x_0)$ and $\varepsilon > 0$, there exists $y \in S(x_0)$ such that $\overline{d}_\alpha(S(y)) < \varepsilon$ for every $\alpha \in (0,1]$.

(*) In the next seven theorems, let (X, d, L, R) be a fuzzy metric space (not necessarily complete, $\phi : X \times X \to (-\infty, \infty]$ be such that (2) is satisfied and the relation $\precsim$ defined by (4) be an order in X satisfying (a) and (b).

Theorem 1. *If (*) is satisfied, then there exists $x^* \in S(x_0)$ such that for all $x \in X \setminus \{x^*\}$*

$$\rho_\alpha(x^*, x) + \phi(x^*, x) > 0 \textit{ for some } \alpha \in (0,1] . \tag{5}$$

Proof: Using assumption (b), we shall form a Cauchy sequence $\{x_n\}_{n\in N}$. Since $x_0 \in S(x_0)$, for $\varepsilon = 1$ there exists $x_1 \in S(x_0)$ such that $\overline{d}_\alpha(S(x_1)) < 1$. If $\varepsilon = \frac{1}{2}$, then there exists $x_2 \in S(x_1)$ such that $\overline{d}_\alpha(S(x_2)) < \frac{1}{2}$. Continuing this process, for $\varepsilon = \frac{1}{n}$ there exists $x_n \in S(x_{n-1})$ such that $\overline{d}_\alpha(S(x_n)) < \frac{1}{n}$ for all $\alpha \in (0,1]$. The sequence $\{x_n\}_{n\in N}$ is nondecreasing ($x_1 \precsim x_2 \precsim \ldots \precsim x_n \precsim \ldots$) and it is Cauchy sequence ($\overline{d}_\alpha(x_n, x_m) < \min\{\frac{1}{n}, \frac{1}{m}\}$ for all $\alpha \in (0,1]$.

By (a) we get that there exists an upper bound $x^* \in X$. Since $x_n \precsim x^*$, this means that $x^* \in S(x_n)$ and $x^* \in \bigcap_{n\in N} S(x_n)$. But $\overline{d}_\alpha(S(x_n)) \to 0$, which implies that $x^* = \lim_{n\to\infty} x_n$. In order to prove (5) we assume that there exists $x \in X \setminus \{x^*\}$ such that

$$d_\alpha(x^*,x)+\phi(x^*,x)\le 0 \text{ for all } \alpha\in(0,1],$$

that is, $x^*\lesssim x$. Then x together with x^* belongs to $S(x_n)$ for all $n\in N$ and $\rho_\alpha(x,x^*)\le$ $\le \bar{d}_\alpha(S(x_n))<\frac{1}{n}$. Putting $n\to\infty$, we get that $\rho_\alpha(x,x^*)=0$ for all $\alpha\in(0,1]$ which means that $x=x^*$. Since we chose x from $X\setminus\{x^*\}$, we have that the assumption $x^*\lesssim x$ is not correct and hence (5) is true.

Theorem 2. *Let (*) be satisfied. If $A\subset X$ has the property that for every $x\in S(x_0)\setminus A$ there exists $y\in S(x_0)\setminus\{x\}$ such that $x\lesssim y$, then there exists $x^*\in S(x_0)\cap A$.*

Proof: From Theorem 1 we know that there exists $x^*\in S(x_0)$ such that $x\not\lesssim x^*$ for all $x\in X\setminus\{x^*\}$. It is obvious that $x^*\in A$, i.e. $x^*\in S(x_0)\cap A$.

Theorem 3. *Let (*) be satisfied. If for every $x\in S(x_0)$ with $\inf_{y\in X}\phi(x,y)<0$ there exists $y\in X\setminus\{x\}$ such that $x\lesssim y$, then there exists $x^*\in S(x_0)$ such that $\phi(x^*,y)\ge 0$ for all $y\in X$.*

Proof: If $A=\{x\in X:\inf_{y\in X}\phi(x,y)\ge 0\}$, then the assumptions from the theorem could be formulated by: for every $x\in S(x_0)\setminus A$ there exists $y\in X\setminus\{x\}$ such that $x\lesssim y$. Now we can apply Theorem 2 which means that there exists $x^*\in S(x_0)\cap A$.

Theorem 4. *If the conditions (*) are satisfied and if $f:X\to X$ is a function satisfying $x\lesssim f(x)$ for all $x\in X$, then f has a fixed point $x^*\in S(x_0)$.*

Proof: By Theorem 1, there exists $x^*\in S(x_0)$ such that $x^*\not\lesssim x$ for every $x\in X\setminus\{x^*\}$. If we suppose that $f(x^*)\ne x^*$, then for some $\alpha\in(0,1]$

$$\rho_\alpha(x^*,f(x^*))+\phi(x^*,f(x^*))>0,$$

that is, $x^*\not\lesssim f(x^*)$. This contradicts the assumption of the theorem that $x\lesssim f(x)$ for all $x\in X$.

Theorem 5. *Let the conditions (*) be satisfied. If $F:X\to 2^X\setminus\{0\}$ is a multivalued mapping such that for every $x\in X$ and every $y\in F(x)$ $x\lesssim y$, then there exists $x^*\in S(x_0)$ such that $F(x^*)=\{x^*\}$.*

Proof: If $f: S(x_0) \to X$ is a selection of F, we can apply Theorem 4 to f, which means that there exists $x^* \in S(x_0)$ such that $f(x^*) = x^*$. If $F(x) \neq \{x\}$ for all $x \in S(x_0)$, then either $x \notin F(x)$ or $x \in F(x)$. The selection formed by $f(x) = y \in F(x) \setminus \{x\}$ has no fixed point, which is a contradiction. This means that there exists $x^* \in S(x_0)$ such that $F(x^*) = \{x^*\}$.

Theorem 6. *Let the conditions (*) be satisfied. If $F: X \to 2^X \setminus \{0\}$ is a multivalued mapping such that for every $x \in S(x_0) \setminus F(x)$ there exists $y \in X \setminus \{x\}$ for which $x \lesssim y$, then there exists $x^* \in S(x_0)$ such that $F(x^*) = \{x^*\}$.*

Proof: Invoking Theorem 1, $x^* \in S(x_0)$ is an element for which $x^* \not\lesssim x$ for all $x \in X \setminus \{x^*\}$. The supposition that $x^* \notin F(x^*)$ means that $x^* \in S(x_0) \setminus F(x^*)$. Then there exists $y \in X \setminus \{x^*\}$ such that $x^* \lesssim y$ which contradicts $x^* \not\lesssim x$ for all $x \in X \setminus \{x^*\}$.

Theorem 7. *The statements of Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5 and Theorem 6 are equivalent.*

Proof: So far we have proved the implications Theorem 1 $\Rightarrow$ Theorem 2, Theorem 2 $\Rightarrow$ Theorem 3, Theorem 1 $\Rightarrow$ Theorem 4, Theorem 4 $\Rightarrow$ Theorem 5 and Theorem 1 $\Rightarrow$ Theorem 6.

The implications Theorem 6 $\Rightarrow$ Theorem 4 and Theorem 5 $\Rightarrow$ Theorem 4 are obvious, since the single-valued mapping is a special case of multivalued mapping.

It only remains to prove that Theorem 3 $\Rightarrow$ Theorem 1 and Theorem 4 $\Rightarrow$ Theorem 1.

To prove that Theorem 3 $\Rightarrow$ Theorem 1, we shall assume that $x^* \in S(x_0)$ from Theorem 3 ($\phi(x^*, x) \geq 0$ for all $x \in X$) is such that (5) does not hold, i.e.

$$\rho_\alpha(x^*, x) + \phi(x^*, x) \leq 0 \text{ for all } \alpha \in (0,1] \text{ and all } x \in X \setminus \{x^*\}. \tag{6}$$

It is obvious that if $\phi(x^*, x) \geq 0$ for all $x \in X$ and $\rho_\alpha(x^*, x) \geq 0$, then together with (6) we get $\rho_\alpha(x^*, x) = 0$ for all $\alpha \in (0,1]$. But we chose x from $X \setminus \{x^*\}$, that is, $\rho_\alpha(x^*, x) > 0$ for some $\alpha \in (0,1]$. This is a contradiction.

To complete the proof, one needs to show that Theorem 4 $\Rightarrow$ Theorem 1. If Theorem 1 does not hold, then for every $x \in S(x_0)$ there exists $y \in X \setminus \{x\}$ such that $x \lesssim y$. We shall form $f: S(x_0) \to X$ by $f(x) = y$.

If Theorem 4 holds, then there $x^* \in S(x_0)$ such that $f(x^*) = x^*$ which contradicts the supposition that $y \in X \setminus \{x\}$.

Lemma 2. *If (X,d,L,R) is a fuzzy metric space and the function $\phi: X\times X \to (-\infty,\infty]$ satisfies (1), (2) and (3), then for any $x\in S(x)$ and $\varepsilon>0$, there exists $y\in S(x)$ such that $\overline{d}_\alpha(S(y))<\varepsilon$ for every $\alpha\in(0,1]$.*

Proof: Let $x\in S(x)$. Then

$$\inf_{z\in S(x)} \phi(x,z) \geq \inf_{z\in S(x)} [\phi(x,z)-\phi(x,x)] \geq$$

$$\geq \inf_{z\in S(x)} \phi(x,z)-\phi(x,x) > -\infty,$$

that is, there exists $a\in R: \inf_{z\in S(x)}\phi(x,z)=a$. We shall choose $y\in S(x)$ such that $\phi(x,y)\leq a+\frac{\varepsilon}{2}$.

Similarly as in the previous case, we get

$$b=\inf_{z\in S(y)} \phi(y,z)\geq \inf_{z\in S(x)} \phi(x,z)-\phi(x,y)\geq -\frac{\varepsilon}{2}.$$

Finally, to finish the proof, it remains to show that $\overline{d}_\alpha(S(y))<\varepsilon$.

If $z_1,z_2\in S(y)$, then for every $\alpha\in(0,1]$,

$$\rho_\alpha(y,z_1)\leq -\phi(y,z_1)\leq\frac{\varepsilon}{2}\Leftrightarrow d(y,z_1)(\frac{\varepsilon}{2})<\alpha$$

$$\rho_\alpha(y,z_2)\leq -\phi(y,z_2)\leq\frac{\varepsilon}{2}\Leftrightarrow d(y,z_2)(\frac{\varepsilon}{2})<\alpha.$$

Since $\lim_{a\to 0} R(a,a)=0$ for every $\alpha\in(0,1]$ there exists $\beta\in(0,1]$ such that $R(\beta,\beta)<\alpha$. Then we get

$$d(z_1,z_2)(\varepsilon)\leq R(d(y,z_1)(\frac{\varepsilon}{2}),d(y,z_2)(\frac{\varepsilon}{2}))\leq R(\beta,\beta)<\alpha$$

which means that $\rho_\alpha(z_1,z_2)<\varepsilon$ for every $\alpha\in(0,1]$. Hence, $\overline{d}_\alpha(S(y))=$ $=\sup_{z_1,z_2\in S(y)}\rho_\alpha(z_1,z_2)\leq\varepsilon$ for all $\alpha\in(0,1]$.

Lemma 3. [5] *If (X,d,L,R) is a complete fuzzy metric space and the function $\phi: X\times X\to(-\infty,\infty]$ satisfies (1) and (2), then every nondecreasing Cauchy sequence has an upper bound in X.*

Theorem 8. *If (X,d,L,R) is a complete fuzzy metric space and the function $\phi: X\times X\to(-\infty,\infty]$ satisfies (1), (2) and (3), then the next six statements are equivalent:*

i. *there exists* $x^* \in S(\tilde{x})$ *such that for all* $y \in X \setminus \{x^*\}$

$\rho_\alpha(x^*, x) + \phi(x^*, x) > 0$ *for some* $\alpha \in (0,1]$,

ii. *if* $A \subset X$ *has the property that for every* $x^* \in S(\tilde{x}) \cap A$ *there exists* $y \in X \setminus \{x\}$ *such that* $x \lesssim y$, *then there exists* $x^* \in S(\tilde{x}) \cap A$,

iii. *if for every* $x \in S(x)$ *with* $\inf_{y \in X} \phi(x, y) < 0$ *there exists* $y \in X \setminus \{x\}$ *such that* $x \lesssim y$, *then there exists* $x^* \in S(\tilde{x})$ *such that* $\phi(x^*, y) \geq 0$ *for all* $y \in X$,

iv. *if* $f : X \to X$ *is a function satisfying* $x \lesssim f(x)$ *for all* $x \in X$, *then* f *has a fixed point* $x^* \in S(\tilde{x})$,

v. *if* $F : X \to 2^X \setminus \{0\}$ *is a multivalued mapping such that for every* $x \in X$ *and every* $y \in F(x)$, $x \lesssim y$, *then there exists* $x^* \in S(\tilde{x})$ *such that* $F(x^*) = \{x^*\}$,

vi. *if* $F : X \to 2^X \setminus \{0\}$ *is a multivalued mapping such that for every* $x \in S(x)$ $F(x)$ *there exists some* $y \in X \setminus \{x\}$ *for which* $x \lesssim y$, *then there exists* $x^* \in S(\tilde{x})$ *with* $x^* \in F(x^*)$.

Proof: Using Theorem 7, Lemma 2 and Lemma 3, we get Theorem 8.

4. THE VARIATIONAL PRINCIPLE AND ITS EQUIVALENTS IN PROBABILISTIC METRIC SPACES

The function $F : \mathbf{R} \to [0,1]$ ($\mathbf{R}$ denotes the set of reals) which is left continuous, nondecreasing with $\sup_{x \in R} F(x) = 1$, is a distribution function. Let D be the set of all distribution functions.

The triplet (X, F, t) where X is any set, $F : X \times X \to D$ is such that

$$F_{x,y} = F_{y,x} \quad \text{for all} \quad x, y \in X,$$
$$F_{x,y}(v) = 1 \quad \text{for all} \quad v > 0 \Leftrightarrow x = y,$$
$$F_{x,y}(0) = 0 \quad \text{for all} \quad x, y \in X,$$
$$F_{x,y}(v+u) \geq t(F_{x,z}(v), F_{z,y}(u)) \quad \text{for all } x, y, z \in X \text{ and } u, v \in R,$$

and $t : [0,1] \times [0,1] \to [0,1]$ is commutative, nondecreasing, associative and $t(a,1) = a$ for all $a \in [0,1]$, is a Menger space.

In [3] it was proved by Kaleva and Seikkala that every Menger space (X,F,t) is a fuzzy metric space (X,d,L,R) where $u_{x,y} = \sup\{v : F_{x,y}(v) = 0\}$ and

$$d(x,y)(u) = \begin{cases} 0 & , \quad u < u_{x,y} \\ 1 - F_{x,y}(u) & , \quad u \geq u_{x,y} \end{cases}$$

$$R(a,b) = 1 - t(1-a,1-b),\ a,b \in [0,1]$$

$$L \equiv 0$$

If $\phi : X \times X \to (-\infty,\infty]$ is a function satisfying (1), (2) and (3), then we define the relation $\preceq$ by

$$x \preceq y \Leftrightarrow F_{x,y}(u) \geq H(u + \phi(x,y)) \text{ for all } u > 0 \qquad (7)$$

where $H(u) = \begin{cases} 0 & u \leq 0 \\ 1 & u > 0 \end{cases}$. For every $x \in X$ by $S(x)$ we denote set $S(x) = \{y \in X : x \preceq y\}$.

Lemma 4. *The relation $\preceq$ defined by (7) is an order in X.*

Theorem 9. *Let (X,F,t) be a complete Menger space such that $\lim_{a \to 1} t(a,a) = 1$ and $\phi : X \times X \to (-\infty,\infty]$ be a function satisfying (1), (2) and (3). Then the statements (ii)-(vi) (from Theorem 8) and*

(i) there exists $x^* \in S(\tilde{x})$ such that for all $x \in X \setminus \{x^*\}$

$$F_{x,y}(u) < H(u + \phi(x,y)) \text{ for some } u \in R^+ ,$$

are equivalent.

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