Yugoslav Journal of Operations Research
8 (1998), Number 1, 25-43

Jovan Petrić memorial volume

ABOUT SOME PROBLEMS OF DISJUNCTIVE PROGRAMMING*

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Dedicated to light memory of Professor Jovan Petrić

Abstract: In this paper we analyse algebra (operations and transformations) and geometry of the class of continuous piecewise linear functions (k*-functions*), in particular, their universal representativity and the algorithms reducing them from one representation to another. For the general piecewise linear programming problem, the dual is formed and the corresponding duality theorem is presented, the method of exact penalty function is grounded, and the saddle point theorems for the disjunctive Lagrangian are proved. It is noted that the logical part of algorithmic tools to solve k-problems can be implemented as a universal computer code allowing the formation and solution of the concluding family of standard linear programs, one of which gives the solution to the original k-problem.

Keywords: Piecewise linear functions, algebra of transformations, saddle point and duality theorems.

1. INTRODUCTION

Piecewise linear programming study [1 - 6] in a natural way leads us to some new settings of optimization problems, namely, to the problems of *disjunctive* programming. We begin with the fact that an arbitrary continuous piecewise linear function (k-function) defined on $\mathbf{R}^n$, allows some standard representation form

$$f(x) = \min_{(j)} \mid A_j x - b_j \mid_{\max}, \tag{1.1}$$

* This research was supported by the Russian Fund for Fundamental Researching, grant numbers 96-01-00116 and 96-15-96247.

where A_j are matrices, $x \in \mathrm{R}^n$, $b_j \in \mathrm{R}^{m_j}$, and $|\, a \,|_{\max}$ denotes the maximal coordinate of the vector a. Thus the inequality $f(x) \leq 0$ determines the set $M = \bigcup_{(j)} M_j$, $M_j = \{\, x \mid A_j x \leq b_j \,\}$.

An arbitrary finite system of k-function inequalities can be reduced (constructively) to a single inequality with a function of the same type (1.1). Therefore, we can write an arbitrary piecewise linear programming problem in the following standard form

$$\max \{\, (c,x) \mid f(x) \leq 0 \,\}, \tag{1.2}$$

where $f(x)$ is as (1.1).

In contrast with the traditional point of view of a feasible set as the *intersection* of a finite number of some sets (halfspaces, simple convex sets and others), the feasible set in (1.2) is the *union* of some sets (polyhedral sets), namely, $M = \{\, x \mid f(x) \leq 0 \,\} = \bigcup_{(j)} M_j$, where $M_j = \{\, x \mid A_j x \leq b_j \,\}$.

In the general framework, let $\{\, M_j \,\}_1^m \subset \mathrm{R}^n$ and $f(x)$ be an arbitrary function defined on R^n. Let us write two problems

$$P_{\wedge} : \max \{\, f(x) \mid x \in \bigcap_{j=1}^{m} M_j \,\}, \tag{1.3}$$

$$P_{\vee} : \max \{\, f(x) \mid x \in \bigcup_{j=1}^{m} M_j \,\}. \tag{1.4}$$

It is natural if the first one is named the *conjunctive* form and the second one is named the *disjunctive* form of the optimization problem. The form (1.3) is very common in mathematical programming. The form (1.4) is rather natural for the piecewise (linear and nonlinear) programming problem (1.2).

In (1.3)-(1.4), let us set $M_j = \{\, x \mid F_j(x) \leq 0 \,\}$, $F_j : \mathrm{R}^n \to \mathrm{R}^{m_j}$, $j = 1,\ldots,m$. The Lagrangian corresponding to $P_{\wedge}$ takes the well-known form

$$\Phi_{\wedge}(x,u) = f(x) - \sum_{j=1}^{m} (u_j, F_j(x)).$$

The Lagrangian for $P_{\vee}$ may be defined as

$$\Phi_{\vee}(x,u) = f(x) - \min_{(j)} (u_j, F_j(x)), \tag{1.5}$$

we refer to it as the *disjunctive* one.

The following scheme illustrates the correspondence between the problems $P_{\cap}$, $P_{\cup}$ and their Lagrangians

$$P_{\cap} \to \Phi_{\cap}(x,u) = f(x) - \sum_{j=1}^{m}(u_j, F_j(x)),$$

$$P_{\cup} \to \Phi_{\cup}(x,u) = f(x) - \min_{(j)}(u_j, F_j(x)).$$

But, to obtain symmetry, it would be more convenient to associate with $P_{\cap}$ another Lagrangian $\Phi(x,u) = f(x) - \max_{(j)}(u_j, F_j(x))$.

Thus, we see that the piecewise linear programming problem (k-problem) in the standard form (1.2) leads us to the disjunctive Lagrangian

$$L(x,u) = (c,x) - \min_{(j)}(u_j, A_j x - b_j). \tag{1.6}$$

Many questions arising in the study of k-problems may be connected with this function. Some of them are investigated in this paper.

The algebra of k-functions and k-problems allows some extensions of problem settings, at least to the so-called *σ-extension* of functional spaces. We shall work with the special algorithmic extension of original functional space $\mathbf{F}_0$, namely, with the extension to minimal functional space $\mathbf{F}$ closed with respect to the *discrete maximum* operation, i.e. $\{ f_j(x) \} \subset \mathbf{F}$ implies $\max_{(j)} f_j(x) \in \mathbf{F}$.

For example, if $\mathbf{F}_0$ is a class of all linear functions then $\mathbf{F}$ is a class of all k-functions, if $\mathbf{F}_0$ is a class of all quadratic functions then $\mathbf{F}$ is a class of piecewise quadratic functions and so on. It makes it possible to study k-programming problems and disjunctive optimization problems connected with them outside the framework of only linear settings.

Although piecewise functions and the corresponding mathematical tools are of importance, works on these topics are rare. We mention papers [1 - 6], especially the first, which contains a rather solid investigation of the algebra and geometry of k-functions.

2. σ-EXTENSIONS OF FUNCTIONAL SPACES

Let $\mathbf{F}_0$ be some functional space whose elements are functions defined on real space X. If $\{ f_j(x) \}_{j \in J} \subset \mathbf{F}_0$, then the function of the *discrete maximum* $f(x) := \max_{j \in J} f_j(x)$ may either belong to $\mathbf{F}_0$ or not. The way we generate $f(x)$ is said to be the *σ-operation*.

Let us consider the minimal extension of space $\mathbf{F}_0$ to some space $\mathbf{F}$ with the property of *σ-closure*:

$$\{ f_j(x) \}_{j\in J} \subset \mathbf{F} \Rightarrow \max_{j\in J} f_j(x) \in \mathbf{F} . \tag{2.1}$$

Of course, in this situation the property of linear closure holds too:

$$\{ f_j^i \mid i\in I, j\in J_i \} \subset \mathbf{F} \Rightarrow \sum_{i\in I}\alpha_i \max_{j\in J_i} f_j^i \subset \mathbf{F} , \tag{2.2}$$

where $\alpha_i \subset \mathbf{R}$, $i\in I$.

We shall call the minimal σ-closed extension of some space simply its *σ-extension*. Obviously, the meaning of such extension implies

$$\mathbf{F} = \bigcup_{k=0}^{+\infty} F_k ,$$

where $F_{k+1} = \{ \sum_{i\in I}\alpha_i \max_{j\in J_i} f_j^i \mid f_j^i \in F_k,\ \alpha_i\in \mathbf{R},\ i\in I,\ j\in J_i, |I| < +\infty,\ |J_i| < +\infty \}$.

As a matter of fact, it is possible to reduce any function from $\mathbf{F}$ to some standard form. The tools for such reducing are the following identities which are valid for an arbitrary set of functions:

$$\max_{j\in J} f_j + \max_{i\in I} g_i = \max_{(j,i)\in J\times I} (f_j + g_i) ; \tag{2.3}$$

$$\sum_{i\in I}\alpha_i \max_{j\in J_i} f_j^i = \max_{s_i\in J_i} \sum_{i\in I_+} \alpha_i f_{s_i}^i - \max_{s_i\in J_i} \sum_{i\in I_-} |\alpha_i|\, f_{s_i}^i , \tag{2.4}$$

where $I_+ = \{ i \mid \alpha_i > 0 \}$, $I_- = \{ i \mid \alpha_i < 0 \}$;

$$\begin{aligned}\max_{j=1,\dots,m} f_j &= [\max_{j=1,\dots,m-1} f_j - f_m]^+ + f_m = \\ &= [f_m - \max_{j=1,\dots,m-1} f_j]^+ + \max_{j=1,\dots,m-1} f_j;\end{aligned} \tag{2.5}$$

$$\begin{aligned}\min_{i=1,\dots,n} f_i &= -[\min_{i=1,\dots,n-1} f_i - f_n]^+ + \min_{i=1,\dots,n-1} f_i = \\ &= -[f_n - \min_{i=1,\dots,n-1} f_j]^+ + f_n;\end{aligned} \tag{2.6}$$

$$[\max_{j\in J} f_j - \max_{i\in I} g_i]^+ = \max_{(j,i)\in J\times I} \{ f_j, g_i \} - \max_{i\in I} g_i ; \tag{2.7}$$

$$\max_{j\in J} f_j - \max_{i\in I} g_i = \min_{i\in I} \max_{j\in J} (f_j - g_i) = \max_{j\in J} \min_{i\in I} (f_j - g_i) \tag{2.8}$$

These identities are of abstract logical sense and can be proved directly.

In a line with **F**, let us consider the space

$$\mathbf{H} = \bigcup_{i=0}^{+\infty} H_i ,$$

where $H_0 = \mathbf{F}_0$, $H_{k+1} = \{ \sum_{i \in I} \alpha_i f_i^+ \mid \alpha_i \in \mathbf{R},\ f_i \in H_k,\ |I| < +\infty \}$.

Note that the positive cut-off function "+", being a particular case of σ-operation, after repeated applications gives the same functional class **F**.

Theorem 2.1.
1) *Any function from* **F** *can be represented in the forms:*

$$\max_{j \in J} f_j - \max_{i \in I} g_i , \tag{2.9}$$

$$\min_{i \in I} \max_{j \in J_i} f_j^i , \tag{2.10}$$

$$\max_{j \in J} \min_{i \in I_j} f_i^j , \tag{2.11}$$

where $\{ f_j, g_i, f_j^i \} \subset \mathbf{F}_0$; (2.9) *means* $F_k = F_1$ *for* $k > 1$, *consequently* $\mathbf{F} = F_1$.

2) *Class* **F** *coincides with* **H**.

3) *Representations* (2.9) - (2.11) *are equivalent*.

Proof: 1) Relation (2.9) means the coincidence of F_k with F_1 for $k > 1$. Clearly, it is sufficient to prove $F_2 = F_1$. Due to (2.4) we may consider only one transformation, namely, the transformation of the function $f(x) = \max_{i \in I} f_i$ with $\{ f_i \}_I \subset F_1$ to the form (2.9).

Let $I = \{1, \ldots, n\}$ and use induction on n. For $n = 1$ it is evident that $f(x) = f_1(x)$ satisfies the required property (2.9). Let $n > 1$. As $f_i \in F_1$, these functions can be represented as

$$f_i = \max_{j \in J_i} f_j^i - \max_{k \in I_i} g_k^i$$

where $\{ f_j^i, g_k^i \} \subset \mathbf{F}_0$. Therefore

$$f(x) \overset{(2.5)}{=} [\max_{i=1,\ldots n-1} f_i - f_n]^+ + f_n .$$

By induction we have $\max\limits_{i=1,\dots,n-1} f_i := \bar{f} \in F_1$. Since $f = [\ \bar{f} - f_n\ ]^+ + f_n$ and $\{\ \bar{f}, f_n\ \} \subset F_1$, due to (2.4) we have: $\bar{f} - f_n \in F_1$, and due to (2.7): $[\ \bar{f} - f_n\ ]^+ \in F_1$. This fact and inclusion $f_n \in F_1$ give us $f \in F_1$. Consequently, $F_2 = F_1$ and therefore $\mathbf{F} = F_1$.

The fact that functions from $\mathbf{F}$ can be represented in the form (2.10) or (2.11) follows from identity (2.8).

2) At first we shall prove inclusion $\mathrm{H} \subset F_1$, i.e. $H_k \subset F_1, \forall k.$ If $f \in H_1$, then due to (2.4): $f \in F_1$. Hence $H_1 \subset F_1$. Let $H_k \subset F_1$. We have to prove $H_{k+1} \subset F_1$. An arbitrary function from H_{k+1} takes the form

$$f = \sum_{i \in I} \alpha_i f_i^+, \quad \{\ f_i\ \} \subset H_k \subset F_1 .$$

This implies that f can be written as a linear combination of discrete maximum functions with generators from $\mathbf{F}_0$. This fact and the relation (2.4) give f the required representation (2.9).

Inversely, let $f \in \mathbf{F}$, i.e. f takes the form (2.9). Let us show that the function of discrete maximum with generators from $\mathbf{F}_0$ belongs to H, i.e. belongs to one from H_k. Thereby, we shall prove the inclusion $f \in \mathrm{H}$.

Let us take

$$f = \max_{j \in J} f_j, \quad \{\ f_j\ \} \subset \mathbf{F}_0, \quad j = 1,\dots m.$$

If $m = 1$, then $f = f_1 \subset \mathrm{H}_0$. Let $m > 1$. Apply the relation (2.5):

$$f = [\ \max_{j=1,\dots,m-1} f_j - f_m\ ]^+ + f_m .$$

According to inductive assumption we have $\bar{f} = \max\limits_{j=1,\dots,m-1} f_j \in H_k$. Therefore, $f = [\ \bar{f} - f_m\ ]^+ + f_m \in H_{k+1}$, which completes the proof of 2).

3) We already mentioned that the function (2.9) can be rewritten as (2.10) with $f_j^i = f_j - g_j$ (see (2.8)). It remains to prove the inverse. Let f have the form (2.10), i.e. $f = \min\limits_{i \in I} \max\limits_{j \in J_i} f_j^i$. If $I = \{1,\dots,n\}$ and $n = 1$, then $f = \max\limits_{j \in J_1} f_j^i \in F_1$. For $n > 1$ the proof can be carried out as above by induction on n. Due to (2.6):

$$f = [\ \min_{i=1,\dots,n-1} \max_{j \in J_i} f_j^i - \max_{j \in J_n} f_j^n\ ]^+ + \max_{j \in J_n} f_j^n .$$

According to the inductive assumption

$$\bar{f} = \min_{i=1,\dots,n-1} \max_{j \in J_i} f_j^i \in F_1 ,$$

i.e. $\bar{f}$ can be represented in the form (2.9). Consequently, using transformations (2.3), (2.4) and (2.7) we can write f as (2.9).

The equivalence of the forms (2.11) and (2.9) can be proved in the same way. The proof of Theorem 2.1 is completed.

We shall call any function f from $\mathbf{F}$ a *σ-function*, or a *σ-piecewise function*. If $\mathbf{F}_0$ is the space of all linear (affine) functions, then $\mathbf{F}$ is the space of all piecewise linear functions.

3. PIECEWISE LINEAR FUNCTIONS

Piecewise linear functions are evidently the σ-functions in the case when $\mathbf{F}_0$ is the space of all linear (affine) functions. One can define the class of piecewise linear functions (below *k-functions*) in one of two ways: either as above or proceeding from some axioms describing such functions. Now let us fix our attention on the second way.

Let $\{ M_j \}_J$ be a finite family of polyhedral sets and $\{ l_j(x) \}_J$ be a family of proper linear functions. We shall say that the system $\{ M_j, l_j(x) \}$ determines a *piecewise linear function* $l(x)$, defined on $\mathbf{X}$, if:

1) $\bigcup_{j \in J} M_j = \mathbf{X}, \quad M_i^0 \cap M_j^0 = \emptyset \quad \text{for} \quad i \neq j;$

2) $l(x) \equiv l_j(x), \quad \forall x \in M_j, \quad \forall j \in J.$

Here M_j^0 is the *algebraic* interior of the polyhedral set M_j, i.e. $y \in M_j^0 \Leftrightarrow y + ts \in M_j$ for all $s \in \mathbf{X}$ and sufficiently small $t \geq 0$. The term polyhedral set, as well as above, denotes the set defined by a finite system of proper linear inequalities

$$(f_j, x) - \alpha_j \leq 0, \quad j \in J.$$

In this definition some of M_j or M_j^0 may be empty.

Let $\mathbf{L}_0$ be the space of all affine functions, and $\mathbf{L}$ be the space of k-functions defined by external manner according to properties 1) and 2). Clearly, functional class $\mathbf{L}$ coincides with $\mathbf{F}$ from the previous section, which is constructed from $\mathbf{F}_0 = \mathbf{L}_0$, i.e. $\mathbf{L}$ is the minimal algebraic extension of the affine functional space, closed with respect to the operation of discrete maximum [7]. Therefore, representation (2.9), as well as both (2.10) and (2.11), are universal forms of piecewise linear functions (k-functions).

To simplify and unify the expressions of k-functions, systems of inequalities with k-functions, problems of piecewise linear functions and so on, we shall assume now $\mathbf{X} = \mathbf{R}^n$. Then

$$\mathrm{L}_0 = \{ \; l(x) = (a,x) - \alpha \;\mid\; a \in \mathbf{R}^n, \;\; \alpha \in \mathbf{R} \; \},$$

$$\mathrm{L} = \{ \; \max_{j \in J} l_j(x) - \max_{i \in I} h_i(x) \;\mid\; \{ \; l_j, h_j \; \} \subset \mathrm{L}_0, \; |J| < +\infty, \; |I| < +\infty \; \}.$$

Let

$$|z|_{\max} = \max_{(i)} z_i, \quad |z|_{\min} = \min_{(i)} z_i,$$

where z is an element of some finite dimensional vector space. If $Ax - b = [\; l_1(x), \ldots, l_m(x) \;]^T$ is a vector of linear functions, then (according to our notations):

$$|Ax - b|_{\max} = \max_{(i)} l_i(x).$$

Representations of piecewise linear functions in the forms (2.9) - (2.11) take unifying forms:

$$|Ax - b|_{\max} - |Bx - d|_{\max}, \tag{3.1}$$

$$\min_i | \; A_i x - b^i \; |_{\max}, \tag{3.2}$$

$$\max_{(j)} | \; A_j x - b^j \; |_{\min}. \tag{3.3}$$

Here are some more obvious properties of the function of discrete maximum:

$$|z|_{\max} = -|z|_{\min};$$

$$\text{for } \alpha > 0: \; |\alpha z|_{\max} = \alpha |z|_{\max}; \text{ for } \alpha < 0: \; |\alpha z|_{\max} = \alpha |z|_{\min}.$$

4. SYSTEMS OF PIECEWISE LINEAR INEQUALITIES AND THEIR GEOMETRIC INTERPRETATION

A finite system of piecewise linear function (k-function) inequalities can be written as

$$\min_{(j)} | \; A_j^t x - b_t^j \; |_{\max} \le 0, \; t = 1, \ldots, T. \tag{4.1}$$

This system can be represented as a single inequality

$$\sum_{t=1}^{T} \min_{(j)} | A_j^t x - b_t^j |_{\max} \le 0,$$

or (according to Theorem 2.1) in the form

$$\min_{j=1,\dots,m} | A_j x - b^j |_{\max} \le 0. \tag{4.2}$$

We assume this form to be standard. Another standard form is

$$| Ax - b |_{\max} - | Bx - d |_{\max} \le 0 \tag{4.3}$$

(see (3.1)). So an arbitrary finite system of piecewise linear inequalities can be written in any of the standard forms (4.1) - (4.3).

Consider the representation form (4.2). Assume that $M_j = \{ x \mid A_j x \le b^j \}$. Then the solution set of the inequality (4.2) is $M = \bigcup_{j=1}^{m} M_j$. On the other hand, if M is an arbitrary polyhedral set (from R^n), i.e. $M = \bigcup_{j=1}^{m} M_j$ and $\{ M_j \}$ are polyhedrons, i.e. M_j are defined by finite systems of linear inequalities: $M_j = \{ x \mid A_j x \le b^j \}$, then M is a solution set of inequality (4.2).

From the same point of view let us look at the inequality (4.3). Let

$$Ax - b = [\, l_1(x), \dots, l_m(x) \,]^T, \quad Bx - d = [\, s_1(x), \dots, s_k(x) \,]^T,$$

$$\{ l_j(x), s_i(x) \}_{1,1}^{m,k} \subset \mathrm{L}_0$$

(L_0 is an affine functional space). Set

$$M_i = \{ x \mid l_j(x) \le s_i(x), \ j = 1, \dots, m \}.$$

Then it is easy to show that $\bigcup_{i=1}^{s} M_i$ coincides with the solution set of inequality (4.3). Thereby, for inequality (4.3) some polyhedrons M_i are pointed. Being united they give us the solution set of inequality (4.3). On the other hand, if polyhedral set M is given by some or other manner, e.g. by a family of linear inequality systems each of which gives a convex polyhedral component of the set M, then to lead us to the representation of the set M in the form of a single inequality of type (4.3) the chain of transformations (2.3) - (2.8) must be applied.

Finally, let us consider a system of piecewise linear inequalities in the form (4.1), which formally is more complicated than (4.2) or (4.3). Let

$$M_j^t := \{ x \mid A_j^t x \le b_t^j \}, \quad M_t := \bigcup_{(j)} M_j^t, \quad M := \bigcap_{(t)} M_t .$$

The set M_t is a solution set of the t-th inequality of system (4.1), so that M is a solution set of the whole system.

Facts contained in 2 - 4 establish the ways of constructive correspondence between polyhedral sets and their analytic representations. Although the accompanying algebra of the transformations may be rather complicated, the logic of such transformations is very simple and in real applications may be carried out by computer.

5. PROBLEMS OF PIECEWISE LINEAR PROGRAMMING

5.1. Preliminary remarks

An arbitrary problem of piecewise linear programming, i.e. the problem of seeking the extremum of some k-function under constraints in the form of a finite system of inequalities with k-functions on the left-hand side, can be written in the universal simple form

$$P : \max\{ (c,x) \mid \min_{j=1,\dots,m} | A_j x - b^j |_{\max} \le 0, \ \ x \ge 0 \}. \tag{5.1}$$

Indeed, the method of reducing a system of k-inequalities to a single k-inequality has already been discussed. Let $f(x)$ be an arbitrary k-function to be optimized (e.g. maximized) under a single k-inequality of the form $g(x) \le 0$ (maybe, with $x \ge 0$). Rewriting the problem $\max\{ f(x) \mid g(x) \le 0 \}$ in the form

$$\max\{ t \mid g(x) \le 0, \ f(x) \ge t \}$$

and transforming the system of two k-inequalities to a single k-inequality, we get the problem of maximizing of some linear functions under a single constraint in the form of the k-inequality. In (5.1) constraint $x \ge 0$ is separated to obtain symmetry in some of the analytic constructions considered below.

The subject under our consideration will be the problem (5.1). Let us write the *partional* problem

$$L_j : \max\{\, (c,x) \mid A_j x \le b^j,\ x \ge 0 \,\}. \tag{5.2}$$

The relation between problems (5.1) and L_j is very simple:

$$\operatorname{opt}(5.1) = \max_{(j:M_j \ne \emptyset)} \operatorname{opt} L_j , \tag{5.3}$$

where $M_j = \{\, x \ge 0 \mid A_j x \le b^j \,\}$.

In spite of the solvability of original problem (5.1), we admit that in (5.2) for some j the sets $M_j = \{\, x \ge 0 \mid A_j x \le b^j \,\}$ may be empty. Therefore, an arbitrary k-problem after reducing to the form (5.1) desintegrates into a finite number of linear programs. Solving them, we find a solution to the original problem. Since the corresponding transformations are constructive, the arbitrary problem of piecewise linear programming can be solved using both such constructivity and some methods of linear programming (e.g., the simplex-method).

5.2. Solvability conditions for the *k*-problem

Many valid properties and theorems can be formulated for k-problems in the terms of linear programming. Some of them are simple consequences from known linear programming facts, others need their own proofs. The following theorem does not need a proof:

Theorem 5.1. *If*

$$\sup\{\, (c,x) \mid \min_{(j)} | A_j x - b^j |_{\max} \le 0,\ x \ge 0 \,\} < +\infty ,$$

then sup *in this problem is attained.*

Let us write the problem which is dual to L_j:

$$L_j^* : \min\{\, (b^j, u^j) \mid A_j^T u^j \ge c,\ u^j \ge 0 \,\}. \tag{5.4}$$

Assume that $M_j^* = \{\, u^j \ge 0 \mid A_j^T u^j \ge c \,\}$.

Theorem 5.2. *The problem* (5.1) *is solvable if and only if*

$$M = \bigcup_{j=1}^{m} M_j \ne \emptyset \qquad \& \qquad M_j \ne \emptyset \Rightarrow M_j^* \ne \emptyset .$$

Proof: Since the conditions $M_j \neq \emptyset$ and $M_j^* \neq \emptyset$ are necessary and sufficient for solvability of the problem L_j, $\max\limits_{j:M_j \neq \emptyset} \operatorname{opt} L_j$ is finite and coincides with opt P. Inversely, if problem P is solvable, then all problems L_j, with $M_j \neq \emptyset$ are solvable too, and according to the well-known dual relations in linear programming we have $M_j^* \neq \emptyset$.

Remark. Another variant of Theorem 5.2 can be formulated just as:

$$(P \text{ is solvable}) \Leftrightarrow (M \neq \emptyset \;\&\; M_j \neq \emptyset \Rightarrow L_j \text{ is solvable}).$$

5.3. Duality

Let us take the original k-problem in the form (5.1), adding to it the following assumption (not so essential): the dimensions of all the vectors $A_j x - b^j$ are equal, i.e. the number of inequalities in all systems $A_j x \leq b^j$ are the same. This condition allows us to denote the dual variable for L_j (i.e. the variable u^j in the problem (5.4)) by the common symbol u. According to this notation problem (5.4) can be rewritten as:

$$\min\{\, (b^j, u) \mid A_j^T u \geq c,\ \ u \geq 0 \,\},\ \ j = 1,\ldots,m. \tag{5.5}$$

Let us formulate the problem

$$P^*:\ \max_{j:M_j^* \neq \emptyset} \min\{\, (b^j, u) \mid A_j^T u \geq c,\ \ u \geq 0 \,\}. \tag{5.6}$$

We shall consider it the dual of problem P. The dual problem for problem P^* does not have symmetric architecture with respect to the setting of the original problem. But if P is rewritten in equivalent form

$$\max_{j \in \{1,\ldots,m\}} \min\{\, (c, x) \mid A_j x \geq b^j,\ x \geq 0 \,\}, \tag{5.7}$$

then (5.6) and (5.7) take a symmetric form.

The following is valid for problem (5.6):

Theorem 5.3. *Problem* (5.6) *is solvable if and only if*

$$\exists j \in \{\, 1,\ldots,m \,\}:\ M_j^* \neq \emptyset \quad \& \quad M_j \neq \emptyset. \tag{5.8}$$

Proof: Indeed, if $J := \{ j \mid M_j \neq \emptyset,\ M_j^* \neq \emptyset \}$, then for any $j \notin J$: $\inf\{ (b^j, u) \mid x \in M_j^* \} = -\infty$, therefore in (5.6) the maximum can be taken only for $j \in J$. This fact implies the solvability of problem P^*. The necessity of conditions (5.8) is obvious: if P^* is solvable, then there exists j' such that the problem L_j^* is solvable too, that is equivalent to (5.8).

Theorems 5.2 and 5.3 as well as dual relations in linear programming lead us to:

Theorem 5.4. *If problem* (5.1) *is solvable, then* (5.6) *is solvable, too, and their optimal values are the same.*

Note that unlike linear programming the property of the simultaneous solvability or unsolvability of problems P and P^* is lost. Let us allow the improper optimal values and write these problems in the forms

$$P : \sup_{(j)} \inf_{x \in M_j} (c, x),$$

$$P^* : \sup_{(j)} \inf_{u \in M_j^*} (b^j, u);$$

The conditions of the simultaneous solvability of P and P^* are described by Theorems 5.2 and 5.3, but it is possible to have the situation: P is unsolvable, P^* is solvable. This reflects the fact that if P^* is solvable then

$$\exists j_0 : M_{j_0} \neq \emptyset \ \ \&\ \ M_{j_0}^* = \emptyset .$$

Then the problem L_{j_0} is improper of the 2nd kind, i.e. $\sup_{x \in M_{j_0}} (c, x) = +\infty$. Therefore $\operatorname{opt} P = +\infty$, i.e. P is unsolvable. The simultaneous unsolvability of the problems P and P^* is realized by the pair of improper linear programming problems L and L^* of the 3rd kind.

6. THE SADDLE POINT PROBLEM FOR DISJUNCTIVE LAGRANGIAN

Consider the problem

$$P_{\cup} : \max\{ f(x) \mid x \in \bigcup_{(j)} M_j \}, \tag{6.1}$$

i.e. the problem (1.4) from the Section 1, and associate with it the function

$$\Phi_{\vee}(x,u) = f(x) - \min_{(j)} (u_j, F_j(x)) , \tag{6.2}$$

which will be called a *disjunctive* Lagrangian for $P_{\vee}$.

We shall call the problem (6.1) *quite regular*, if all the problems

$$\max\{ \, f(x) \mid F_j(x) \le 0, \; x \ge 0 \, \} \tag*{(6.1)$_j$}$$

are solvable. Such property is equivalent to the solvability of problem (6.1) and the nonemptiness of all the sets $M_j : M_j \neq 0, j = 1,...m$.

Let $\Phi_j(x,u_j) = f(x) - (u_j, F_j(x))$ be the Lagrangian for (6.1)$_j$; here $x \in \mathrm{R}^n$, $u_j \in \mathrm{R}^{m_j}$. The well-known fact (e.g., see [7]) holds:

Lemma 6.1. *If* $[\,\bar{x},\bar{u}\,] \ge 0$ *is a saddle point for the Lagrangian* $\Phi(x,u) = f(x) - (u, F(x))$ *of the problem*

$$\max\{ \, f(x) \mid F(x) \le 0, \;\; x \ge 0 \, \}, \tag{6.3}$$

then $(\bar{u}, F(x)) = 0$ *and* $\bar{x} \in \operatorname{Arg}(6.3)$.

Lemma 6.2. *Let all the functions* $\Phi_j(x,u_j)$ *have saddle points* $[\,\bar{x}_j, \bar{u}_j\,] \ge 0$, $j = 1,...,m$. *If* $\bar{x}$ *is the value of* $\bar{x}_j$ *which gives us* $\max_{(j)} f(\bar{x}_j)$, *then*

$$\Phi_{\vee}(x,\bar{u}) \underset{\forall x \ge 0}{\le} f(\bar{x}), \tag{6.4}$$

where $\Phi_{\vee}(x,\bar{u}) = f(x) - \min_{(j)} (\bar{u}_j, F_j(x))$.

Proof: According to Lemma 6.1: $(\bar{u}_j, F_j(\bar{x}_j)) = 0$, $j = 1,..,m$ and $f(x) - (\bar{u}_j, F_j(x)) \le f(\bar{x}_j) \;\; (\le f(\bar{x}))$, $\forall x \ge 0$. Hence

$$\max_{(j)} [\, f(x) - (\bar{u}_j, F_j(x)) \,] \le f(\bar{x}).$$

But since the left side of this inequality is equal to

$$f(x) - \min_{(j)} (\bar{u}_j, F_j(x)) \;\; (= \Phi_{\vee}(x,\bar{u})),$$

the desired inequality is valid.

Lemma 6.3. *If* $[\ \bar{x},\bar{u}\ ]\geq 0$ *is a saddle point for* $\Phi_{\cup}(x,u)$, *then*

$$\min_{(j)}(\bar{u}_j,F_j(\bar{x}))=0$$

and $\bar{x}\in \text{Arg }(6.1)$.

Proof: According to the saddle point definition

$$\Phi_{\cup}(x,\bar{u}) \underset{\forall x\geq 0}{\leq} \Phi_{\cup}(\bar{x},\bar{u}) \underset{\forall u\geq 0}{\leq} \Phi_{\cup}(\bar{x},u),$$

or

$$f(x)-\min_{(j)}(\bar{u}_j,F_j(x)) \underset{\forall x\geq 0}{\leq} f(\bar{x})-\min_{(j)}(\bar{u}_j,F_j(\bar{x})), \tag{6.5}$$

$$f(\bar{x})-\min_{(j)}(\bar{u}_j,F_j(\bar{x})) \underset{\forall u\geq 0}{\leq} f(\bar{x})-\min_{(j)}(u_j,F_j(\bar{x})). \tag{6.6}$$

Rewrite inequality (6.6) as

$$\min_{(j)}(\bar{u}_j,F_j(\bar{x})) \underset{\forall u\geq 0}{\geq} \min_{(j)}(u_j,F_j(\bar{x})). \tag{6.7}$$

At first we prove that $\bar{x}\in M=\bigcup_{(j)} M_j$, i.e. $\exists_{j_0}:F_{j_0}(\bar{x})\leq 0$. Indeed, if $F_j(\bar{x})\nleq 0,\ \forall_j$, then choosing a suitable $u\geq 0$ one can obtain an arbitrary large right-hand side of (6.7). But it conflicts with (6.7). The fact proved gives us: $\alpha:=\min_{(j)}(\bar{u}_j,F_j(\bar{x}))\leq 0$. More precisely, $\alpha=0$. Indeed, if $\alpha<0$, then relation (6.7) with $u=0$ implies the contradictory $0>-|\alpha|\geq 0$. Therefore, relation $\min_{(j)}(\bar{u}_j,F_j(\bar{x}))$ is proved.

Next it is necessary to prove the optimality of the vector $\bar{x}$ for the problem (6.1), i.e. $f(x)\leq f(\bar{x}),\ \forall x\in M$. Let us turn to the relation (6.5). We write it now as

$$f(\bar{x})-f(x)\geq -\min_{(j)}(\bar{u}_j,F_j(x)). \tag{6.8}$$

If $x\in M$, i.e. $x\in M_{j_0}$ for some j_0, then $\min_{(j)}(\bar{u}_j,F_j(x))\leq 0$. Due to (6.8) we have $f(\bar{x})-f(x)\geq 0,\ \forall x\in \bigcup_{(j)} M_j$. The proof is completed.

Lemma 6.4. *Let the problem* (6.1) *be quite regular, i.e. the problem*

$$L_{\cup}:\max\{\ (c,x)\ \big|\ x\in \bigcup_{j=1}^{m} M_j\ \},$$

where $M_j = \{\, x \mid A_j x \le b^j,\ x \ge 0 \,\}$, *is solvable and* $M_j \ne \emptyset, \forall_j$. *Then the function* $L_{(\ )}(x,u) = (c,x) - \min\limits_{(j)} (u_j, A_j x - b^j)$ *has a saddle point* $[\, \bar{x}, \bar{u} \,]$, *and its components* $\bar{x}$ *and* $\bar{u} = [\, \bar{u}_1, \ldots, \bar{u}_m \,]$ *satisfy:*

$$\bar{x} = \arg\max_{(j)} (c, \bar{x}_j), \qquad \bar{x}_j \in \operatorname{Arg}\max_{x \in M_j} (c, x),$$

$$\bar{u}_j \in \operatorname{Arg}\min_{u \in M^*} (b^j, u), \qquad j = 1, \ldots, m.$$

Proof: If we show that $\alpha := \min\limits_{(j)} (\bar{u}_j, A_j x - b^j) = 0$, then due to Lemma 6.2 the left inequality in the definition of the saddle point for $L_{(\ )}(x,u)$ will be valid. Since the vector $\bar{x} \ge 0$ satisfies at least one of the systems $A_j x \le b^j$, we have $\alpha \le 0$. But $\alpha \ge 0$, because

$$\min_{(j)} (\bar{u}_j, A_j \bar{x} - b^j) =$$
$$= \min_{(j)} [\ -(b^j, \bar{u}_j) + (c, \bar{x}) + (A_j^T \bar{u}_j - c, \bar{x})\ ] \ge \min_{(j)} [\ (c, \bar{x}) - (c, \bar{x}_j)\ ] \ge 0 .$$

Here we use the relations: $A_j^T \bar{u}_j - c \ge 0$; $(b^j, \bar{u}_j) = (c, \bar{x}_j)$ according to dual relations in linear programming; $(c, \bar{x}) \ge (c, \bar{x}_j),\ \forall j$.
It remains to prove the right inequality in the saddle point definition for $L_{(\ )}(x,u)$, i.e.

$$L_{(\ )}(\bar{x}, \bar{u}) \underset{\forall u \ge 0}{\le} L_{(\ )}(\bar{x}, u) .$$

Since $\alpha = 0$, the inequality above takes the form

$$0 \underset{\forall u_j \ge 0}{\le} -\min_{(j)} (u_j, A_j \bar{x} - b^j) . \tag{6.9}$$

But $\forall j_0 :\ A_{j_0} \bar{x} - b^j \le 0$, and for all $u \le 0$: $\min\limits_{(j)} (u_j, A_j \bar{x} - b^j) \le 0$, consequently (6.9) is valid. The proof is completed.

Lemmas 6.3 and 6.4 imply

Theorem 6.1. *Let all the systems* $A_j x \le b^j,\ x \ge 0,\ j = 1, \ldots, m$ *be consistent. Then problem* (6.1) *is solvable if and only if its disjunctive Lagrangian*

$$L_{(\ )}(x,u) = (c,x) - \min_{(j)} (u_j, A_j x - b^j)$$

has a saddle point $[\, \bar{x}, \bar{u} \,] \ge 0$, *and*

1) *if* $L_{\vee}$ *is solvable and* $\bar{x}_j \in \operatorname{Arg} L_j$, $\bar{u}_j \in \operatorname{Arg} L_j^*$, $\bar{x} = \arg\max_{(j)}(c,\bar{x}_j)$, *then* $[\ \bar{x},\bar{u}\ ]$ *is a saddle point;*

2) *if* $[\ \bar{x},\bar{u}\ ]$ *is a saddle point for* $L_{\vee}(x,u)$, *then* $\{\ \bar{x},\bar{u}_j\ \}$ *satisfy all the relations from* 1).

The existence of a saddle point $[\ \bar{x},\bar{u}\ ] \geq 0$ for the function $\Phi(x,u)$ written in the form (6.5) - (6.6) is equivalent to equalities

$$\max_{x\geq 0}\min_{u\geq 0}\Phi(x,u) = \min_{u\geq 0}\max_{x\geq 0}\Phi(x,u) = \Phi(\bar{x},\bar{u})\,. \tag{6.10}$$

In the game interpretation of mathematical programming duality it is natural to formulate statements similar to Theorem 6.1 in the form of relation (6.10).

Theorem 6.2. *Let the problem* (6.1) *be solvable and* $M_j \neq \emptyset,\ \forall j$. *Then for* $\bar{x} \in$ Arg (6.1) *there exist such* $\bar{u}_j \geq 0,\ j = 1,\dots,m$ *that the vector* $[\ \bar{x},\bar{u}\ ]$ *satisfies the relation* (6.10) *with*

$$\Phi(x,u) = L_{\vee}(x,u) = (c,u) - \min_{(j)}(\bar{u}_j, A_j x - b^j)\,.$$

7. METHOD OF EXACT PENALTY FUNCTIONS FOR PIECEWISE LINEAR PROGRAMMING PROBLEMS

Let us consider the general problem of piecewise linear programming in a canonical setting (5.1), i.e.

$$P_{\vee} : \max\{\ (c,x) \mid \min_{j=1,\dots,m} |\ A_j x - b^j\ |_{\max} \leq 0,\ x \geq 0\ \}. \tag{7.1}$$

We are interested in its equivalent reduction to some problem of the same type but without the main constraint in (7.1). Associate with $L_{\vee}$ the following k-problem

$$\sup_{x\geq 0}[\ (c,x) - \min_{(j)}(R_j,(A_j x - b^j)^+)\ ], \tag{7.2}$$

where R_j is a nonnegative vector parameter of dimension m_j, i.e. m_j is the number of inequalities in the system $A_j x - b^j \leq 0$.

As before, we shall use the following notations: L_j denotes the problem $\max\{\ (c,x) \mid A_j x \leq b^j,\ x \geq 0\ \}$, L_j^* denotes the dual, $\bar{u}_j = \arg L_j^*$, $\bar{u} = [\bar{u}_1,\dots,\bar{u}_m\ ]$, $M_j = \{\ x \geq 0 \mid A_j x \leq b^j\ \}$.

Theorem 7.1. *Let problem* (7.1) *be solvable,* $M_j \neq 0$, $\forall j$; $\overline{u}_j \in \operatorname{Arg} L_j^*$. *If* $R_j \geq R_0 \overline{u}_j$, $R_0 > 1$, *then the optimal values and optimal sets of problems* (7.1) *and* (7.2) *are the same, i.e.*

$$\operatorname{opt}(7.1) = \operatorname{opt}(7.2), \tag{7.3}$$

$$\operatorname{Arg}(7.1) = \operatorname{Arg}(7.2). \tag{7.4}$$

Proof: We denote the goal function in (7.2) by $\Phi_R(x)$ and the part subtracted from (c,x) by $\Phi_0(x)$. At first let us prove equality (7.3). Since $\overline{x} \in \operatorname{Arg}(7.1)$, we have $\Phi_R(\overline{x}) = (c,\overline{x}) = \operatorname{opt}(7.1)$, consequently

$$\operatorname{opt}(7.2) = \sup_{x \geq 0} \Phi_R(x) \geq \operatorname{opt}(7.1).$$

Inversely, due to Lemmas 6.4 and 6.3:

$$(c,x) - \min_{(j)} (\overline{u}_j, A_j x - b^j) \leq (c,\overline{x}), \quad \forall x \geq 0.$$

Taking this inequality into account we can evaluate $\Phi_R(x)$ for $x \geq 0$ as follows:

$$\left.\begin{aligned}
&\Phi_R(x) \leq (c,\overline{x}) + \min_{(j)} (\overline{u}_j, A_j x - b^j) - \Phi_0(x) \leq \\
&\leq \operatorname{opt}(7.1) + \min_{(j)} (\overline{u}_j, (A_j x - b^j)^+) - \Phi_0(x) \leq \\
&= \operatorname{opt}(7.1) + \frac{1}{R_0} (R_j, (A_j x - b^j)^+) - \Phi_0(x) = \\
&= \operatorname{opt}(7.1) - \frac{R_0 - 1}{R_0} \min_{(j)} (R_j, (A_j x - b^j)^+) \leq \operatorname{opt}(7.1).
\end{aligned}\right\} \tag{7.5}$$

Hence $\sup_{x \geq 0} \Phi_R(x) \leq \operatorname{opt}(7.1)$. Therefore, equality (7.3) is proved. From it, in particular, follows the inclusion $\operatorname{Arg}(7.1) \subset \operatorname{Arg}(7.2)$, which makes it possible to write max instead of sup in problem (7.2).

Now prove the inverse inclusion. Let $\overline{x} \in \operatorname{Arg}(7.2)$. According to (7.5) we have:

$$\operatorname{opt}(7.1) = \Phi_R(\overline{x}) \leq \operatorname{opt}(7.1) - \frac{R_0 - 1}{R_0} \min_{(j)} (R_j, (A_j \overline{x} - b^j)^+).$$

Hence

$$\min_{(j)}(R_j,(A_j x - b^j)^+) = 0\,,$$

and we have $\exists j_0 : (R_{j_0},(A_{j_0}\bar{x} - b^{j_0})^+) = 0$, which with $R_{j_0} > 0$ implies $A_{j_0}\bar{x} \le b^{j_0}$. Consequently, $\bar{x} \in M_{j_0} \subset M = \bigcup_{j=1}^{m} M_j$. The feasibility of the vector $\bar{x}$ for problem (7.1) and equality $(c,\bar{x}) = \text{opt}\,(7.1)$ imply $\bar{x} \in \text{Arg}\,(7.1)$. Therefore $\text{Arg}\,(7.2) \subset \text{Arg}\,(7.1)$. The proof of equality (7.4) is complete too.

Construction of the proof can be repeated in more general cases of problem (6.2) and its equivalent reduction to the problem

$$\sup_{x \ge 0}\left[\, f(x) - \min_{(j)}(R_j, F_j^+(x)) \,\right]. \tag{7.6}$$

Namely, the following Theorem is valid:

Theorem 7.2. *Let all the problems* $\max\{\, f(x) \mid F_j(x) \le 0,\ x \ge 0 \,\}$ *have the saddle points* $[\, \bar{x}_j, \bar{u}_j \,]$. *Then for* $R_j > R_0 \bar{u}_j$, $R_0 > 1$ *problems* (6.2) *and* (7.6) *are equivalent, i.e. their optimal values and optimal sets are the same.*

Indeed, according to Lemma 6.3 we can write the inequality

$$f(x) \le f(\bar{x}) + \min_{(j)}(\bar{u}_j, F_j(x))\,, \qquad \forall x \ge 0,$$

and then perform all steps according to (7.5).

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