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PSEUDO-DELTA FUNCTIONS AND SEQUENCES IN THE OPTIMIZATION THEORY

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Dedicated to the memory of Professor Jovan Petrić

Abstract: Pseudo-delta sequences and pseudo-delta functions in the framework of pseudo-analysis are presented. In this paper an important property of pseudo-delta sequences with respect to pseudo-convolution is proved. The study of such sequences is of interest since we can use them in approximaton of a class of operators that appear in optimization problems.

Key words: Pseudo-addition, pseudo-multiplication, decomposable measure, pseudo-integral, pseudo-convolution, pseudo-delta sequence.

1. INTRODUCTION

Classical mathematical analysis is based on the field of reals $(\mathbf{R},+,\cdot)$. This has implications on the corresponding linear algebra, measure theory and integration theory, which are corner stones in many applications on ordinary and partial differential equations and difference equations (mostly linear). Many types of non-additive measures and corresponding integrals ([6], [8], [18], [25]) have been investigated, stimulated by many different problems in practice, mostly by modeling different uncertainties in the theory of fuzzy systems, which are bases for the decision theory and artificial intelligence, the system theory, and the game theory. An important subclass of non-additive measures contains decomposable measures with respect to some semiring $([a,b],\oplus,\otimes), -\infty \leq a < b \leq +\infty$ ([5], [7]). There are many different applications of analysis based on such semirings (usually called pseudo-analysis) in optimization theory, nonlinear equations of different types, decision theory, etc. ([2], [8], [12], [18], [20], [25]).

In the second Section we present the basic definitions and examples related to semirings on reals. In Section Three we briefly present the corresponding measure and integration theory and introduce the notion of pseudo-convolutions, a generalization of the classical convolution of functions. Section Four contains the basic results on pseudo-delta functions and pseudo-delta sequences and in Section Five we prove a theorem on the pseudo-convolution of pseudo-delta sequences. In Section Six we present some applications in the optimization theory.

2. SEMIRINGS ON AN INTERVAL OF EXTENDED REALS

Let $[a,b]$ be a closed subinterval of $[-\infty,+\infty]$ (in some cases we will also take semiclosed subintervals). The partial order on $[a,b]$ will be denoted by: $\leq$. The symbol $<$ has the usual meaning: for any $x,y\in[a,b]$, $x<y$ if and only if $x\leq y$ and $x\neq y$.

The structure $([a,b],\oplus,\otimes)$ or $[a,b]^{\oplus,\otimes}$ is a semiring in which the operations $\oplus$ and $\otimes$ have the following properties:

The operation $\oplus$ (pseudo-addition) is a function $\oplus:[a,b]\times[a,b]\to[a,b]$ which is commutative, nondecreasing (with respect to $\leq$) associative and either a or b is a zero element, denoted by $\mathbf{0}$, i.e., for each $x\in[a,b]$ $\mathbf{0}\oplus x=x$ holds.

The pseudo-addition of n elements is defined by $\oplus_{i=1}^{n} x_i = x_n \oplus(\oplus_{i=1}^{n-1} x_i)$. We define further: $\oplus_{i=1}^{\infty} x_i = \lim_{n\to\infty} \oplus_{i=1}^{n} x_i$.

Pseudo-addition $\oplus$ is idempotent if for any $x\in[a,b]$, $x\oplus x=x$ holds.

Let $[a,b]_+ = \{x : x\in[a,b], x\geq\mathbf{0}\}$.

The operation $\otimes$ (pseudo-multiplication) is a function $\otimes:[a,b]\times[a,b]\to[a,b]$ which is commutative, positively nondecreasing, i.e. $x\leq y$ implies $x\otimes z\leq y\otimes z$, $z\in[a,b]_+$, associative and for which there exists a unit element $\mathbf{1}\in[a,b]$, i.e., for each $x\in[a,b]$ $\mathbf{1}\otimes x=x$.

We suppose, further, $\mathbf{0}\otimes x=\mathbf{0}$ and that $\otimes$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x\otimes(y\oplus z)=(x\otimes y)\oplus(x\otimes z)$.

Let f and h be functions defined on X and with values in a semiring $[a,b]^{\oplus,\otimes}$. Then, we define, for any $x\in X$, $(f\oplus h)(x)=f(x)\oplus h(x)$, $(f\otimes h)(x)=f(x)\otimes h(x)$ and for any $\lambda\in[a,b]$ $(\lambda\otimes f)(x)=\lambda\otimes f(x)$.

In this paper we will consider the following semirings:

I a) (i) The semiring $(-\infty,+\infty\,]^{\min,+}$:

$$x \oplus y = \min\{x, y\}, \quad x \otimes y = x + y, \quad x, y \in (-\infty, +\infty\,].$$

We have $\mathbf{0} = +\infty$ and $\mathbf{1} = 0$. The idempotent operation min induces a partial (full) order in the following way: $x \leq y$ if and only if $\min(x, y) = y$, which is the opposite of the usual order on the interval $(-\infty,+\infty\,]$.

I a) (ii) The semiring $[-\infty,+\infty\,)^{\max,+}$:

$$x \oplus y = \max\{x, y\}, \quad x \otimes y = x + y, \quad x, y \in [-\infty, +\infty\,).$$

We have $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$. The idempotent operation max induces a partial (full) order in the following way: $x \leq y$ if and only if $\max(x, y) = y$. Hence this order is the usual order on the interval $[-\infty,+\infty\,)$.

I b) (i) The semiring $(\,0,+\infty\,]^{\min,\cdot}$:

$$x \oplus y = \min\{x, y\}, \quad x \otimes y = x \cdot y, \quad x, y \in (\,0, +\infty\,].$$

We have $\mathbf{0} = +\infty$ and $\mathbf{1} = 1$. The idempotent operation min induces a partial (full) order in the following way: $x \leq y$ if and only if $\min(x, y) = y$, which is the opposite of the usual order on the interval $(\,0,+\infty\,]$.

I b) (ii) The semiring $[\,0,+\infty\,)^{\max,\cdot}$:

$$x \oplus y = \max\{x, y\}, \quad x \otimes y = x \cdot y, \quad x, y \in [\,0, +\infty\,).$$

We have $\mathbf{0} = 0$ and $\mathbf{1} = 1$. The idempotent operation max induces a partial (full) order in the following way: $x \leq y$ if and only if $\max(x, y) = y$, which is the usual order on the interval $[\,0,+\infty\,)$.

II The semiring $[0,+\infty]^{\min,\max}$.

$$x \oplus y = \min\{x, y\}, \quad x \otimes y = \max\{x, y\}, \quad x, y \in [0, +\infty].$$

We have $\mathbf{0} = +\infty$ and $\mathbf{1} = 0$. The idempotent operation min induces a partial (full) order in the following way: $x \leq y$ if and only if $\min(x, y) = y$. Hence this order is the opposite of the usual order on the interval $[0,+\infty]$. We also consider a semiring with operations $\oplus = \max$ and $\otimes = \min$.

III A nonidempotent case in which the pseudo-operations are defined by monotone and continuous generator g [15], [18].

By Aczel's representation theorem for each strict pseudo-addition $\oplus$ (i.e. continuous and monotone on its domain), there exists a monotone function g (generator for $\oplus$), $g : [a,b] \to [0,\infty]$ such that $g(\mathbf{0}) = 0$ and

$$u \oplus v = g^{-1}(g(u) + g(v)).$$

If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(a) = \infty$ and $g(b) = 0$. Hence g is an isomorphism of semigroup $([a,b],\oplus)$ with the semigroup $([0,\infty],+)$.

Using a generator g of strict pseudo-addition $\oplus$, we can define pseudo-multiplication $\otimes$:

$$u \otimes v = g^{-1}(g(u)\, g(v)).$$

This is the only way to define pseudo-multiplication $\otimes$, which is distributive with respect to $\oplus$ generated by the function g (see [13]). The operation $\otimes$ has all the properties of the pseudo-multiplication from the previous Section. It can be easily seen that $g(\mathbf{1}) = 1$.

We will denote by S the domain of a semiring of the type **I**$-$**III**.

3. MEASURES, INTEGRALS AND CONVOLUTION IN PSEUDO-ANALYSIS

The goal of this Section is to define the integrals based on $\sigma - \oplus$ decomposable measures (see the definition below). For that purpose we omit the details and refer the reader to [18] for a more detailed study of the subject (see also [2], [11], [12], [24]).

Let X be a non-empty set. Let Σ be a $\sigma-$ algebra of subsets of X.

Definition 1. *A set function* $m : \Sigma \to [a,b]$ *(or semiclosed interval) is a* $\oplus-$*decomposable measure if it holds that* $m(\emptyset) = \mathbf{0}$ *(if* $\oplus$ *is idempotent we do not always suppose this condition);* $m(A \cup B) = m(A) \oplus m(B)$ *for* $A, B \in \Sigma$ *such that* $A \cap B = \emptyset$. *A* $\oplus-$*decomposable measure m is* $\sigma - \oplus -$ *decomposable if*

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$$

holds for any sequence $\{ A_i \}$ *of pairwise disjoint sets from* Σ.

Remark that in the case when $\oplus$ is idempotent it is possible that m is not defined on an empty set. Let m be a $\sigma-\oplus$ – decomposable measure. Some function $f: X \to [a,b]$ is measurable if for any $c \in [a,b]$ the set $\{ x : f(x) \leq c \}$ belongs to Σ.

We suppose further that $([a,b],\oplus)$ and $([a,b],\otimes)$ are complete lattice ordered semigroups. A complete lattice means that for each set $A \subset [a,b]$ bounded from above (below) there exists $\sup A\ (\inf A)$. Further, we suppose that $[a,b]$ is endowed with a metric d compatible with sup and inf, i.e., $\lim\sup x_n = x$ and $\lim\inf x_n = x$ imply $\lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

(a) $d(x \oplus y, x' \oplus y') \leq d(x,x') + d(y,y')$

(b) $d(x \oplus y, x' \oplus y') \leq \max\{ d(x,x'), d(y,y') \}$.

We suppose further the monotonicity of the metric d, i.e., $x \leq z \leq y$ implies $d(x,y) \geq \max\{ d(y,z), d(x,z) \}$.

For example, on the semiring $(-\infty,+\infty]^{\min,+}$ the metric $d(x,y) = | e^{-\max(x,y)} - e^{-\min(x,y)} |$ satisfies all of the preceding conditions.

We define the characteristic function with values in a semiring by

$$\chi_A(x) = \begin{cases} \mathbf{0}, & x \notin A \\ \mathbf{1}, & x \in A. \end{cases}$$

The mapping $e: X \to [a,b]$ is an elementary (measurable) function if it has the following representation

$$e = \bigoplus_{i=1}^{\infty} a_i \otimes \chi_{A_i} \text{ for } a_i \in [a,b]$$

and $A_i \in \Sigma$ is disjoint if $\oplus$ is not idempotent.

The pseudo-integral of an elementary function is defined by

$$\int_X^{\oplus} e \otimes dm = \bigoplus_{i=1}^{\infty} a_i \otimes m(A_i).$$

For any measurable function $f: X \to [a,b]$ one can construct a sequence $\{ \varphi_n \}$ of elementary functions such that, for each $x \in X$, $d(\varphi_n(x), f(x)) \to 0$ uniformly as $n \to \infty$ (see [18]). Using this fact we give the following

Definition 2. *The pseudo-integral of a bounded measurable function* $f : X \to [a,b]$ *is defined by*

$$\int_X^{\oplus} f \otimes dm = \lim_{n\to\infty} \int_X^{\oplus} \varphi_n(x) \otimes dm ,$$

where $\{ \varphi_n \}$ *is the sequence of elementary functions constructed in the above-mentioned theorem.*

Some elementary properties (e.g., linearity) and applications of the introduced pseudo-integral can be found in [14], [18], [16], [17], [19].

Definition 3. *Let* $B(X,S)$ *denote the semimodule of all functions from* X *into the semiring* $(S,\oplus,\otimes)$ *such that*

$$\int_X^{\oplus} f(x) dm \in S ,$$

where m is a $\sigma - \oplus$ *– decomposable measure.*

In the rest of the paper we take that $X = \mathbf{R}$.

Definition 4. *The pseudo-convolution of two functions* $f : \mathbf{R} \to [a,b]$ *and* $h : \mathbf{R} \to [a,b]$ *with respect to a* $\oplus$*-decomposable measure m is given in the following way*

$$(f * h)_m(x) = \int_{\mathbf{R}}^{\oplus} f(x-t) \otimes dm_h ,$$

where $m_h = m$ *in the case of* sup*-decomposable measure* $m(A) = \sup_{x\in A} h(x)$ *(and in the case of* inf*-decomposable measure* $m(A) = \inf_{x\in A} h(x)$ *), and* $dm_h = h \otimes dm$ *in the case of* $\oplus$*–decomposable measure m, where* $\oplus$ *has an additive generator g and* $g \circ m$ *is the Lebesgue measure (g-calculus).*

It is easy to check that a pseudo-convolution is a commutative operation. This follows by the equality

$$(f * h)_m(x) = \int_{\mathbf{R}}^{\oplus} f(x-t) \otimes dm_h = \int_{\mathbf{R}}^{\oplus} h(x-t) \otimes dm_f = (h * f)(x) .$$

It is also an associative operation.

The following examples show the explicit forms of pseudo-integral and pseudo-convolution for special important cases (see also [20], [21]).

I a) (i) For any real valued function h bounded from below we can define a σ - sup - decomposable measure $m(A) = \inf_{x \in A} h(x) \quad (A \subset \mathbf{R})$. By taking $\oplus = \min = \inf$, $\otimes = +$, we obtain

$$\int_{\mathbf{R}}^{\oplus} f \otimes dm = \inf_{x \in \mathbf{R}} (f(x) + h(x)),$$

for f bounded below. We will denote by $B(X, (-\infty, +\infty]^{\min,+})$ the semiring of all functions bounded from below (with respect to the usual order). In this case the pseudo convolution becomes

$$(f * h)_m(x) = \int_{\mathbf{R}}^{\oplus} f(x-t) \otimes dm_h = \int_{\mathbf{R}}^{\oplus} h(x-t) \otimes dm_f = \inf_{t \in \mathbf{R}} (f(t) + h(x-t)).$$

I a) (ii) For any real valued function h bounded from above we can define a σ - sup - decomposable measure $m(A) = \sup_{x \in A} h(x) \quad (A \subset \mathbf{R})$, Taking $\oplus = \max = \sup$, $\otimes = +$, we obtain

$$\int_{\mathbf{R}}^{\oplus} f \otimes dm = \sup_{x \in \mathbf{R}} (f(x) + h(x)),$$

for f bounded above. We will denote by $B(X, [-\infty, +\infty)^{\max,+})$ the semiring of all functions bounded from above.

$$(f * h)_m(x) = \int_{\mathbf{R}}^{\oplus} f(x-t) \otimes dm_h = \sup_{\mathbf{R}} h(x-t) \otimes dm_f = \sup_{t \in \mathbf{R}} (f(t) + h(x-t)).$$

Cases **I b) (i)** and **I b) (ii)** are treated similarly, taking $\cdot$ instead of $+$.

II For bounded functions f with values in semiring $[0, +\infty]^{\min,\max}$ the pseudo-integral based on inf-decomposable measure m, $m(A) = \inf_{x \in A} h(x)$, is given by

$$\int_{\mathbf{R}}^{\oplus} f \otimes dm = \inf_{x \in \mathbf{R}} (\max(f(x), h(x))),$$

and the pseudo-convolution of the functions f and h will be

$$(f * h)_m(x) = \int_{\mathbf{R}}^{\oplus} f(x-t) \otimes dm_h = \inf_{t \in \mathbf{R}} (\max(f(t), h(x-t))).$$

III If $\oplus$ is a strict pseudo-addition with a monotone generator g, $g \circ m : \Sigma \to [0, g(c)]$ with $c \in [a, b]$ is a measure and f is a measurable function, we have

$$\int_X^{\oplus} f \otimes dm = g^{-1}\left(\int_X (g \circ f) \cdot dx\right),$$

where $dx = d(g \circ m)$ is the Lebesgue measure and $u \otimes v = g^{-1}(g(u) \cdot g(v))$. In the sequel, generators g will be monotone and continuous functions, so $g^{-1}(g(s)) = g(g^{-1}(s)) = s$.

The pseudo-convolution in the sense of the g-integral is given by

$$(f * h)(x) = g^{-1}\left(\int g(f(t)) \cdot g(h(x-t))\, dt\right). \tag{1}$$

Remark. Using continuously differentiable generators g, the g-derivative of a differentiable function f can be defined. This definition has local character and the function f must have the same monotonicity as the generator g. The properties of the g-derivative and g-integral and the corresponding applications can be found, e.g. in [18], [15], [14], [22], [20].

4. PSEUDO-DELTA SEQUENCES

4.1. The 'delta function' and delta convergent sequences – the classical case

About sixty years ago, Paul Dirac introduced the famous 'delta function' while solving problems in quantum-mechanics. Although the formal use of this object has become an efficient tool in quantum mechanics calculus, this 'function' has contradictory properties: it differs from zero in only one point, but its integral equals one. This contradiction has been overcome by the construction of various delta sequences which in a certain sense converge to a 'delta function' and also by defining a 'delta function' as a functional. The classical 'delta function' is introduced as a 'function' defined on the real line as follows

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$

with an additional property $\int \delta(x)\, dx = 1$. We have further that for a test function ϕ (infinitely differentiable function with compact support)

$$\int_{\mathbf{R}} \delta(t-x)\phi(t)\, dt = \phi(x) \quad \text{holds.}$$

A classical delta sequence $\{\delta_n\}$ is a sequence of functions which converges to a 'delta function' in the following sense

$$\lim_{n\to\infty} \int_{\mathbf{R}} \delta_n(t-x)\phi(t)\, dt = \phi(x).$$

and at the same time the equation

$$\lim_{n\to\infty}\int_a^b \delta_n(t-x)\,dt = \begin{cases} 0 & \text{if } b < x \text{ or } x < a, \\ 1 & \text{if } a \le x \le b. \end{cases} \tag{2}$$

is satisfied.

By (2) it follows that $\lim_{n\to\infty}\int_{\mathbf{R}} \delta_n(x)\,dx = 1$, which explains the equation $\int_{\mathbf{R}} \delta(x)\,dx = 1$.

Some well known examples (see [9], [23]) of delta sequences are:

$$\left\{\frac{1}{\pi}\frac{\sin nx}{x}\right\},\ \left\{\frac{1}{\pi}\frac{n}{1+n^2x^2}\right\},\ \left\{\sqrt{\frac{n}{4\pi}}\,e^{-nx^2/4}\right\}.$$

4.2. Idempotent analysis and g-calculus

If the pseudo-addition $\oplus$ is either max or min, the pseudo-delta function will be (see also [12], [21])

$$\delta^{\oplus,\otimes}(x) = \begin{cases} \mathbf{1} & \text{if } x = 0, \\ \mathbf{0} & \text{if } x \neq 0. \end{cases} \tag{3}$$

As usual $\mathbf{0}$ is the zero element for pseudo-addition, and $\mathbf{1}$ is the unit element for pseudo-multiplication. The pseudo-delta function defined by (3) is the unit element for the 'convolution' in the sense of a pseudo-integral. As in the classical theory, we have

$$\int^{\oplus} \delta^{\oplus,\otimes}(t) \otimes dm = \mathbf{1}.$$

Since $\delta^{\oplus,\otimes}$ may not be a function in the usual sense, we have investigated the existence of the sequences of functions which converge to $\delta^{\oplus,\otimes}$.

Definition 5. *Pseudo-delta sequence* $\{\delta_n^{\oplus,\otimes}\}$ *is a sequence of functions* $\delta_n^{\oplus,\otimes} \in B(X,[a,b]^{\oplus,\otimes})$ *with the properties:*

1. $\lim_{n\to\infty} (\delta_n^{\oplus,\otimes} * f)(x) = f(x)$.
2. $\lim_{n\to\infty} \int_X^{\oplus} \delta_n^{\oplus,\otimes}(x)\,dx = \mathbf{1}$.

For simplicity we suppose that the elements of delta sequences are even functions. We gave the following characterizations ([21]):

I a) (i) For the semiring $(-\infty,+\infty\,]^{\min,+}$, from (3) it follows:

$$\delta^{\min,+}(x)=\begin{cases}\mathbf{1}\quad(=0) & \text{if } x=0,\\ \mathbf{0}\quad(=\infty) & \text{if } x\neq 0.\end{cases}$$

The pseudo-delta sequence will be a sequence which converges to this pseudo-delta function which is the unit element for the pseudo-convolution. Hence for the pseudo-delta sequence $\{\,\delta_n^{\min,+}\,\}$ the following equation must hold

$$\lim_{n\to\infty}(f*\delta_n^{\min,+})(x)=\lim_{n\to\infty}\int_{\mathbf{R}}^{\oplus}\delta_n^{\min,+}(x-t)\otimes dm_f=$$
$$\lim_{n\to\infty}\inf_{t\in\mathbf{R}}(f(t)+\delta_n^{\min,+}(x-t))=f(x)\ .$$

Therefore we have:

Theorem 1. *The sequence of functions* $\{\,\delta_n^{\min,+}\,\}$ *is a pseudo-delta sequence in* $B(\,\mathbf{R},(-\infty,+\infty\,]^{\min,+}\,)$ *if the following three conditions are satisfied*

1. $\delta_n^{\min,+}(0)=0$;
2. $\delta_n^{\min,+}(x)>0$ *if* $x\neq 0$;
3. $\delta_n^{\min,+}(x)\to\infty$ *when* $n\to\infty$ *for* $x\neq 0$.

An example of $\{\,\delta_n^{\min,+}\,\}$ is the sequence $\{\,n\cdot x^{2m}\,\}$, for an arbitrary but fixed $m\in\mathbf{N}$.

I a) (ii) For the semiring $[-\infty,+\infty\,)^{\max,+}$, the pseudo-delta function has the following form

$$\delta^{\max,+}(x)=\begin{cases}\mathbf{1}\quad(=0) & \text{if } x=0,\\ \mathbf{0}\quad(=-\infty) & \text{if } x\neq 0.\end{cases}$$

Pseudo-delta sequence $\{\,\delta_n^{\max,+}\,\}$ is a sequence for which the following equation holds

$$\lim_{n\to\infty}(f*\delta_n^{\max,+})(x)=\lim_{n\to\infty}\int_{\mathbf{R}}^{\oplus}\delta_n^{\max,+}(x-t)\otimes dm_f=$$
$$\lim_{n\to\infty}\sup_{t\in\mathbf{R}}(f(t)+\delta_n^{\max,+}(x-t))=f(x)\ .$$

We now have:

Theorem 2. *The sequence of functions* $\{ \delta_n^{\max,+} \}$ *is a pseudo-delta sequence if the following three conditions are satisfied*

1. $\delta_n^{\max,+}(0) = 0$;
2. $\delta_n^{\max,+}(x) < 0$ *if* $x \neq 0$;
3. $\delta_n^{\max,+}(x) \to -\infty$ *when* $n \to \infty$ *for* $x \neq 0$.

Here, we can take the following example $\delta_n^{\max,+}(x) = -n \cdot x^{2m}$, $m \in \mathbf{N}$.

Cases **I b) (i)** and **I b) (ii)** can be treated similarly, taking into account that, in these cases, we have $\mathbf{1} = 1$. Case **II** is analogous to case **I a) (i)**.

III We start with a counterexample which shows that definition (3) is not adequate if the $\oplus$ is defined by a monotone and continuous generator *g*.

Example 1. Let us assume that the *g*-delta function $\delta^{\oplus}$ is defined by (3), and let the generator *g* be

$$g(x) = \frac{x}{1-x}, \quad x \in [0,1].$$

Then we have

$$(f * \delta^{\oplus})(x) = \int^{\oplus} f(t) \otimes \delta^{\oplus}(x-t) \otimes dm =$$
$$g^{-1}(\int g(g^{-1}(g(f(t)) \cdot g(\delta^{\oplus}(x-t)))dt) = g^{-1}(\int g(f(t))dt).$$

We obtained the *g*-integral of function *f*, which, in general, differs from function *f*. For example, (see ([22])) for $f(x) = x$ we obtain

$$\int^{\oplus} x \otimes dm = \frac{x + \ln(1-x)}{x + \ln(1-x) - 1} \oplus C.$$

Hence $(f * \delta^{\oplus})(x) \neq f(x)$.

For that reason we give the following definition.

Definition 6. *A g-delta function is the mapping* $\delta^g : \mathbf{R} \to [a,b]$ *defined by*

$$\delta^g(x) = \begin{cases} b & if\ x = 0, \\ a & if\ x \neq 0 \end{cases} \tag{4}$$

in the case of $a = \mathbf{0}$. *If* $b = \mathbf{0}$ *the g-delta function is:*

$$\delta^{g}(x)=\begin{cases} a & \text{if } x=0, \\ b & \text{if } x\neq 0. \end{cases} \tag{5}$$

We can easily show that we again have $\int^{\oplus}\delta^{\oplus,\odot}(t)\otimes dm=\mathbf{1}$. Recall that the pseudo-convolution is given by $(f*h)(x)=g^{-1}(\int g(f(t))\cdot g(h(x-t))dt)$.

The g-delta sequence $\{\ \delta_n^g\ \}$, is a sequence of functions which converges to δ^g, i.e.,

$$\lim_{n\to\infty} g^{-1}(\int g(f(t))\cdot g(\delta_n^g(x-t))dt)=f(x). \tag{6}$$

An almost immediate consequence of the (6) is the following

Theorem 3. *A sequence $\{\ \delta_n^g\ \}$ of functions is a g-delta sequence if and only if* $\delta_n^g=g^{-1}\circ\delta_n$, *where* $\{\ \delta_n\ \}$ *is a classical delta-sequence with the property* $\delta_n(x)\geq 0\ \ (x\in\mathbf{R})$.

Example 2. Let $g(t)=e^{-t}$, $g^{-1}(u)=-\ln u$. Generator g is a decreasing function, $t\in[a,b]=[-\infty,\infty]$, and $b=\mathbf{0}$, $0=\mathbf{1}$. From (3) it follows: $\delta_n^g(x)=-\ln(\delta_n(x))$. If we put $\delta_n(x)=\frac{n}{\pi}e^{-nx^2}$, we obtain $\delta_n^g(x)=nx^2-\ln(n/\pi)$.

Example 3. Let $g(t)=-\ln(1-t)$, $g^{-1}(u)=1-e^{-u}$. Generator g is an increasing function, $t\in[a,b]=[0,1]$, and $a=0=\mathbf{0}$, $1-e^{-1}=\mathbf{1}$. The g-delta sequence is $\delta_n^g(x)=1-e^{-\delta_n(x)}$.

5. CONVOLUTION OF PSEUDO-DELTA SEQUENCES

We now introduce a special subclass of pseudo-delta sequences.

Definition 7. *A pseudo-delta sequence* $\{\ \delta_n^{\oplus,\odot}\ \}$ *is* asymptotically admissible *if* $\lim_{x\to\pm\infty}\delta_n^{\oplus,\odot}>\mathbf{1}$, *for every* $n\in\mathbf{N}$.

Example 4. Pseudo-delta sequence $\{\ \delta_n^{\min,+}\ \}$ is *asymptotically admissible* if it satisfies the following condition: $\lim_{x\to\pm\infty}\delta_n^{\min,+}(x)>\varepsilon>0$ for some $\varepsilon>0$.

Example 5. Pseudo-delta sequence $\delta_n^{\max,+}$ is *asymptotically admissible* if it satisfies the following condition: $\lim_{x\to\pm\infty}\delta_n^{\max,+}(x)<-\varepsilon<0$ for some $\varepsilon>0$.

Theorem 4. a) *The pseudo-convolution for idempotent cases* (**I** *and* **II**) *of two asymptotically admissible pseudo-delta sequences is an asymptotically admissible pseudo-delta sequence* (*for cases* **I** *and* **II**, respectively).
b) *The pseudo-convolution of two g-delta sequences is a g-delta sequence.*

Proof: a) We show that the convolution of two asymptotically admissible pseudo-delta sequences for Case **I a) (i)** is again an asymptotically admissible pseudo-delta sequence for the Case **I a) (i)**. The other idempotent cases can be proved in an analogous way.

Let $\{ \delta_{1,n}^{\min,+} \}$ and $\{ \delta_{2,n}^{\min,+} \}$ be two pseudo delta sequences. We show that their pseudo-convolution $(\delta_{1,n}^{\min,+} * \delta_{2,n}^{\min,+})(x) = \inf_{t\in\mathbf{R}} (\delta_{1,n}^{\min,+}(t) + \delta_{2,n}^{\min,+}(x-t))$ is a pseudo-delta sequence, i.e., it satisfies conditions 1 – 3 in Theorem 1.

1. Let $x = 0$. For $t = 0$ we have $\inf_{t\in\mathbf{R}} (\delta_{1,n}^{\min,+}(0) + \delta_{2,n}^{\min,+}(0)) = 0$.

   If $t \neq 0$ we have $\delta_{1,n}^{\min,+}(t) + \delta_{2,n}^{\min,+}(-t) > 0$. Hence

   $$(\delta_{1,n}^{\min,+} * \delta_{2,n}^{\min,+})(0) = \inf_{t\in\mathbf{R}} (\delta_{1,n}^{\min,+}(t) + \delta_{2,n}^{\min,+}(-t)) = 0 .$$

2. Since $\{ \delta_{1,n}^{\min,+} \}$ and $\{ \delta_{2,n}^{\min,+} \}$ are both asymptotically admissible pseudo-delta sequences, it is easy to see that their pseudo-convolution, $\inf_{t\in\mathbf{R}} (\delta_{1,n}^{\min,+}(t) + \delta_{2,n}^{\min,+}(x-t))$ is strictly positive for all $x \neq 0$.

3. For every arbitrary but fixed x we have $\lim_{n\to\infty} \inf_{t\in\mathbf{R}} (\delta_{1,n}^{\min,+}(t) + \delta_{2,n}^{\min,+}(x-t)) = \infty$.

We shall prove that $\{ \delta_{1,n}^{\min,+} * \delta_{2,n}^{\min,+} \}$ is an asymptotically admissible pseudo-delta sequence. By definition we have

$$\lim_{x\to\pm\infty} \delta_{1,n}^{\min,+}(x) > \varepsilon_1 > 0 \quad \text{for some} \quad \varepsilon_1 > 0 \quad \text{and}$$

$$\lim_{x\to\pm\infty} \delta_{2,n}^{\min,+}(x) > \varepsilon_2 > 0 \quad \text{for some} \quad \varepsilon_2 > 0 .$$

Now

$$\lim_{n\to\infty} \inf_{t\in\mathbf{R}} (\delta_{1,n}^{\min,+}(t) + \delta_{2,n}^{\min,+}(x-t)) =$$

$$\lim_{n\to\infty} (\delta_{1,n}^{\min,+}(t_0) + \delta_{2,n}^{\min,+}(x-t_0)) \geq \delta_{1,n}^{\min,+}(t_0) + \varepsilon_2 > 0 ,$$

where t_0 denotes the point in which the infimum is reached.

Thus, the first part of the theorem is proved.

b) Let $\delta^g_{1,n}(x)$ and $\delta^g_{2,n}(x)$ be pseudo-delta sequences. Their convolution is defined by $(\delta^g_{1,n} * \delta^g_{2,n})(x)$. So we have

$$g^{-1}(\int g(\delta^g_{1,n}(t)\cdot g(\delta^g_{2,n}(x-t))dt) = g^{-1}(\int g(g^{-1}(\delta_{1,n}(t)))\cdot g(g^{-1}(\delta_{2,n}(x-t)))dt) =$$
$$g^{-1}(\int \delta_{1,n}(t)\cdot \delta_{2,n}(x-t)dt) = g^{-1}(\delta_{1,n} * \delta_{2,n}(x)) = g^{-1}(\delta_n(x)) = \delta^g_n(x),$$

where $\{\delta_{1,n}\}$, and $\{\delta_{2,n}\}$ and $\{\delta_n\}$ are classical nonnegative delta sequences. Here we used the fact that the convolution of two delta sequences is a delta sequence (see [1] p. 117).

6. PSEUDO-INTEGRAL REPRESENTATION: APPLICATIONS IN THE OPTIMIZATION THEORY

In optimization problems the operator known as the Bellman operator often occurs ([3]). Let X and Y be arbitrary non-empty subsets of **R**. Let $k \in C(X \times Y)$. Then the operator $B : C(Y) \to C(X)$ is defined by

$$(Bf)(x) = \max_{y \in Y}(k(x,y) + f(y)).$$

Using the pseudo-integral with respect to $\oplus = \max$ and $\otimes = +$, we can rewrite the preceding operator in the form

$$(Bf)(x) = \int_Y^{\oplus} k(x,y)\,dm_f = \int_Y^{\oplus} k(x,y) \otimes f(y)\,dy.$$

This non-linear operator is pseudo-linear in the following sense

Definition 8. *A mapping $T : B(Y,S) \to B(X,S)$ is pseudo-linear if the following conditions are satisfied*

$$T(\mathbf{0}) = \mathbf{0}\,;\ T(f \oplus h) = T(f) \oplus T(h)\,;\ T(\lambda \otimes f) = \lambda \otimes T(f) \quad (\lambda \in S,\ f,h \in B(Y,S)).$$

The Bellman operator B can be extended over the whole $B(Y, S^{\max,+})$ by

$$(Bf)(x) = \sup_{y \in Y}(k(x,y) + f(y)).$$

In paper [21] we found a condition which allows pseudo-integral representation for a class of pseudo-linear operators. Namely, the following theorem is proved:

Theorem 5. *Let S be one of the semirings of the type* **I–II**, *with the property that for every bounded subset* $\{ a_\alpha \}$ *of* S *and* $\lambda \in S$ *we have* $\oplus_\alpha(\lambda \otimes a_\alpha) = \lambda \otimes \oplus_\alpha a_\alpha$. *If* $T : B(Y,S) \to B(X,S)$ *is a pseudo-linear operator, then it satisfies the condition*

$$T(\oplus_\alpha f_\alpha) = \oplus_\alpha T(f_\alpha)$$

if and only if there exists a unique function $k \in B(X \times Y, S)$ *such that*

$$(Tf)(x) = \int_Y^{\oplus} k(x,y) \otimes f(y)\,dy\,.$$

In the proof of the theorem (see [21]) we used a pseudo-delta function to obtain the kernel of the pseudo-integral representation.

We can now use a pseudo-delta sequence $\{ (\delta_y^{\oplus,\otimes})_n \}$, where $(\delta_y^{\oplus,\otimes})_n(\cdot) = \delta_n^{\oplus,\otimes}(\cdot - y)$, to construct a sequence $\{ k_n \}$ of pseudo-kernels as follows

$$k_n(x,y) = (T(\delta_y^{\oplus,\otimes})_n)(x)\,.$$

Then the sequence $\{ T_n f \}$ of pseudo-linear operators defined by

$$(T_n f)(x) = \int_Y^{\oplus} k_n(x,y) \otimes f(y)\,dy$$

tends pseudo-weakly to Tf, i.e.,

$$(T_n f, \varphi)_\oplus(x) = \int_X^{\oplus}\Big(\int_Y^{\oplus} k_n(x,y) \otimes f(y)\,dy\Big) \otimes \varphi(x)\,dx$$

tends to

$$(Tf, \varphi)_\oplus(x) = \int_X^{\oplus}\Big(\int_Y^{\oplus} k(x,y) \otimes f(y)\,dy\Big) \otimes \varphi(x)\,dx$$

as $n \to \infty$. In such a way we can approximate pseudo-linear operators.

The representation of a pseudo-linear operator, in the form of a Bellman operator, occurs in many fields. Let us give only two examples.

The Shortest Path. In discrete optimization, the pseudo-additive Bellman operator occurs in trajectory problems ([2], [12]) with the variable time of motion. Namely, let (V,A) be a graph with the set of vertices V and the set of arcs A. The problem of the shortest path is the construction of the shortest path from the fixed point $x_0 \in V$ to any point $x \in V$, with the motion beginning at time t_0. If we denote by $h(a,t)$ the time of motion on the arc $a \in A$ beginning at time t, then the corresponding Bellman operator B is defined on the set of functions

$$\{ f | \ f : V \to \{ \ t \in \mathbf{R} | \ t \geq t_0 \ \} \}$$

by

$$(Bf)(x_i) = \min_{a \in A} \min_{\tau \geq f(x_i)} (\tau + h(a,\tau)) ,$$

where $\min_{a \in A}$ is taken over all arcs $a \in A$ entering x_i. Then the shortest time $T(x), T : V \to [a,b]$, for the motion from x_0 to x satisfies the following Bellman equation $T = (BT) \oplus (t_0 \otimes \delta_{x_0})$, where for

$$\delta_{x_0}(x) = \begin{cases} \mathbf{1} & \text{if } x = x_0 , \\ +\infty & \text{if } x \in V \setminus \{ \ x_0 \ \}. \end{cases}$$

Antagonistic multistep games. Let $X = \{ \ 1,2,\ldots,n \ \}$ and let M be the metric space of the strategies of two players. Let $p_{ij}(a,b)$ be the probability of transition from the state i to the state j, if the first player chooses the strategy $a \in M$ and the second one the strategy $b \in M$. If we denote by $I_{ij}(a,b)$ the income of the first player from the given transition, then the game is called a game with value if the following equality holds for all $\mathbf{x} = (x^1, x^2, \ldots, x^n) \in \mathbf{R}^n$

$$\min_a \max_b \sum_{j=1}^{n} p_{ij}(a,b)(x^j + I_{ij}(a,b)) = \max_a \min_b \sum_{j=1}^{n} p_{ij}(a,b)(x^j + I_{ij}(a,b)) .$$

The operator $B : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$B_i(x) = \min_a \max_b \sum_{j=1}^{n} p_{ij}(a,b)(x^j + I_{ij}(a,b))$$

is the Bellman oparator for the game. It has the following properties

$$B(\mathbf{c} + \mathbf{x}) = \mathbf{c} + B(\mathbf{x}) \quad \mathbf{x} \in \mathbf{R}^n, \quad \mathbf{c} = (c,c,\ldots c) \in \mathbf{R}^n ; \tag{7}$$

$$| \ B(\mathbf{x}) - B(\mathbf{y}) \ | \leq | \mathbf{x} - \mathbf{y} | \quad \mathbf{x} \in \mathbf{R}^n, \quad \mathbf{y} \in \mathbf{R}^n , \tag{8}$$

where $| \mathbf{x} | = \max_i x^i$.

It can be proved, using dynamic programming [4], that the value of the r-step game given by the initial position i and the terminal income $\mathbf{x} \in \mathbf{R}^n$ of the first player always exists and is given by $B_i^r(\mathbf{x})$. The representation of any operator B which satisfies (7) and (8) has the form of the Bellman operator of the game. For details see [10].

7. CONCLUSIONS

We presented a part of even developing pseudo-analysis which serves as a mathematical base for many different fields such as fuzzy logic, decision theory, system theory, nonlinear equations, optimization, and control theory. Pseudo-delta functions and pseudo-delta sequences in nonlinear analysis play an analogous role to delta-functions and delta sequences in linear analysis.

REFERENCES

[1] Antosik, P., Mikusiński, J., Sikorski, R., *Theory of Distributions*, Elsevier Sci. Publ. Comp., Amsterdam, PWN-Polish Sci. Publ., Warszawa, 1973.

[2] Baccelli, F., Cohen, G., Olsder, G.J., and Quadrat, *Synchronization and Linearity: an Algebra for Discrete Event Systems*, John-Wiley and Sons, New York, 1992.

[3] Bellman, R., Karush, W., "On a new functional transform in analysis: the Maximum Transform", *Bull. Amer. Math. Soc.*, 67 (1961) 501-503.

[4] Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, NJ, 1960.

[5] Cuninghame-Green, R.A., *Minimax Algebra, Lecture Notes in Economics and Math. Systems 166*, Springer-Verlag, Berlin-Heidelberg- New York, 1979.

[6] Denneberg, D., *Non-Additive Measure and Integral*, Kluwer Academic Publishers, Dordrecht, 1994.

[7] Golan, J.S., *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Sciences*, *Pitman Monog. and Surveys in Pure and Appl. Maths. 54*, Longman, New York, 1992.

[8] Grabisch, M., Nguyen, H.T., Walker, E.A., *Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference,* Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[9] Kanwal, R., *Generalized Functions: Theory and Technique*, Academic Press, 1983.

[10] Kolokoltsov, V.N., "On linear, additive, and homogenous operators in idempotent analysis", in: V.P. Maslov and S.N. Samborskij (eds.), *Idempotent Analysis, Advances in Soviet Mathematics 13*, Amer. Math. Soc., Providence, Rhode Island, 1992, 87-101.

[11] Maslov, V.P., "On a new superposition principle in optimization theory", *Uspehi mat. nauk,* 42 (1987) 39-48.

[12] Maslov, V.P., and Samborskij S.N. (eds.), *Idempotent Analysis, Advances in Soviet Mathematics 13*, Amer. Math. Soc., Providence, Rhode Island, 1992.

[13] Mesiar, R. and Rybárik, J., "Pseudo-arithmetical operations", *Tatra Mountains Math. Publ.*, 2 (1993) 185-192.

[14] Pap, E., "Integral generated by decomposable measure", *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, 20 (1) (1990) 135-144.

[15] Pap, E., "*g*-calculus", *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, 23 (1) (1993) 145-150.

[16] Pap, E., "Solution of nonlinear differential and difference equations", *Proc. of EUFIT '93*, 1 (1993) 498-504.

[17] Pap, E., "Nonlinear difference equations and neural nets", *BUSEFAL*, 60 (1994) 7-14.

[18] Pap, E., *Null-Additive Set Functions*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[19] Pap, E., "Decomposable measures and nonlinear equations", *Fuzzy Sets and Systems,* 92 (1997) 205-221.

[20] Pap, E., Ralević, N., "Pseudo-Laplace transform", *Nonlinear Analysis* (to appear).

[21] Pap, E., Teofanov, N., "Pseudo-delta sequences" (preprint).

[22] Ralević, N., "Some new properties of *g*-calculus", *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, 24,1 (1994) 139-157.

[23] Schwartz, L., *Théorie des distributions I, II*, Paris, 1950, 1951.

[24] Sugeno, M., and Murofushi, T., "Pseudo-additive measures and integrals", *J. Math. Anal. Appl.*, 122 (1987) 197-222.

[25] Wang Z., and Klir, G.J., *Fuzzy Measure Theory*, Plenum Press, New York, 1992.