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CAPACITY AND MAXIMAL VALUE OF THE NETWORK FLOW WITH MULTIPLICATIVE CONSTRAINTS

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Abstract: A class of network flows, called multiplicative or M-flows is investigated in this paper. M-flows are subject to multiplicative capacity constraints. These constraints are sums of products with positive coefficients of flow function values on the arcs of subsets of the network arcs.
A definition is given to the flow capacity of a cutting set. Maximality conditions for multiplicative flow optimality are obtained. A theorem, analogous to the mincut-maxflow theorem for the classical network flow is proved.

Keywords: Network flow, generalized flow, side constraints.

1. INTRODUCTION

Classical network flows defined by L. Ford and D. Fulkerson [4] have been extended to flows with capacity constraints on the arc flow functions. In [1, 2, 3, 6] a class of network flows called L-flow is investigated. In this model side linear constraints are added to the capacity constraints on the arc flow function.

In [7] another class of network flows has been defined, for which the capacity constraints on the arc flow function are replaced by multiplicative capacity constraints. Thus, principally new properties are added. This flow will be called in short a multiplicative, or M-flow. The model is a natural theoretical generalization of the L-flow model. It appears when modelling some classes of generalized decision making Markov processes (MDMP). In [5] a MDMP class with a finite set of states S is described. To each state $i \in S$ a set of actions $K_i = \{1, 2, \ldots, K(i)\}$ is associated. The principle opportunity for presenting MDMP as a flow on a graph is given. In this case the transition probability from state i to another state j may be presented as flow $G(i, j)$ on the arc (i, j), subject to the constraints

$$0 \le G(i,j) \le 1, \quad j \in S;$$

$$\sum_{j \in K_i} G(i,j) = 1.$$

This model becomes complicated when $G(i,j)$ depends on the flow on another arc (conditional probability) or when a constraint is imposed on the probability of reaching state k from state i. This leads to a multiplicative constraint to arcs of the chain connecting the nodes i and j of the respective graph.

2. PROBLEM FORMULATION

Let $G(N,U)$ be a direct network with a finite number of nodes N, $|N|=n$ and arcs U, $|U|=m$. The M-flow function $f(x,y)$ on the arcs of this network is defined by (1), (2) and (3):

$$f(x,N) - f(N,x) = \begin{cases} v, \text{ if } x = s, \\ 0, \text{ if } x \ne s,t, \\ -v, \text{ if } x = t, \end{cases} \tag{1}$$

$$\sum_{j \in A_i} b_{ij} \prod_{(x,y) \in D_{ij}} f(x,y) \le C_i, \quad i \in I_k, \tag{2}$$

$$f(x,y) \ge 0, \quad (x,y) \in U. \tag{3}$$

where I_k is the set of indices of the constraints (2), $I_k = \{1,2,...,k\}$;

$C_i, i \in I_k$ - a set of real non-negative numbers;

$A_i \subseteq I_n, i \in I_k$;

D_{ij} - subsets of U;

$$D_i = \bigcup_{j \in A_i} D_{ij}, \quad \bigcup_{i \in I_k} D_i = U; \tag{4}$$

s and t - source and sink of the flow;
v and f - flow value and arc flow function;
$f(x,N) = \sum_{y \in N} f(x,y); \quad f(N,x) = \sum_{y \in N} f(y,x); \quad x \in N;$
b_{ij} - non-negative coefficients for which

$$b_{ij} > 0, \text{ if } i \in I_k, \ j \in A_i. \tag{5}$$

Let for every arc of the network at least one subset $D_{ij} \neq \varnothing$ be found, such that

$$D_{ij} = \{(x,y)\}, \quad (x,y) \in U.$$

In [7] it has been proved that if requirements (1)-(6) are satisfied, there always exists a non-zero multiplicative flow.

The maximal multiplicative flow can be defined by the following nonlinear programming problem:

$$\max \quad v \tag{7}$$

subject to the multiplicative constrains (1) - (3).

In [7] a theorem is proved on the necessary, but not sufficient condition for maximality of the multiplicative flow f. The condition reduces to the existence of the cut $(X, \overline{X})$ between the source and the sink, for which

$$\Delta f(x,y) = 0, \quad (x,y) \in (X, \overline{X}) \tag{8}$$

$$f(\overline{X}, X) = 0. \tag{9}$$

where

$\Delta f(x,y) = 0$ if $i \in I_k$ exists, such that $(x,y) \in D_{ij}$ for some $j \in A_i$ and

$$\sum_{j \in A_i} b_{ij} \prod_{(x,y) \in D_{ij}} f(x,y) = C_i$$

Each optimization nonlinear programming problem with multiplicative constraints and with a linear objective function can be formulated as an optimal M-flow problem.

Let Nm be the described nonlinear problem:

$$Nm: \ \max \sum_{i \in I_n} x_i$$

$$\text{s.t. } g_i(x_1, x_2, \dots, x_n) \le d_i, \quad i \in I_k;$$

$$x_i \ge 0, \quad i \in I_n.$$

Let k variables $f(x_{j_1}), \dots, f(x_{j_k})$ be juxtaposed to each variable x_j, $j \in I_n$. The sequence $x_{j_1}, x_{j_2}, \dots, x_{j_k}$ makes the j-th path in the graph $G_1 = (N_1, U_1)$, $|N_1| = (k-1)n + 2$, $|U_1| = kn$. The graph consists of n nonintersecting paths from node s to node t, corresponding to the variables $x_1, x_2, \dots, x_n$ (Fig. 1).

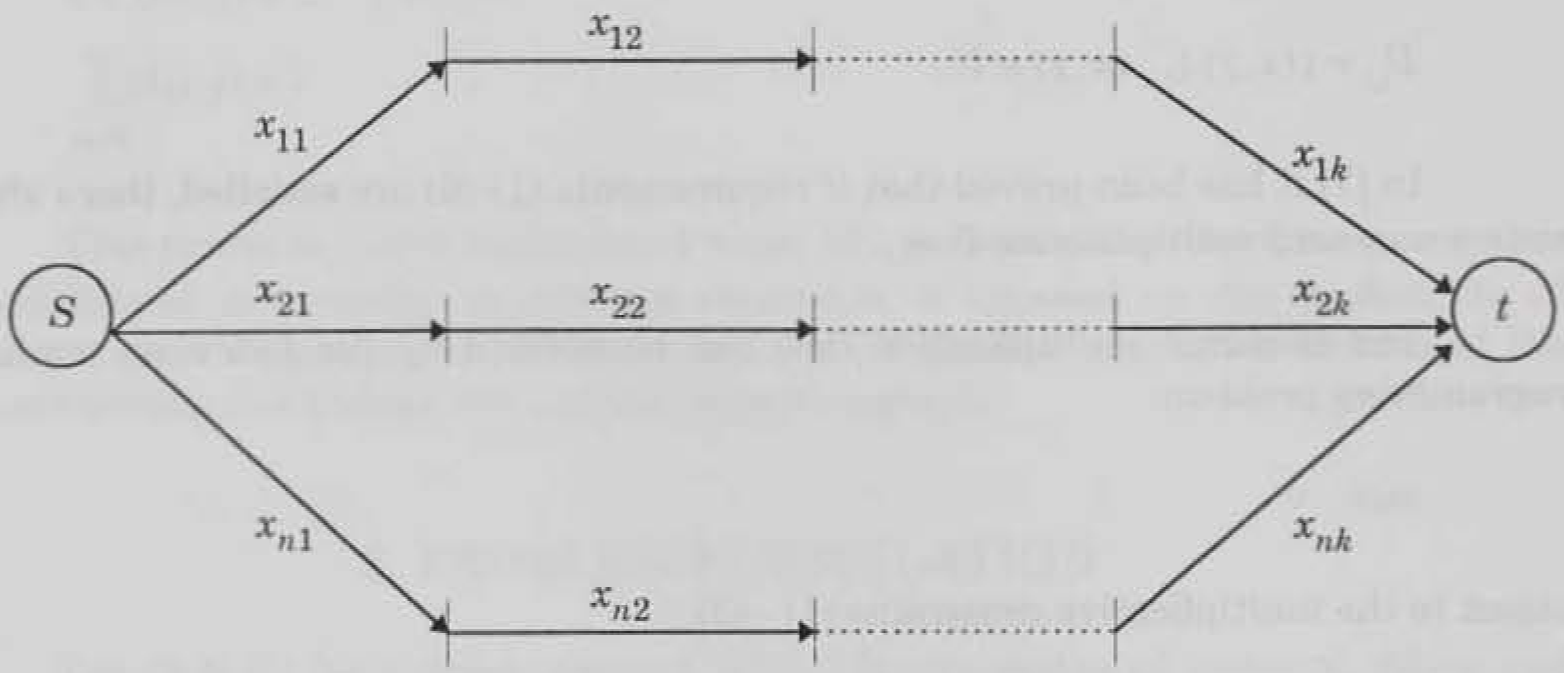

Figure 1. Graph $G_1 = (N_1, U_1)$

Then the Nm problem is reduced to the following M-flow problem on graph G

$$\max\ v$$

$$\text{s.t. } f(s, N_1) - f(N_1, t) = 0;$$

$$f(x_{ij}) - f(x_{ij+1}) = 0, \quad i \in I_n, j \in I_{k-1};$$

$$g_i(f(x_{1i}), f(x_{2i}), ..., f(x_{ni})) \le d_i, \quad i \in I_k;$$

$$f(x_{ij}) \ge 0.$$

The multiplicative constraints are imposed on disjoint sets of arcs (cuts).

3. MAXIMALITY CONDITIONS

A definition of a cutting set is necessary for further investigation of the multiplicative flow. A cutting set is determined by a set of arcs that blocks all paths from source s to sink t [4]. The class of cuts is a subclass of the class of cutting sets.

The set $U(r)$ denotes the arcs of the cutting set $r \in I_R$, where I_R is the set of the indices of all cutting sets. Consider cutting sets consisting of only the union of arc subsets D_i. These arc subsets correspond to different constraints (2) with indices from I_k. Hence any cutting set $U(r)$ is represented in the following way: for each $r \in R$

$$U(r) = \bigcup_{i \in I(r)} D_i \qquad (10)$$

where

$$I(r) \subseteq I_k \tag{11}$$

The capacity $C(r)$ of cutting set r is defined by the following nonlinear programming problem:

$$\max\ C(r) = v \tag{12}$$

subject to the relations (1) and (3) and the following multiplicative constraints:

$$\sum_{j \in A_i} b_{ij} \prod_{(x,y) \in D_{ij}} f(x,y) \le C_i; \quad i \in I(r) \tag{13}$$

The next theorem is of great importance for estimating the capacity of the different cutting sets:

Theorem 1: If two cutting sets $U(r)$ and $U(p)$ are given, such that,

$$U(p) \subseteq U(r) \tag{14}$$

then

$$C(p) \ge C(r) \tag{15}$$

Proof: A. The coincidence of the two sets $U(r)$ and $U(p)$ is a trivial case leading to equality of the corresponding capacities $C(p)$ and $C(r)$.
B. Let us study the second possibility:

$$U(p) \subset U(r) \tag{16}$$

Assume

$$C(p) < C(r) \tag{17}$$

It follows from (11), (13) and (16) that

$$I(p) \subset I(r) \tag{18}$$

The optimal solution to nonlinear programming problem (12) is the flow $\{f^*(x,y) / (x,y) \in U\}$ with value v^*, where $C(r) = v^*$. Considering relation (16) this flow is a feasible solution to the problem of finding $C(p)$. According to (17)

$$C(r) = v^* > C(p) \tag{19}$$

The last inequality contradicts the requirement for $C(p)$ maximality.

1.2. Consider an example network of Fig. 2.

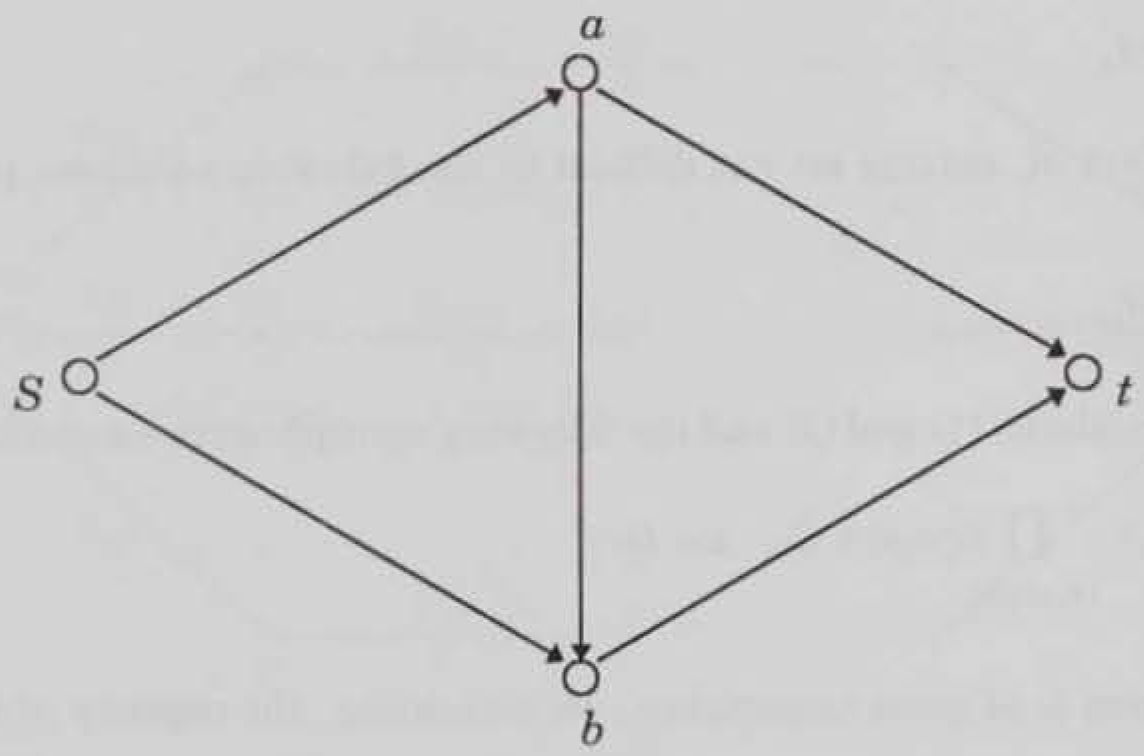

Figure 2. An example network

The M-flow on this network is defined by relations (1), (3) and the following two multiplicative constraints:

$$f(a,t)+f(b,t)+f(s,a)f(a,b)\le C_{i_1}; \tag{20}$$

$$f(s,a)+f(s,b)+f(a,t)f(a,b)\le C_{i_2}; \tag{21}$$

where

$$C_{i_2}=kC_{i_1}, \tag{22}$$

and k is a coefficient, for which

$$k>1. \tag{23}$$

Let the two cutting sets with indices r and p be defined in the following way

$$I(r)=\{i_1,i_2\};\ \ I(p)=\{i_2\}. \tag{24}$$

Hence

$$U(p)\subset U(r) \text{ and } I(p)\subset I(r). \tag{25}$$

Consider the two M-flow problems to determine the capacities $C(r)$ and $C(p)$ of the two cutting sets. Obviously, the maximal M-flow values will be reached at a null value of $f(a,b)$. Hence

$$C(r)=C_{i_1};\ \ C(p)=C_{i_2}=kC_{i_1}. \tag{26}$$

From (22), (23) and (26) it follows that in the case discussed

$$C(r) < C(p) \quad (27)$$

1.3. Considering the previous case assume

$$k = 1, \quad (28)$$

then it follows from (21), (26) and (28) that

$$C(r) = C(p). \quad (29)$$

Definition 1: A cutting set r^*, for which

$$U(r^*) = U \quad (30)$$

is called a complete cutting set.

Definition 2: If for two cutting sets r and p

$$U(p) \subseteq U(r) \quad (31)$$

and

$$C(p) = C(r) \quad (32)$$

and cutting set p is the minimal with respect to the properties (31) and (32), then the cutting set p is called r-minimal. The r^* - minimal cutting set is called minimal.

For each cutting $U(r)$ one r - minimal cutting set at least exist and it is the set $U(r)$.

Lemma 2: The minimal value ν of the multiplicative flow, defined by relations (1) - (3), is equal to the flow capacity of the complete cutting set, i.e.

$$\nu = C(r^*). \quad (33)$$

Proof: The two nonlinear programming problems for the determination of $C(r^*)$ and for the maximal M-flow ν coincide.

Theorem 3: (For maximal multiplicative flow and minimal cutting set): For each network the maximal value of the M-flow ν from source s to sink t is equal to the capacity of the minimal cutting set.

This result follows directly from Lemma 2 and Definition 2.

Equality (33) is the necessary and sufficient condition for the multiplicative flow maximality.

Lemma 4: If ν is a value of multiplicative flow on a network, then the existence of a cutting set $U(r)$, for which

$$\nu = C(r) \tag{34}$$

is a sufficient condition for this flow maximality.

Proof: Let cutting set $U(r)$ be given, for which requirement (34) is satisfied. Since according to Definition 2 - $U(r) \subseteq U(r^*)$, it follows from Theorem 1 that

$$C(r) > C(r^*). \tag{35}$$

The sufficiency of (34) for multiplicative flow maximality follows from Lemma 2 and (35).

4. CONCLUSIONS

The following general properties of the multiplicative flow investigated can be noted:

1. The multiplicative flow defined by relations (1) - (3) is in fact a nonlinear network flow with specific properties. It can be regarded as a further generalization of the classical network flow [4] if $|D_i| = 1;\ i \in I_k,\ b_{ij} = 1$ and also of the linear flow [6] if $|D_{ij}| = 1$.

2. The functional $C(r)$, determining the capacity of the cutting set $U(r)$, is of a more general type than the cut capacity, used in the classical networks flow.

In the most general case multiplicative flow capacities are not defined by capacity constraints on the arc flow functions (as for classical network flows). Only some general multiplicative nonlinear constraints are present on subsets of network arcs. This adds principally new combinatorial properties to the multiplicative flow.

3. Theorem 3 proves equality between the maximal multiplicative flow and the minimal cutting set capacity. It can be considered as a generalization of the famous mincut-maxflow theorem of Ford and Fulkerson for multiplicative flow.

4. Conditions (8) and (9) are only necessary, but they are not sufficient conditions for multiplicative flow maximality. In the case of a classical flow they are simultaneously necessary and sufficient conditions for the flow maximality [4].

5. In the multiplicative and classical flow case several properties are combined - network, linear and nonlinear. For classical flow the network properties dominate and this enables the development of efficient optimization algorithms. In the case of a multiplicative flow the nonlinear properties are numerous and this creates considerable difficulties in finding specific efficient algorithms, considering the network properties of the problems. Problems remain open regarding the designing of specific algorithms - different from the general methods of nonlinear programming that are able to solve efficiency the maxflow-problem and the mincost-maxflow problem for this kind of flow.

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