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CORE AND COALITION - POWER IN A PUBLIC GOOD ECONOMY

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Abstract: The paper represents a public good economy where any deviating coalition is allowed to impose any tax-system on nonmembers. This institutional environment shows that the core of the public good economy has been reduced though it may be empty. Furthermore, a new core concept is introduced in the paper and its existence and stability properties are examined. The overall conclusion is that there is a trade-off between existence and stability of the core allocations for these type of economies.

Keywords: Cooperative games, core, coalition, public good economy.

1. INTRODUCTION

The core as the set of coalitionally stable allocations, emerges as a natural equilibrium concept in cooperative games; specifically, the core consists of that set of allocations such that no individual or group of individuals can improve their position by forming an alternative coalition.

In economies with private goods only, it has been shown that under certain assumptions the set of core allocations shrinks as the number of individuals increases. However, if public goods are present in the economy, the core may expand instead of shrinking as the number of individuals increases (Champsaur et. al., 1975). This is due to the fact that the usual definition of the core is itself problematic in the presence of public goods. For an initial allocation to be improved upon ("blocked"), a coalition must exist such that its members, by suitable reallocation among them selves of their own initial resources, can do better for themselves by comparison with the initial allocation. In the presence of a public good, the restriction to reallocations of their own resources means that any coalition has to rely entirely on its own members' contribution towards the production of the public good, assuming that nonmembers are free riders. Thus when a (small) coalition "blocks" a given allocation, it has to provide itself with an amount of the public good equivalent to what is being offered by the whole society, based on its resources.

Many have tried to modify the blocking power of coalitions to make blocking easier for small coalitions, modifying thus the core definition. Champsaur, Roberts and Rosenthal (1975) define "fiscal laws" which specify the set of forced contributions which a given coalition can legally extract from its complement for use in the production of public goods. Nakayama (1977) allowed coalitions to tax their complements using only proportional taxation. Both studies failed to show any shrinking effect of their modified core. In the present paper we consider a modification of the usual definition of the core by allowing deviating coalitions to impose any tax on their complements to making new proposals concerning production levels of the public commodity. However, the complementary coalitions may offset the power of the deviating coalitions by destroying their endowments. Then we develop a new notion of a core set for this institutional environment. We show that the new core is a subset of the old one and it has stronger stability properties. However, its existence requires quite strict assumptions, i.e. convexity on preferences, the existence of a common preferable set for members of the society, etc.

The model is presented in Section 2. Section 3 provides the basic results, and the conclusions are given in Section 4.

2. ASSUMPTIONS AND DEFINITIONS

We consider an economy with one private good, y, and one public good, x. The production set of the economy is characterized by a cost function $C(x)$ which associates with every quantity of public good the minimum cost (in terms of input of the private good) needed in order to produce x. Let $N=\{1,\ldots,n\}$ be the set of agents (consumers) in the public good economy. The set of all coalitions in the economy is denoted by $\Pi(N):=\{S \mid S\subseteq N\}$. Every coalition has access to the aggregate production set characterized by the cost function. An agent $i\in N$ is characterized by his consumption set, by his preferences on the consumption set, which are represented by a complete preordering and by his resources of the private good $w_i\in R_{++}$. The total endowment of the economy with respect to the private good is denoted by $W:=\sum_{i\in N} w_i$. We make the following assumptions:

Assumption 1: For each agent the preference ordering over consumption bundles $(x,y_i)\in R^2_{++}$ is continuous and strictly monotonic. The preference relation is represented by a continuous, quasi-concave and strictly monotonic utility function $U_i: R^2_+ \to R,\ i\in N$, which satisfies: $U_i(0,w_i)>U_i(x,0)\,\forall\, x\in R_+$.

Assumption 2: The production function is characterized by a decreasing returns to scale and furthermore the cost function $C(x)$ is continuous and strictly increasing with $C(0)=0$.

We now make the following definitions:

Definition 1. The set of all S-feasible allocations for any coalition $S \subseteq N$, is denoted by:

$$F(S) := \{(x, y_i)_{i \in S} \mid \sum_{i \in S} t_i \geq C(x), t_i = w_i - y_i, x \geq 0, 0 \leq y_i \leq w_i \ \forall\, i \in S\}.$$

where t_i is the contribution of agent i to the production of the public good.
Therefore, by requiring $t_i \leq w_i$, we are ruling out the possibility that agents are allowed to borrow, and the restriction $t_i \geq 0$, implies that the agents are not subsidized to consume the public good.

Definition 2. An allocation $(x, y_i) \in F(N)$ is "individually rational" if:

$$U_i(x, y_i) \geq U_i(0, w_i) \ \forall\, i \in N.$$

A feasible allocation is "blocked" if there is a coalition $S \subseteq N$ and an allocation $(x', y_i') \in F(S)$ such that: $U_i(x', y_i') > U_i(x, y_i) \ \forall\, i \in S$. The core, $K(E)$, is the set of all individually rational allocations which are not blocked by any coalition.

We describe a particular negotiation process associated with our public good economy. It is assumed, therefore that the agents are negotiating with respect to the amount of the public good produced in the economy. The negotiation process takes place as follows: an amount x of the public good is proposed to be produced; then either all individuals agree and the amount $\dot{x}$ is produced, or there is a coalition which has an objection against the proposed outcome and makes a new proposition, say x'. This deviating coalition is entitled with the power to tax the complementary coalition to cover the cost associated with the amount x'. However, to avoid expropriation or, alternatively, to offset the increasing power of the deviating coalitions, the complementary coalitions have the right to abandon the society or alternatively to "destroy" their endowments. Let us define the set: $E(N) := \{(x, y_i)_{i \in N} \mid t_i = t_j$ if $w_i = w_j \ \forall\, i \neq j,\ (x, y_i) \in F(N)\}$. Then, a set of "destroy-free" allocations proposed by a certain coalition S, may be given by:

$$\Theta(S) := \{(x', y_i') \in E(N) \mid \sum_{i \in S} t_i' > \sum_{i \in S} t_i \text{ if } x' > x,\quad \sum_{i \in N \setminus S} t_i' < \sum_{i \in N \setminus S} t_i \text{ if } x' < x,\ \forall (x, y_i) \in F(N)\}$$

We now give some definitions associated with the above-mentioned negotiation-renegotiation process.

Definition 3. Given an allocation $X \in F(N)$, the proposal $(S', X') \in \Pi(N) x\, \Theta(S)$ is said to be an "objection" to X if:

$$U_i(X') > U_i(X) \ \forall\, i \in S, \text{ where } U_i(X') := U_i(x', y').$$

Definition 4. A proposal $(S'', X'') \in \Pi(N) x\, \Theta(S'')$ is a "counter-objection" to objection

X' if:

$$U_i(X'') \geq U_i(X') \quad \forall i \in S'' \cap S' \neq \emptyset, \text{ and } U_i(X'') > U_i(X) \quad \forall i \in S'' \setminus S'.$$

Definition 5. An objection is said to be "justified" if there is no counter-objection to it.

Definition 6. The modified core $MC(E)$, (bargaining set, $MBS(E)$) is the set of all feasible allocations for which there does not exist any objection (justified objection) against them. That is:

$$MC(E) := \{X \in F(N) \mid \exists \text{ an objection to it}\}$$
$$MBS(E) := \{X \in F(N) \mid \exists \text{ a justified objection to it}\}.$$

3. THE MAIN RESULTS

The set of all allocations that every member of a coalition S does not prefer to any reference allocation X, is denoted by:

$$\Delta_s(X) := \{X' \in \Theta(S) \mid U_i(X) \geq U_i(X') \, \forall i \in S\}.$$

The modified core is then given by:

$$MC(E) = F(N) \setminus \bigcup_{S \subseteq N} \Delta_s(F) = \bigcap_{S \subseteq N} \overline{\Delta_s(F)}, \text{ where } \Delta_s(F) := \bigcup_{X \in F(N)} \Delta_s(X) \text{ and } \overline{\Delta_s(F)}$$

is the complementary set of $\Delta_s(F)$, which of course is convex, open, bounded and may be empty.

Proposition 1. $MC(E) \subseteq K(E)$.

Proof: We know that

$$K(E) := F(N) \setminus \bigcup_{S \subseteq N} D_s(F), \text{ where } D_s(F) := \bigcup_{X \in F(S)} \{X' \in F(N) \mid U_i(X) \geq U_i(X') \, \forall i \in S\}.$$

Since $\bigcup_{S \subseteq N} D_s(F) \subseteq \bigcup_{S \subseteq N} \Delta_s(F)$ then $F(N) \setminus \bigcup_{S \subseteq N} \Delta_s(F) \subseteq F(N) \setminus \bigcup_{S \subseteq N} D_s(F)$ or $MC(E) \subseteq K(E)$.

Therefore Proposition 1 implies that the modified core is always within the usual core, which means that it is a sharper solution than the usual core. Next we provide a sufficient condition for the existence of the modified core.

Proposition 2. If $\overline{\Delta_{N/i}(X)} \cap \overline{\Delta_{N/j}(X)} \neq \emptyset \; \forall i, j \in N, X \in F(N)$, then $MC(E) \neq \emptyset$.

Proof: If $\overline{\Delta_{N/i}(X)} \cap \overline{\Delta_{N/j}(X)} \neq \emptyset \; \forall i, j \in N, X \in F(N)$, then $\overline{\Delta_{N/i}(F)} \cap \overline{\Delta_{N/j}(F)} \neq \emptyset$, $\forall i, j \in N$.

From Helly's theorem we obtain:

$$\bigcap_{i\in N}\overline{\Delta_{N/i}(F)}\neq\varnothing \tag{1}$$

Since $\forall\, S\subset T$ we have $\overline{\Delta_T(X)}\subset\overline{\Delta_S(X)}$ then (1) implies that $\bigcap_{S\in\Pi(N)}\overline{\Delta_s(F)}\neq\varnothing$ which implies that $MC(E)\neq\varnothing$.

The relation between the modified core and the modified bargaining set is given below.

Proposition 3. If the modified core exists, then $MC(E)=MBS(E)$.

Proof: Let an allocation $X\in MBS(E)\setminus MC(E)$. Then, there exists an objection (S',X') against X and therefore $\overline{\Delta_{S'}(X)}\neq\varnothing$. There is also, a counter-objection (S'',X''), $S'\cap S''\neq\varnothing$, against X', which implies $\overline{\Delta_{S''}(X)}\neq\varnothing$. From the existence of the modified core we have: $\overline{\Delta_{S'}(X)}\cap\overline{\Delta_{S''\setminus S'}(X)}\neq\varnothing$. Therefore there is an allocation $Z\in F(N)$ such that $Z\in\overline{\Delta_{S'\cup S''}(X)}$. The $(S'\cup S'',Z)$ is a justified objection against X, which implies $X\notin MBS(E)$, a contradiction.

When a coalition S forms it has to confront the following threat: its members coordinating their actions with nonmembers to find a better allocation and therefore abandoning coalition S. In that case we can say that coalition S lacks credibility. However, the definition of MBS requires that the objections be credible in the sense that a deviating coalition has to take into account the possibility of future objections. Therefore Proposition 3 implies that if the MC exists then it has strong credibility and stability properties.

4. CONCLUSION

In the pure public goods model the usual definition of the core is inadequate since small coalitions are very weak in the sense that they can not "object" to allocations proposed by large coalitions. Thus, the usual core is "too large", containing allocations which might not agree with one's intuitive notion of social stability. If we allow coalitions to tax their complements, then the power of the smaller coalitions increases so much that many allocations may be blocked such that the modified core may even be empty. Therefore, one core definition leads to a very large core while the other one may lead to no core at all. However, it is clear from the results that the modified core has stronger stability properties than the usual one. This implies that there is a trade-off between stability and existence. That is, if we give power to small coalitions in economies with pure public goods, we end up with allocations which are intuitively appealing from a social stability point of view but their existence is questionable.

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