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AN INTERVAL ARITHMETIC ALGORITHM FOR MULTIVARIATE CONSTRAINED GLOBAL OPTIMIZATION USING CORD-SLOPE FORMS OF TAYLOR'S EXPANSION*

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Dedicated to the memory of Professor Jovan Petrić

Abstract: A new algorithm, based on interval analysis, is proposed for global optimization of constrained nonlinear nonconvex functions of several variables. It exploits the cord-slope form of Taylor's expansion in several ways. Computational results, including a comparison with results of Sengupta are reported.

Keywords: Global optimization, interval arithmetic, Taylor expansion, cord-slope form.

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1. INTRODUCTION

Consider the problem

$$\min \quad f(x_1, x_2, x_3, \ldots, x_n) \tag{1}$$

$$\text{subject to} \quad g_i(x_1, x_2, x_3, \ldots, x_n) \le 0 \quad i = 1, 2, \ldots, m; \tag{2}$$

$$\ell_j \le x_j \le u_j \quad j = 1, 2, \ldots, n \tag{3}$$

where the objective function $f(x)$ and the constraint left-hand sides g_i are assumed to be continuous and twice differentiable on the region defined by (3). Let

$$S = \{x = (x_1, x_2, \ldots, x_n) \mid g_i(x) \le 0, i = 1, \ldots, m; \ell_j \le x_j \le u_j, j = 1, 2, \ldots, n\} \tag{4}$$

be the feasible region and $f^* = \min_{x \in S} f(x)$. Globally minimizing $f(x)$ subject to the constraints $x \in S$ may be defined as finding $\bar{x} \in S$ such that

$$| f(\bar{x}) - f^* | \le \varepsilon \tag{5}$$

where ε is a given small positive number.

Let us call this problem P. Several algorithms using interval arithmetic have been designed for this general problem. Fujii, Ichida, and Ozasa [1] propose to solve it by first converting all inequality constraints into equality constraints and introducing Lagrange multipliers to reduce the problem to an unconstrained one, then applying interval Newton's method to locate the globally optimal solution. A more sophisticated algorithm is due to Hansen [2], Hansen and Sengupta [3] and Sengupta [10]. They apply line searches to obtain a good incumbent solution, linearization of constraints, interval Newton's methods to solve a system of interval linearized inequalities and finally monotonicity tests and convexity tests to find the globally optimal solution. Other algorithms are described in Chapter 4 of Ratschek and Rokne [9].

We propose in this paper a new algorithm which uses the *cord-slope form of Taylor's expansion*, defined below, and interval arithmetic. The basic idea is to find linear functions in one variable that bound the function $f(x)$ and $f'(x)$ respectively. Those linear functions can be used to eliminate parts of the box that cannot contain any globally optimal solution. Similar bounding linear functions are found for the constraint functions $g_i(x)$ and used to eliminate parts of the box that are not feasible. This is done by applying various tests which consist in checking if a sufficient condition for some proposition about the problem (or the current subproblem) to hold is satisfied or not, and making use of this proposition in the former case. The tests can be classified as follows (Hansen, Jaumard, and Lu [6]). A *direct test* is such that the information provided when the sufficient condition is satisfied suffices to solve the current subproblem. This is the case when it can be shown: (i) that a known solution x^* is the globally optimal solution of the current subproblem (*direct solution test*); (ii) that the subproblem has no solution better than the incumbent (*direct optimality test*); (iii) that the subproblem has no feasible solution (*direct feasibility test*). A *conditional test* is such that when the sufficient condition holds, part of the feasible domain of the current subproblem can be eliminated.

The proposed algorithm can be considered as an extension of the algorithm for global minimization of univariate functions given in [4].

This paper is organized as follows: Background on interval arithmetic is recalled in Section 2. Cord-slope forms and various tests used in the algorithm are introduced in Section 3. The algorithm itself is presented in Section 4. An extension of the algorithm designed to locate all globally optimal solutions is described in Section 5. Computational results are reported in Section 6.

2. INTERVAL ARITHMETIC

Interval arithmetic was introduced by Moore [7] as a basic tool for control of numerical errors in machine computations. Instead of approximating a real value x by a machine representable number, as is done in real arithmetic, a pair of machine representable numbers is used representing an interval in which x lies. Arithmetic operations for intervals are defined as follows:

$$[a,b]+[c,d]=[a+c,\ b+d]; \tag{6}$$

$$[a,b]-[c,d]=[a-d,\ b-c]; \tag{7}$$

$$[a,b]\cdot[c,d]=[\min\{ac,ad,bc,bd\},\max\{ac,ad,bc,bd\}]; \tag{8}$$

$$[a,b]/[c,d]=[\min\{a/c,a/d,b/c,b/d\},\max\{a/c,a/d,b/c,b/d\}]. \tag{9}$$

These definitions are readily used to compute intervals containing the range of a rational function $f(x)$ for x belonging to an interval X. The simplest procedure is *natural extension* of $f(x)$. It consists in replacing each occurrence of variable x by the interval X and then applying the rules of interval arithmetic. Special operations for bounding trigonometric and transcendental functions allow to extend this procedure to analytical functions.

The bounds so obtained are not always precise, but often better ones can be obtained by exploiting Taylor's expansion, see Ratschek and Rokne [8] for a thorough survey and discussion.

While interval arithmetic was not initially designed for global optimization, it was soon understood that it could be effectively used to solve such problems. The many efforts made in that direction are summarized in the recent book of Ratschek and Rokne [9].

3. CORD-SLOPE FORMS OF TAYLOR'S EXPANSION AND THEIR USE IN GLOBAL OPTIMIZATION

3.1. Cord-slope Forms of Taylor's Expansion

Let X^0 denote the initial box. As we proceed with the algorithm, we will dynamically subdivide this box into subboxes. Let $X = X_1 \times X_2 \times \ldots \times X_n$ be the current subbox (initially $X = X^0$) and $c = (c_1, c_2, \ldots, c_n)$ the middle point of this subbox. Take any $i \in \{1, 2, \ldots, n\}$; using Taylor's expansion, we have:

$$\begin{aligned} f(x) &= f(x_1, \ldots, x_n) - f(c_1, x_2, \ldots, x_n) \\ &\quad + f(c_1, x_2, \ldots, x_n) - f(c_1, c_2, x_3, \ldots, x_n) \\ &\quad \vdots \\ &\quad + f(c_1, c_2, \ldots, c_{n-1}, x_n) - f(c_1, c_2, \ldots, c_n) \\ &\quad + f(c_1, c_2, \ldots, c_n) \\ &= f(c_1, c_2, \ldots, c_n) + f_1'(\xi_1, x_2, \ldots, x_n)(x_1 - c_1) \\ &\quad + f_2'(c_1, \xi_2, x_3, \ldots, x_n)(x_2 - c_2) + \cdots + f_n'(c_1, c_2, \ldots, \xi_n)(x_n - c_n) \\ &\quad (\text{for some } \xi_1 \in X_1, \xi_2 \in X_2, \ldots, \xi_n \in X_n) \qquad (10) \\ &= r_{f_i}^1(x) + (x_i - c_i) q_{f_i}^1(x) \qquad (11) \end{aligned}$$

where

$$r_{f_i}^1(x) = f(c) + \sum_{k=1, k \neq i}^{n} f_k'(c_1, \ldots, c_{i-1}, \xi_i, x_{i+1}, \ldots, x_n), \quad q_{f_i}^1(x) = f_i'(c_1, \ldots, c_{i-1}, \xi_i, x_{i+1}, \ldots, x_n).$$

The second-order Taylor's expansion gives:

$$f(x) = f(c) + \sum_{k=1}^{n} f_k'(c)(x_k - c_k) + \frac{1}{2}(x - c)H(x, c, \eta)(x - c)^T \qquad (12)$$

where $x - c = (x_1 - c_1, x_2 - c_2, \ldots, x_n - c_n)$ and the elements of $H(x, c, \eta) = (h_{ij}(x, c, \eta))$ are defined as:

$$h_{ij} = \begin{cases} \partial^2 f / \partial x_i^2 (c_1, c_2, \ldots, c_{j-1}, \eta_{ij}, x_{j+1}, \ldots, x_n) & \text{if } i = j \quad (i = 1, \ldots, n), \\ 2\partial^2 f / \partial x_i \partial x_j (c_1, c_2, \ldots, c_{j-1}, \eta_{ij}, x_{j+1}, \ldots, x_n) & \text{if } j < i \quad (i = 1, \ldots, n;\ j = 1, \ldots, i-1), \\ 0 & \text{otherwise;} \end{cases}$$

where $\eta_{ij} \in X_j$. Grouping the terms that contain the factor $(x_i - c_i)$ together, we have

$$f(x) = f(c) + \sum_{k=1,k\neq i}^{n} (x_k - c_k)(f_k'(c) + \frac{1}{2}h_{kk}(x_k - c_k))$$
$$+ \sum_{k=1,k\neq i}^{n} \sum_{j=1,j\neq i}^{k-1} h_{kj}(x_k - c_k)(x_j - c_j)$$
$$+(x_i - c_i)(f_i'(c) + \frac{1}{2}h_{ii}(x_i - c_i) + \sum_{j=1,j\neq i}^{n} h_{ij}(x_j - c_j)).$$
$$= r_{f_i}^2(x) + (x_i - c_i)q_{f_i}^2(x) \tag{13}$$

where

$$r_{f_i}^2(x) = f(c) + \sum_{k=1,k\neq i}^{n} (x_k - c_k)(f_k'(c) + \frac{1}{2}h_{kk}(x_k - c_k))$$
$$+ \sum_{k=1,k\neq i}^{n} \sum_{j=1,j\neq i}^{k-1} h_{kj}(x_k - c_k)(x_j - c_j)$$
$$q_{f_i}^2(x) = f_i'(c) + \frac{1}{2}h_{ii}(x_i - c_i) + \sum_{j=1,j\neq i}^{n} h_{ij}(x_j - c_j)$$

Both (11) and (13) express the function $f(x)$ in the form:

$$f(x) = r(x) + (x_i - c_i)q(x). \tag{14}$$

We will call the expression (14) a *cord-slope form* of $f(x)$, and in particular call (11) the *first order cord-slope* form of $f(x)$ and (13) the *second order cord-slope* form of $f(x)$.

Similarly we have the first order cord-slope form of $f'(x)$:

$$f_i'(x) = f_i'(c_1,c_2,\ldots,c_n) + \sum_{j=1}^{n} f_{ij}''(c_1,c_2,\ldots,c_{j-1},\xi_{ij},x_{j+1},\ldots,x_n)(x_j - c_j)$$
$$= r_{f_{ik}'}^1(x) + (x_k - c_k)q_{f_{ik}'}^1(x) \tag{15}$$

and the first and second order cord-slope forms of $g_i(x)$ for $i = 1,2,\ldots,m$.

3.2. Use of the Cord-slope Forms in Global Optimization

The cord-slope forms can be readily used to eliminate parts of the current subbox that cannot contain the global minimum. In favorable cases these parts will in fact be the whole current subbox. The basic results are the following:

Theorem 1. *Suppose $f(x)$ can be expressed in a cord-slope form as:*

$$f(x) = r(x) + (x_i - c_i)q(x) \tag{16}$$

over the box X. Let $[r_\ell, r_u] = r(X)$ *be an inclusion interval of* $r(x)$ *over X and* $[q_\ell, q_u]$ *be an inclusion interval of* $q(x)$ *over X, then for any fixed number* $\bar{f}$*, we have:*

Case 1 $q_\ell \geq 0$,

i) if $\bar{f} - r_\ell \geq 0$, $f(x) > \bar{f}$ *for all x such that* $x_i \in \bar{X}_i = X_i \cap (c_i + \frac{\bar{f} - r_\ell}{q_\ell}, +\infty]$ *if* $q_\ell > 0$;

ii) if $\bar{f} - r_\ell \leq 0$, $f(x) > \bar{f}$ *for all x such that* $x_i \in \bar{X}_i = X_i \cap (c_i + \frac{\bar{f} - r_\ell}{q_u}, +\infty]$;

iii) if $\bar{f} - r_u \geq 0$, $f(x) < \bar{f}$ *for all x such that* $x_i \in \hat{X}_i = X_i \cap [-\infty, c_i + \frac{\bar{f} - r_u}{q_u})$,

iv) if $\bar{f} - r_u \leq 0$, $f(x) < \bar{f}$ *for all x such that* $x_i \in \hat{X}_i = X_i \cap [-\infty, c_i + \frac{\bar{f} - r_u}{q_\ell})$ *if* $q_\ell > 0$;

Case 2 $q_u \leq 0$,

i) if $\bar{f} - r_\ell \geq 0$, $x) > \bar{f}$ *for all x such that* $x_i \in \bar{X}_i = X_i \cap [-\infty, c_i + \frac{\bar{f} - r_\ell}{q_u})$,

ii) if $\bar{f} - r_\ell \leq 0$, *then* $f(x) > \bar{f}$ *for all x such that* $x_i \in \bar{X}_i = X_i \cap [-\infty, c_i + \frac{\bar{f} - r_\ell}{q_\ell})$;

iii) if $\bar{f} - r_u \geq 0$, $f(x) < \bar{f}$ *for all x such that* $x_i \in \hat{X}_i = X_i \cap (c_i + \frac{\bar{f} - r_u}{q_\ell}, +\infty]$,

iv) if $\bar{f} - r_u \leq 0$, $f(x) < \bar{f}$ *for all* $x_i \in \hat{X}_i = X_i \cap (c_i + \frac{\bar{f} - r_u}{q_u}, +\infty]$ *if* $q_u < 0$;

Case 3 $q_\ell < 0 < q_u$.

i) if $\bar{f} - r_\ell \leq 0$, $f(x) > \bar{f}$ *for all x such that* $x_i \in \bar{X}_i = X_i \cap (c_i + \frac{\bar{f} - r_\ell}{q_u}, c_i + \frac{\bar{f} - r_\ell}{q_\ell})$;

ii) if $\bar{f} - r_u \geq 0$, *then* $f(x) < \bar{f}$ *for all x such that* $x_i \in \hat{X}_i = X_i \cap (c_i + \frac{\bar{f} - r_u}{q_u}, c_i + \frac{\bar{f} - r_u}{q_\ell})$.

This theorem can be proved in a straightforward way. For example, in *i)* of **Case 1**, when $x_i \in \bar{X}_i = X_i \cap (c_i + \frac{\bar{f} - r_\ell}{q_\ell}, +\infty]$ we have:

$$\begin{aligned} f(x) &= r(x) + (x_i - c_i)q(x) \\ &> r_\ell + \frac{\bar{f} - r_\ell}{q_\ell} q_\ell \\ &= \bar{f} \end{aligned}$$

Before we state all the tests using the cord-slope form, we define first what we call a *strictly feasible subbox*. We say that a subbox X is strictly feasible, if we have found that the upper bounds $(g_i)_u$ on the constraint functions $g_i(X)$ are negative for all $i \in \{1,2,\ldots,m\}$ and all the boundaries of the current subbox differ from the boundaries of the original box X^0.

Theorem 1 can be applied to find the global optimum of $f(x)$ in the following ways:

i) use the cord-slope forms of $f(x)$ and take $\bar{f}$ to be the incumbent optimal value; $\overline{X}_i$ can be eliminated from X_i since for all x such that $x_i \in \overline{X}_i$ the function value is larger than $\bar{f}$;

ii) if the current subbox X is strictly feasible, use the cord-slope forms of $f'_k(x)$ and take $\bar{f}'_k$ to be zero, both $\overline{X}_i$ and $\hat{X}_i$ can be eliminated from X_i since the derivative $f'_k(x)$ cannot be zero for all x such that $x_i \in \overline{X}_i$ or $x_i \in \hat{X}_i$;

iii) finally use the cord-slope forms of the left hand side constraint functions $g_i(x)$ and take $\bar{g}_i$ to be zero, $\overline{X}_i$ can be eliminated from X_i since all x such that $x_i \in \overline{X}_i$ are infeasible.

Now we state all the tests using the cord-slope form specifically as follows:

1. **Direct Optimality Test.** Compute the values $f'_i(c)$, $i = 1,2,\ldots,n$ and the inclusion intervals $F'_i = f'_i(c_1,c_2,\ldots,c_{i-1},X_i,\ldots,X_n)$, $H_{ij} = h_{ij}(c_1,c_2,\ldots,c_{j-1},X_j,\ldots,X_n)$, $j = 1,2,\ldots,i$, $i = 1,2,\ldots,n$. Compute the range $[f_l^0, f_u^0] = f(X)$, $[f_l^1, f_u^1] = f(c_1,c_2,\ldots,c_n) + \sum_{i=1}^{n} F'_i(X_i - c_i)$ and $[f_l^2, f_u^2] = f(c) + \sum_{k=1}^{n}(f'_k(c) + \frac{1}{2}H_{kk}(X_k - c_k))(X_k - c_k) + \sum_{i=1}^{n}\sum_{j=1}^{k} H_{ij}(X_i - c_i)(X_j - c_j)$, let $f_l = \max\{f_l^0, f_l^1, f_l^2\}$. If $f_l > f_{opt} - \varepsilon$, the entire box can be eliminated from further consideration.

   Proof: Because of (10) and (12), f_l is a lower bound of $f(x)$ on box X. Therefore if $f_l > f_{opt} - \varepsilon$, $f(x) > f_{opt} - \varepsilon$ for all x in X, so the entire box can be eliminated.

2. **Direct Feasibility Test.** Apply the same procedure (with f replaced by g_i) as in 1 to obtain an inclusion interval $[(g_i)_l, (g_i)_u]$ for each constraint left-hand side g_i. If $(g_i)_l > 0$ for some i, the entire box X can be eliminated.

   Proof: Similarly as in 1, $(g_i)_l$ is a lower bound of $g_i(x)$ on X. Therefore if $(g_i)_l > 0$, there is no feasible point in the box X, so it can be eliminated from further consideration.

3. **Monotonicity Tests (for Strictly Feasible Subboxes).** If X is strictly feasible, for all $i \in \{1,2,\ldots,n\}$, find the inclusion intervals $[(f_i')_\ell, (f_i')_u]$ on the derivatives $f_i'(x)$. If $(f_i')_\ell > 0$ or $(f_i')_u < 0$, the subbox X can be eliminated.

   Proof: If $(f_i')_\ell > 0$ or $(f_i')_u < 0$, the function is monotonous on this subbox with respect to x_i, since the subbox is strictly feasible, it cannot contain any globally optimal solution, so it can be deleted.

4. **First Conditional Optimality Test (for All Subboxes).** For all i in $\{1,2,\ldots,n\}$, compute the inclusion intervals, $[(r_f^1)_\ell, (r_f^1)_u]$, $[(q_f^1)_\ell, (q_f^1)_u]$, $[(r_f^2)_\ell, (r_f^2)_u]$ and $[(q_f^2)_\ell, (q_f^2)_u]$, and let $\bar{f} = f_{opt} - \varepsilon$. Then the subinterval $\overline{X}_i$ as defined in Theorem 1 can be eliminated from X_i.

   Proof: Because of Theorem 1, for all x such $x_i \in \overline{X}_i, f(x) > \bar{f} = f_{opt} - \varepsilon$, therefore $\overline{X}_i$ can be eliminated.

5. **Second Conditional Optimality Test (for Strictly Feasible Subboxes).** If X is strictly feasible, for all $k \in \{1,2,\ldots,n\}$, we can further compute inclusion intervals $[(r_{f_k'}^1)_\ell, (r_{f_k'}^1)_u]$ and $[(q_{f_k'}^1)_\ell, (q_{f_k'}^1)_u]$. Let $\bar{f}_k' = 0$. Both $\overline{X}_i$ and $\hat{X}_i$ as defined in Theorem 1 with f replaced by f_k' can be eliminated from X_i.

   Proof: Because of Theorem 1, for all x such that $x_i \in \overline{X}_i$ or $x_i \in \hat{X}_i, f_k'(x)$ cannot be zero, therefore x cannot be a global optimum and can be eliminated.

6. **Conditional Feasibility Test.** For all $i \in \{1,2,\ldots,n\}$, apply the same procedure as in 4 with f replaced by g_i and $\bar{f}$ replaced by 0, then eliminate $\overline{X}_i$ from X_i.

   Proof: From Theorem 1, for all x such that x_i in $\overline{X}_i, g_i(x) > 0$, therefore $\overline{X}_i$ cannot contain any feasible point and can be eliminated.

4. ALGORITHM

Rules of the algorithm are as follows. All steps are performed in sequential order except when branching takes place.

Step 1. Start with the initial box X^0. Let $c = (c_1, c_2, \ldots, c_n)$ be the middle point of this subbox, and compute the values $f_i'(c), i = 1,2,\ldots,n$ and the inclusion intervals $F_i' = f_i'(c_1, c_2, \ldots, c_{i-1}, X_i^0, \ldots, X_n^0)$, $H_{ij} = h_{ij}(c_1, c_2, \ldots, c_{j-1}, X_j^0, \ldots, X_n^0)$, $j = 1,2,\ldots,i$, and $i = 1,2,\ldots,n$. Compute the range $[f_\ell^0, f_u^0] = f(X^0)$, $[f_\ell^1, f_u^1] = f(c_1, c_2, \ldots, c_n) + \sum_{i=1}^{n} F_i'(X_i^0 - c_i)$ and $[f_\ell^2, f_u^2] = f(c) + \sum_{k=1}^{n} (f_k'(c) + \frac{1}{2} H_{kk}(X_k^0 - c_k))$

$(X_k^0 - c_k) + \sum_{i=1}^{n} \sum_{j=1}^{k} H_{ij}(X_i^0 - c_i)(X_j^0 - c_j)$, let $f_\ell = \max\{f_\ell^0, f_\ell^1, f_\ell^2\}$. Store the pair (X^0, f_ℓ) into a list L, initialize $x_{opt} = 0$ and $f_{opt} = \infty$;

Step 2. If L is empty, stop; otherwise take the last pair from list L, denote it by (X, f_X), and delete it from the list. Let $X = X_1 \times X_2 \times \cdots \times X_n = [l_1, u_1] \times [l_2, u_2] \times \ldots \times [l_n, u_n]$, and $c = (c_1, c_2, \ldots, c_n)$ be the middle point of X. If $f_X > f_{opt} - \varepsilon$, stop and report x_{opt} and f_{opt}; otherwise compute $f(c)$, check if c is feasible; if c is feasible and $f(c) < f_{opt}$, update $x_{opt} = c$ and $f_{opt} = f(c)$.

Step 3. Apply the Direct Optimality Test described in the previous section. If X is eliminated, return to **Step 2**.

Step 4. Apply the Direct Feasibility Test. If X is eliminated, return to **Step 2**.

Step 5. Apply the First Conditional Optimality Test to eliminate part of X_i for all $i = 1, 2, \ldots, n$.

Step 6. Apply the Conditional Feasibility Test to eliminate part of X_i for all $i = 1, 2, \ldots, n$.

Step 7. Check if the subbox X is strictly feasible. If it is, apply the Second Conditional Optimality Test to eliminate part of X_i for all $i = 1, 2, \ldots, n$.

Step 8. If for some i, the entire X_i has been eliminated, return to **Step 2**; otherwise partition the subbox X into subboxes $X^1, X^2, \ldots, X^k$ as described below. For each subbox X^t, $t = 1, 2, \ldots, k$, let $c = (c_1, c_2, \ldots, c_n)$ be the middle point and compute the values $f_i'(c)$, $i = 1, 2, \ldots, n$ and the inclusion intervals $F_i' = f_i'(c_1, c_2, \ldots, c_{i-1}, X_i^t, \ldots, X_n^t)$, $H_{ij} = h_{ij}(c_1, c_2, \ldots, c_{j-1}, X_j^t, \ldots, X_n^t)$, $j = 1, 2, \ldots, i$, $i = 1, 2, \ldots, n$. Compute the range $[f_\ell^0, f_u^0] = f(X^t)$, $[f_\ell^1, f_u^1] = f(c_1, c_2, \ldots, c_n) + \sum_{i=1}^{n} F_i'(X_i^t - c_i)$ and $[f_\ell^2, f_u^2] = f(c) + \sum_{k=1}^{n} (f_k'(c) + \frac{1}{2} H_{kk}(X_k^t - c_k))(X_k^t - c_k) + \sum_{i=1}^{n} \sum_{j=1}^{k} H_{ij}(X_i^t - c_i)(X_j^t - c_j)$, let $f_\ell = \max\{f_\ell^0, f_\ell^1, f_\ell^2\}$. If $f_\ell > f_{opt} - \varepsilon$, discard X^t; otherwise insert the pair (X^t, f_ℓ) into the list L so that the second member of all elements in L is in a descending order, return to **Step 2**.

The partitioning step in **Step 8** is carried out as follows: suppose that after elimination, the box X has been reduced to X'. If no gaps are present in all components of X', let X_k be the component of X' with largest width; bipartition X_k

into two and consequently bipartition the box X' into two subboxes. If gaps are present in some components in X', take the component such that the total length of the two largest gaps in this component are the largest and partition the box X' according to the two largest gaps in this component. In this way, we will only divide one component and generate at most three subboxes at one iteration, thus preventing generation of too many small subboxes.

To assure finite convergence, the following rule is also applied on the partition step. We define the pth parent of a subbox X (where p is an integer greater than or equal to one) as follows: a subbox X' is called the pth parent of subbox X if X is obtained from X' after applying p iterations of the tests. At each iteration, the largest width $w(X)$ of the subbox X under consideration is compared with the width $w(X')$ of its pth parent X'; if $w(X) \geq \frac{1}{2}w(X')$, we bipartition X along the direction parallel to which X has an edge of maximum length into two subboxes.

5. CONVERGENCE OF THE ALGORITHM

Assume that the functions $f(x)$ and $g_i(x)$, their first and second derivatives are all bounded. Also assume that the problem is feasible and has only finitely many local minima. At iteration k, let $X_{k,1}, X_{k,2}, \ldots, X_{k,n_k}$ be the intervals in the list L and $Y_{k,1}, Y_{k,2}, \ldots, Y_{k,n_k}$ be the corresponding ranges for the function values obtained in the algorithm. Let $X^k = \cup_{j=1}^{n_k} X_{k,j}$ be the union of all the boxes left and X_{k,n'_k} be the last box in the list L that contains at least one feasible point, then:

Lemma 1. $f^* \in Y_{k,n'_k}$ *for all* k.

Proof: As proved in Section 3, none of the tests remove the globally optimal solution x^* from consideration, therefore $f^* = \min_{x \in X^k \cap S} f(x)$ where S is the feasible set. Let $Y_{k,n'_k} = [y_{kl}, y_{ku}]$ and $\hat{x} \in X_{k,n'_k}$ be a feasible point, we have $f^* \leq f(\hat{x}) \leq y_{ku}$. Since there is no feasible point in $X_{k,j}$ when $j > n'_k$ and the lower endpoints of $Y_{k,j}, j = 1, \ldots, n_k$ are in a descending order, $y_{kl} \leq \min_{x \in X^k \cap S} f(x) = f^*$. Therefore $f^* \in [y_{kl}, y_{ku}] = Y_{k,n'_k}$.

Lemma 2. $\lim_{k \to \infty} w(X_{k,n'_k}) = 0$

Proof: Let T be the set of all subboxes obtained during application of the algorithm, i.e. $T = \{X_{i,j}, i = 1,2,\ldots, j = 1,2,\ldots,n_i\}$. Suppose this lemma is not true. Then for some constant $d > 0$, we have infinitely many subboxes in T with width larger than or equal to d. Let T' be the list of all those subboxes ordered according to their order of appearance, i.e. the subboxes obtained in earlier iterations listed before the subboxes obtained in later iterations. Let $T'' = \{X^{i_1}, X^{i_2}, \ldots, X^{i_j}, \ldots\}$ be a sublist of T' so that X^{i_j} is the $(p+1)$ th parent of $X^{i_{j+1}}$, then we have $d(X^{i_{n+1}}) \leq \frac{1}{2} w(X^{i_1})$. This can be proved as follows. Let $l_1 \geq l_2 \geq \cdots \geq l_n$ be the width of each component of X^{i_1} and let $X^{i'_2}$ be the first parent of X^{i_2}. Then X^{i_1} is the pth parent of $X^{i'_2}$. If $d(X^{i'_2}) \leq \frac{1}{2} d(X^{i_1})$, we have $d(X^{i_{n+1}}) \leq d(X^{i_2}) \leq d(X^{i'_2}) \leq \frac{1}{2} d(X^{i_1})$ since the width of a subbox cannot be greater than that of its parents. If $d(X^{i'_2}) \geq \frac{1}{2} d(X^{i_1})$, according to our rule of partitioning, $X^{i'_2}$, is bipartitioned in the direction parallel to which it has an edge of maximum length, therefore X^{i_2} as a subbox obtained from such a bipartition, has the property that $d(X^{i_2}) \leq \max\{\frac{l_1}{2}, l_2, \ldots, l_n\}$. Using induction, it can be proved that $d(X^{i_{n+1}}) \leq \max\{\frac{l_1}{2}, \frac{l_2}{2}, \ldots, \frac{l_n}{2}\} = \frac{l_1}{2} = \frac{w(X^{i_1})}{2}$. Therefore we have $w(X^{i_{(n+1)j}}) \leq \frac{1}{2^j} w(X^{i_1})$. This contradicts the assumption that $w(X^{i_{(n+1)j}}) > d$.

Lemma 3. $\lim_{k \to \infty} w(Y_{k,n'_k}) = 0$

Proof: Since $f(x)$ is continuous, we have

$$\lim_{k \to \infty} w(f(X_{k,n'_k})) = 0.$$

The inclusion interval Y_{k,n'_k} obtained in the algorithm is at least as good as the inclusion interval obtained by using the mean value form (Ratschek and Rokne [9]) which has convergence order 2; we have

$$\lim_{k \to \infty} (w(Y_{k,n'_k}) - w(f(X_{k,n'_k}))) = 0.$$

which proves the lemma.

The following theorem follows immediately from Lemma 1 and Lemma 3:

Theorem 2. *The algorithm converges in finitely many iterations to a globally ε-optimal solution.*

6. BOUNDING ALL GLOBALLY OPTIMAL SOLUTIONS

Let $f^* = \min_{x \in S} f(x)$ be the minimum function value and $X^* = \{y \mid y \in S, f(y) = f^*\}$ be the set of all globally optimal solutions. The algorithm presented in the previous section aims at finding only one such solution. However, it can be easily modified to find all of them in the sense of solving one of the following two problems (assuming that the number of globally optimal solutions is finite).

Problem Q : Find disjoint subboxes $I^k, k \in K$ such that $X^* \subseteq \bigcup_{k \in K} I^k$ and

$$\sum_{k \in K} w(I^k) \leq \varepsilon' .$$

Problem Q′ : Find disjoint subintervals $I^k, k \in K'$, containing only globally ε'-optimal points and such that

$$X^* \subseteq \bigcup_{k \in K'} I'_k .$$

To solve these two problems, we need to set the parameter ε in the algorithm to be zero, so that no globally optimal solutions will be eliminated. Other than that, we only need to change the termination criteria as follows:

To solve Problem Q, the termination criteria shall be changed so that the algorithm stops when the summation of the widths of the remaining intervals is less that ε'.

To solve Problem Q', the algorithm works in two phases. The first phase stops when a globally $\frac{\varepsilon'}{2}$-optimal solution is found, i.e., when the difference of the incumbent optimal solution and the lower bound obtained on the function value is less than $\frac{\varepsilon'}{2}$. In the second phase, at each iteration we put aside the subintervals on which the upper bounds of the function is at most $\frac{\varepsilon'}{2}$ away from the value of the best solution found so far. Those intervals contain only globally ε'-optimal solution and can be taken as part of the solution set. We continue evaluating the remaining intervals until all of them have been included in the solution set.

The second phase can be modified to work in another way: after a globally $\frac{\varepsilon'}{2}$-optimal solution is found, we can store the remaining intervals in a list so that the upper bound of the function on the intervals are in a descending order, and we always choose the remaining interval with largest function upper bound to iterate until all the upper bounds of the function on the remaining intervals are at most $\frac{\varepsilon'}{2}$ away from the optimal solution.

These ways to solve Problems Q and Q' parallel those proposed by Hansen, Jaumard and Lu [5] for global optimization of univariate Lipschitz functions.

7. COMPUTATIONAL EXPERIENCE

The algorithm has been implemented in Fortran 77 and tested on a SPARC station 2 with a 28.5 mips central processor. The test problems used are the constrained optimization problems from Sengupta ([10]):

Problem COP1: Three-hump Camel-back Function ([10])

$$\text{minimize} \quad f(x) = 2x_1^2 - 1.05x_1^4 + \frac{1}{6}x_1^6 + x_1x_2 + x_2^2 \tag{17}$$

$$\begin{aligned} \text{subject to} \quad & 16x_1^2 + 25x_1^2 \le 400 \\ & 13x_1^3 - 145x_1 + 84x_2 \le 252 \\ & x_1x_2 \le 4 \\ & -1 \le x_i \le 2,\ i = 1,2 \end{aligned}$$

Problem COP2: Rosenbrock or Banana Function ([10])

$$\text{minimize} \quad f(x) = \alpha(x_2 - x_1^2)^2 + (1 - x_2)^2 \tag{18}$$

$$\begin{aligned} \text{subject to} \quad & x_1^2 + x_2^2 \le 4 \\ & x_1x_2 \le 3 \\ & -4 \le x_i \le 4,\ i = 1,2 \end{aligned}$$

where α is a parameter set to be 1.0 during our test.

Problem COP3: Hansen's problem ([10])

$$\text{minimize} \quad f(x) = x_1^2 - x_2^2 \tag{19}$$

$$\begin{aligned} \text{subject to} \quad & -x_1^2 - (x_2 - 1.7)^2 + 1 \le 0 \\ & 30x_1 - 4(x_2 - 2)^2 - 10 \le 0 \\ & 0 \le x_i \le 1,\ i = 1,2 \end{aligned}$$

Problem COP4: Hansen's Variable Dimensional Problem

$$\text{minimize} \quad f(x) = \sum_{k=1}^{m} (k(x_k - 1)^2 + (x_k - 1)^4) \tag{20}$$

$$\begin{aligned} \text{subject to} \quad & 2x_i - 3 - \sum_{k=1}^{2} x_k^2 \le 0 \\ & -4 \le x_i \le 4,\ i = 1,2,\ldots,m \end{aligned}$$

where m is set to be 2 for our test.

Our algorithm has three options, corresponding to whether we want to solve Problem P or Problem Q or Problem Q'. We compared our algorithm for problem Q with Sengupta's algorithm, which also solves problem Q. The tolerance for the function value is set to be 10^{-5}, i.e., the final solution value f_{opt} shall be at most 10^{-5} away from f^*; the widths of all the remaining subboxes are required to be smaller than 10^{-4}. Computational results are listed in Table 1, together with those reported by Sengupta [10]. Sengupta's computations were done on a CDC CYBER 174 machine.

Table 1: Computational Results on Sengupta's Test Problems

Problem	New Algorithm		Sengupta's Algorithm	
	No. of Iterations	Computing Time	No. of Iterations	Computing Time
COP1	114	0.71	101	19
COP2	257	1.79	388	83
COP3	35	0.18	64	11
COP4	67	0.64	49	15

Observe that computing times are much smaller than those of Sengupta (although the difference of computers used must be taken into account; note that SPARC is a work station while CDC CYBER 174 is a large mainframe).

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