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THE PROCEDURE TO DETERMINE MICROSTATE PROBABILITIES IN A TRUNK GROUP SERVING OVERFLOW TRAFFIC

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Abstract: In this paper the procedure to determine microstate probabilities in a trunk group with sequential searching and offered overflow traffic is developed. Generating function technique is used to evaluate the common form of the solution for the statistical equilibrium equation system. A multistep iterative procedure to calculate unknown coefficients in common solution to the system is proposed.

Keywords: Owerflow traffic, microstate probability, sequential serving, switching.

1. INTRODUCTION

For a serving system consisting of a fully available trunk group with losses and offered Poisson traffic, the probability of a certain number of trunks being occupied, i.e. the macrostate probability, can be evalueted from the Erlang distribution. The microstate probability, i.e. the probability of a specific trunk state, with random searching, can be calculated by dividing the macrostate probability with the total number of microstate combinations. Microstate probability recognition is useful for the valuation of blocking calculation methods in multistage switching networks, as well as for more sophisticated traffic parameter analysis in circuit switching telecommunication networks.

When there is sequential searching it is easy to evaluate the traffic served in different trunks, but the problem of determining microstate probabilities is rather complicated and should be numerically solved.

The problem statement of the serving model for correlated overflow traffic formed from the request of one of the lost Poisson traffic in the primary trunk group is given. The model generalizes some simpler traffic cases. The solution approach is based on the technique of unifying the statistical equilibrium equations by generating functions.

The overflow traffic basic model is analytically solved in several different ways. The procedure commonly used as basis for more complex problem analysis is given in detail in [1]. In [2] the problem of two overflow components has been solved but with no explicit or recursive solution form. The problem of serving correlated overflow components, generalized with the assumption of changed serving intensity, is solved explicitly in [3].

Based on experience from the above references, the procedure to determine microstate probabilities in a trunk group serving correlated overflow traffic in the case sequential searching is developed in this paper. The procedure can be applied to evaluate microstate probabilities in the first two models [1,2] and can also be adapted for the case of changed serving intensity as in the third model [3].

By evaluating the general solution form for the system of $(c+1)2^s$ equations of stochastic equilibrium, where c is the number of trunks in the primary trunk group and s in the secondary one, the problem is reduced to the numerical solving of 2^s unknown coefficients, and by implementing type [3] solutions it can be reduced to $2^s - 2$.

Appreciation of the connection between part of unknown coefficients and the coefficients in a smaller secondary trunk group is enabled by the step by step calculating procedure until the final step, when $2^{s-1} - 1$ equations are to be solved.

It can be noted that in [4] the technique for solving the special forms of the grading model that can be adapted to our problem is evalueted. But the author outlines the complexity of state probability determination and gives only the analytical solution for losses.

2. THE STATISTICAL EQUILIBRIUM EQUATION SYSTEM

The system analyzed in this paper is shown in Fig. 1. It can be assumed to be general because it assumes that the overflow traffic originates from part of the Poisson traffic offered to a primary trunk group.

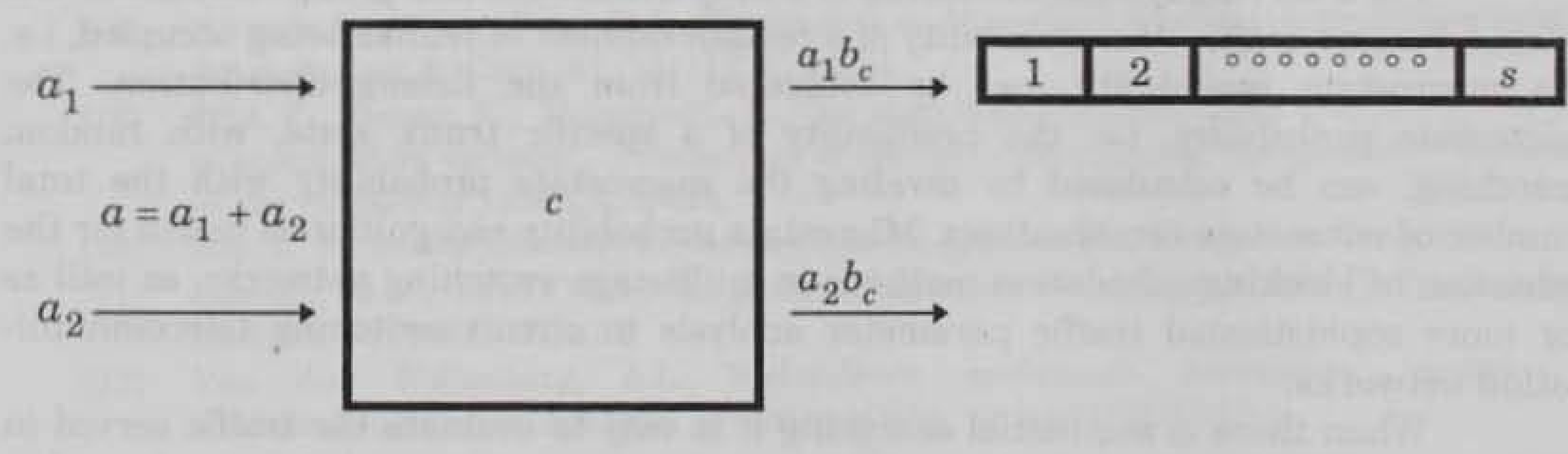

Figure 1. The analyzed sequential serving system

The legend in Fig. 1. is:

c - the number of trunks in a primary group,
s - the number of trunks in a secondary group,
$a = a_1 + a_2$ - the Poisson traffic intensity in the primary trunk group,
$b_c = B(c, a)$ - the loss in the primary trunk group (first Erlang formula),
$a_1 b_c$ - the overflow traffic intensity in the secondary trunk group,
$a_2 b_c$ - the lost part of traffic a_2.

For the given model the statistical equlibrium equations system is:

for

$$0 \le m < c, \qquad n_i = 0,1, \qquad i = 1,\dots,s,$$

$$(a + m + \sum_{j=1}^{s} n_j)p(m, n_1,\dots,n_s) = ap(m-1, n_1,\dots,n_s) + (m+1)p(m+1, n_1,\dots,n_s) +$$
$$+ \sum_{j=1 (n_j=0)}^{s} (n_j + 1)p(m, n_1,\dots,n_j + 1,\dots,n_s); \tag{1}$$

for

$$m = c, \qquad n_1, n_2,\dots,n_i = 1, \qquad n_{i+1} = 0, \qquad i = 0,1,\dots,s-1,$$

$$(a_1 + c + \sum_{j=1}^{s} n_j)p(c, n_1,\dots,n_s) = a\, p(c-1, n_1,\dots,n_s) + a_1 \sum_{j=1}^{i+1} p(c,\dots,n_j - 1,\dots,n_s) +$$
$$+ \sum_{j=i+1 (n_j=0)}^{s} (n_j + 1)p(c,\dots,n_j + 1,\dots,n_s); \tag{2}$$

and for

$$m = c, \qquad n_1,\dots,n_s = 1,$$

$$(c+s)p(c,1,\dots,1) = ap(c-1,1,\dots,1) + a_1 \sum_{j=1}^{s} p(c,\dots,n_j - 1,\dots). \tag{3}$$

Any of $(c+1)2^s$ equations in the previous system is dependent and the system should be completed with the normalizing condition

$$\sum_{m=0}^{c} \sum_{n_1=0}^{1} \cdots \sum_{n_s=0}^{1} p(m, n_1,\dots,n_s) = 1. \tag{4}$$

The meaning of the parameters used in (1-4) is:

m - the number of occupied trunks in group c,

n_i - the state (0 or 1) of the i^{th} trunk in group s,

$p(m, n_1,\dots,n_s)$ - the probability that m trunks in group c are occupied and individual trunk states in group s.

3. THE ANALYTICAL SOLUTION TO THE EQUATION SYSTEM

The generating function technique will be used for analytical evaluation of the state probability of equation system (1-4). For the supplementary equations system of type (1), with $m \ge 0$, the generating function is defined as

$$P(x,y_1,y_2,\ldots,y_s)=P=\sum_{m=0}^{\infty}\sum_{n_1=0}^{1}\cdots\sum_{n_s=0}^{1}p(m,n_1,\ldots,n_s)x^m y_1^{n_1}\cdots y_s^{n_s}. \tag{5}$$

Based on (5), from the supplementary system, the Lagrange linear partial differential equations of the first order is obtained

$$(1-x)\frac{\partial P}{\partial x}+\sum_{j=1}^{s}(1-y_j)\frac{\partial P}{\partial y_j}=a(1-x)P. \tag{6}$$

The solution to this equation can be evaluated through characteristic equation and with development of the Taylor series the form is

$$P=\sum_{m=0}^{\infty}\sum_{n_1=0}^{1}\cdots\sum_{n_s=0}^{1}\sigma_{n_{1s}}(m)C(n_1,\ldots,n_s)x^m(1-y_1)^{n_1}\cdots(1-y_s)^{n_s}. \tag{7}$$

where $n_{ij}=n_i+\ldots+n_j$ and

$$\sigma_n(m)=\begin{cases}\dfrac{a^m}{m!}, & n=0\\[2ex] \displaystyle\sum_{i=0}^{m}\binom{n+i-1}{i}\frac{a^{m-i}}{(m-i)!}, & n>0\end{cases} \tag{8}$$

Expression (7) can be written in the form

$$P=\sum_{m=0}^{\infty}\sum_{n_1=0}^{1}\cdots\sum_{n_s=0}^{1}(-1)^{n_{1s}}\sum_{k_1=n_1}^{1}\cdots\sum_{k_s=n_s}^{1}\sigma_{k_{1s}}(m)C(k_1,\ldots,k_s)x^m y_1^{n_1}\cdots y_s^{n_s}. \tag{9}$$

Hence, compared with (5) it can be concluded that

$$p(m,n_1,\ldots,n_s)=(-1)^{n_{1s}}\sum_{k_1=n_1}^{1}\cdots\sum_{k_s=n_s}^{1}\sigma_{k_{1s}}(m)C(k_1,\ldots,k_s). \tag{10}$$

The next step is the determination of 2^s constants $C(k_1,\ldots,k_s)$, to satisfy Eds. (2-4). If we let $m=c$ in the supplementary system (1) and the subtract from it (2) and (3) a simpler system will be obtained

$$m=c,\qquad n_1,n_2,\ldots,n_i=1,\qquad n_{i+1}=0,\qquad i=0,1,\ldots,s-1,$$

$$a_1\sum_{j=1}^{i+1}p(c,\ldots,n_j-1,\ldots)+a_2p(c,n_1,\ldots,n_s)=(c+1)p(c+1,n_1,\ldots,n_s) \tag{11}$$

and

$$m = c, \qquad n_1, \ldots, n_s = 1,$$

$$a_1 \sum_{j=1}^{s} p(c, \ldots, n_j - 1, \ldots) + ap(c,1,\ldots,1) = (c+1)p(c+1,1,\ldots,1), \tag{12}$$

where $p(c+1, n_1, \ldots, n_s)$ are supplementary probabilities not reflecting the state of the system.

Multiplying each equation by the corresponding $y_1^{n_1} y_2^{n_2} \cdots y_s^{n_s}$ and then summing we get

$$\begin{aligned} a_1 \sum_{i=1}^{s} \Big[y_i \sum_{n_i=0}^{1} \cdots \sum_{n_s=0}^{1} p(c,1,\ldots,1,n_i,\ldots,n_s) y_1 \cdots y_{i-1} y_i^{n_i} \cdots y_s^{n_s} - \\ - y_i \sum_{n_{i+1}=0}^{1} \cdots \sum_{n_s=0}^{1} p(c,1,\ldots,1,n_{i+1},\ldots,n_s) y_1 \cdots y_i y_{i+1}^{n_{i+1}} \cdots y_s^{n_s} \Big] + \\ + ap(c,1,\ldots,1) y_1 \cdots y_s = (c+1) \sum_{n_1=0}^{1} \cdots \sum_{n_s=0}^{1} p(c+1, n_1, \ldots, n_s) y_1^{n_1} \cdots y_s^{n_s}. \end{aligned} \tag{13}$$

Using the expressions for generating functions (9,7), the marginal generating function, equalities [1,2]

$$y_i = 1 - (1 - y_i) = \sum_{n_i=0}^{1} (-1)^{n_i} (1 - y_i)^{n_i} \tag{14}$$

and

$$a\sigma_n(c) - (c+1)\sigma_n(c+1) = -n\sigma_{n+1}(c) \tag{15}$$

with further notation $\sigma_n(c) = \sigma_n$, the following condition is evaluated

$$\begin{aligned} -\sum_{n_1=0}^{1} \cdots \sum_{n_s=0}^{1} n_{1s} \sigma_{n_{1s+1}} C(n_1, \ldots, n_s)(1-y_1)^{n_1} \cdots (1-y_s)^{n_s} = \\ = a_1 \sum_{i=1}^{s} \Big[(-1)^{i-1} \sum_{n_1=1}^{1} \cdots \sum_{n_i=0}^{1} \cdots \sum_{n_s=0}^{1} (-1)^{n_{1(i-1)}} \sigma_{(i-1)+n_{is}-1} C(1,\ldots,1,n_i - 1, n_{i+1}, \ldots, n_s)(1-y_1)^{n_1} \cdots \\ \cdots (1-y_s)^{n_s} + (-1)^i \sum_{n_1=0}^{1} \cdots \sum_{n_i=1}^{1} \cdots \sum_{n_s=0}^{1} (-1)^{n_{1i}} \sigma_{i+n_{(i+1)s}} C(1,\ldots,1,n_{i+1},\ldots,n_s)(1-y_1)^{n_1}(1-y_s)^{n_s} \Big], \end{aligned} \tag{16}$$

where $n_{10} = 0$ and $n_{(s+1)s} = 0$.

The equation system for unknown variables $C(n_1,\dots,n_s)$ is of the form

$$n_1,\dots,n_s = 0,1,$$

$$\begin{aligned} -n_{1s}\sigma_{n_{1s}+1}C(n_1,\dots,n_s) = \\ = a_1\sum_{i=1}^{s}\Big[(-1)^{i-1+n_{1(i-1)}}\sigma_{(i-1)+n_{is}-1}C(1,\dots,1,n_i-1,n_{i+1},\dots,n_s) + \\ +(-1)^{i+n_{1i}}\sigma_{i+n_{(i+1)s}}C(1,\dots,1,n_{i+1},\dots,n_s)\Big] \end{aligned} \tag{17}$$

For $n_{1s}=0$, using solution (10) and the normalizing condition, it we can get

$$C(0,\dots,0)^{-1} = \sum_{m=0}^{c}\sigma_0(m) = \sigma_1 . \tag{18}$$

When $n_{1s}=s$ there is also an analytical solution for $C(1,\dots,1)$ that can be obtained through macrostate analysis (formula (38) from [3], for $\mu=1$)

$$C(1,\dots,1) = \frac{\dfrac{(-a_1)^s}{s!\sigma_s}\prod_{i=0}^{s}\dfrac{\sigma_i}{\sigma_{i+1}}}{\sum_{n=0}^{s}\dfrac{(a_1)^{s-n}}{(s-n)!}\prod_{i=n+1}^{s}\dfrac{\sigma_i}{\sigma_{i+1}}} . \tag{19}$$

The problem is reduced because it is necessary to solve the system of 2^s-2 type (17) equations.

4. AN EXAMPLE

Equations system (17) with 2^{s-2} equation together with evaluated expressions (8, 10,18,19) can be solved either by standard programs or by a certain iterative procedure developed as it was here. The microstate probability $P(m,n_1,n_2,n_3)$, as well as coefficients $C(n_1,n_2,n_3)$ and macrostate probability P_i, in the rather simple case for $a_1=3$ Erl, $a_2=0$ and trunk groups $c=1, s=3$, are given in Table 1.

Table 1.

n_1,n_2,n_3	$P(0,n_1,n_2,n_3)$	$P(1,n_1,n_2,n_3)$	$C(n_1,n_2,n_3)$	i	P_i
0,0,0	0.0611	0.0973	0.2500	0	0.0611
0,0,1	0.0159	0.0296	-0.0840	1	0.1832
0,1,0	0.0255	0.0517	-0.1100	2	0.2748
0,1,1	0.0153	0.0420	0.0496	3	0.2748
1,0,0	0.0446	0.1248	-0.1324	4	0.2061
1,0,1	0.0186	0.0584	0.0529	$a_1 = 3$ Erl	
1,1,0	0.0349	0.1400	0.0692	$a_2 = 0$	
1,1,1	0.0344	0.2061	-0.0344	$c = 1, s = 3$	

5. THE IMPROVED SOLUTION METHOD

Based on the fact that

$$p(c,n_1,\dots,n_{j-1}) = p(c,n_1,\dots,n_{j-1},0) + p(c,n_1,\dots,n_{j-1},1) =$$
$$= (-1)^{n_{1(j-1)}} \sum_{k_1=n_1}^{1} \cdots \sum_{k_{j-1}=n_{j-1}}^{1} \sigma_{k_{1(j-1)}} C(k_1,\dots,k_{j-1},0), \tag{20}$$

it can be concluded that the equality $C(n_1,\dots,n_{j-1},0) = C(n_1,\dots,n_{j-1})$ holds. For $j = 2,3,\dots,s$ the system of $2^{j-1} - 2$ equations can be solved. They are of type

$$-n_{1j}\sigma_{n_{1j}+1} C(n_1,\dots,n_{j-1},1) =$$
$$= a_1 \sum_{i=1}^{j} \left[(-1)^{i-1+n_{1(i-1)}} \sigma_{(i-1)+n_{ij}-1} C(1,\dots,1,n_i - 1, n_{i+1},\dots,n_j) + \right.$$
$$\left. + (-1)^{i+n_{1i}} \sigma_{i+n_{(i+1)j}} C(1,\dots,1,n_{i+1},\dots,n_j) \right] \tag{21}$$

This approach uses the multistep iterative procedure. This procedure requires solving $2^{s-1} - 1$ equations for unknown coefficients in the last step which, compared to the former case, makes only half of the equations. Regardess of the necessity to solve the system in the previous steps of the procedure it is clear that this will reflect on the calculating time and convergence which are commonly the problems caused by bigger s.

6. CONCLUSION

The relations derived in this paper, the specially improved solution method, facilitate the complex calculating of overflow microstate probability in a full availability trunk group compared to solving the statistical equilibrium equation system. The procedure can be used to analyze switching and telecommunication network. It is also of fundamental interest as a mathematical problem.

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