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A REGULARIZED CONTINUOUS LINEARIZATION METHOD OF THE FOURTH ORDER

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Abstract: For the minimization problem with inaccurately specified objective function and set, a regularization method based on the continuous linearization method of the fourth order is proposed. Sufficient conditions for convergence are given and the regularizing operator is constructed.

Keywords: Continuous linearization method, regularization, Tichonov function.

1. INTRODUCTION

Consider the minimization problem

$$J(u) \to inf, \quad u \in U = \{u \in U_0: g_i(u) \leq 0, i = 1,\dots,m\}, \tag{1}$$

where U_0 is a given convex closed set of a certain Hilbert space H and the functions $J(u)$, $g_1(u),\dots,g_m(u)$ are defined and Frechet differentiable on H. The scalar product of two elements $u, v \in H$ will be denoted by $<u, v>$; $\| u \| = <u, u>^{1/2}$ is the norm of an element $u \in H$.

Suppose that

$$J_* = \inf_{u \in U} J(u) > -\infty, \quad U_* = \{u \in U: \ J(u) = J_*\} \neq \emptyset \tag{2}$$

As we know [1], [5], problem (1) is unstable with respect to perturbations of the initial data $J(u)$, $g_i(u)$ and regularization methods must be employed to solve it. We propose and investigate a regularization method established on the continuous linearization method of the fourth order (see [4] for solving problem (1) with inaccurate initial data). In practical problems the functions $g_i(u), J'(u), g_i'(u)$ are known inexactly, so instead of them we have only their approximations $g_i(u,t), J'(u,t), g_i'(u,t), u \in H, i=1,\dots,m$, depending on the parameter $t \geq 0$. Let $u=u(t), t \geq 0$ be the solution to the differential equation

$$\beta_4(t)\, u^{iv}(t) + \beta_3(t)\, u'''(t) + \beta_2(t)\, u''(t) + u'(t) + u(t) = \\ = P_{U(u(t),t)}[u(t) - \gamma(t) T'(u(t),t)], \quad t \geq 0, \tag{3}$$

with the initial values

$$u(0) = u_o, \quad u'(0) = u_1, \quad u''(0) = u_2, \quad u'''(0) = u_3, \tag{4}$$

where

$$U(u,t) = \{ z \in U_0 : g_i(u,t) + \langle g_i'(u,t), z-u \rangle \leq \\ \leq \theta(t)\ (1+\|u\|^2), \ i=1,\dots,m \}; \tag{5}$$

$P_{U(u(t),t)}(z)$ is the projection of point z on set $U(u(t),t)$; $u_0, u_1, u_2, u_3 \in H$ are arbitrary points; $T'(u,t) = J'(u,t) + \alpha(t) u$, $u \in H$, $t \geq 0$ is the approximation of the gradient $Q'(u,t) = J'(u) + \alpha(t) u$ of the Tichonov function $Q(u,t) = J(u) + \frac{1}{2}\alpha(t)\|u\|^2$; $\alpha(t), \beta_i(t), \gamma(t), \theta(t)$, are parameters of the method. The derivatives $u^{(i)}(t)$, $i=1,\dots,4$ of the function $u(t)$, $t \geq 0$, take the values in H, and are understood in the sense [2], Ch. 4.

2. THE CONDITIONS FOR CONVERGENCE

We will consider the behaviour of the solution $u(t)$ of the differential equation (3), (4) when $t \to \infty$ and prove that the method (3)-(5) has regularization properties. Before that we can remark that the method (3)-(5) for $m=0$ in (5) (i.e. $U = U(u(t),t) = U_0$) reduces to the regularized continuous projection-gradient method of the fourth order treated in [10]; for $m=0$, $\beta_4(t) \equiv \beta_3(t) \equiv 0$ - to the regularized continuous projection-gradient method of the second order presented in [9].

The following theorem gives sufficient conditions for the convergence of the trajectory $u=u(t)$, $t \geq 0$ of the differential equation (3), (4).

Theorem 1. Suppose that

1) U_0 is a convex closed set in the Hilbert space H, the functions $J(u)$, $g_1(u),\dots,g_m(u)$ are convex, Frechet differentiable on H and their gradients satisfy the Lipschitz condition

$$\max\{\|J'(u)-J'(v)\|;\ \max_{1\le i\le m}\|g_i'(u)-g_i'(v)\|\}\le \tag{6}$$

$$\le L\,\|u-v\|,\quad u,v\in H;$$

and furthermore condition (2) and Slater's condition

$$\exists u_c\in U_0,\quad g_i(u_c)<0,\quad i=1,\dots,m; \tag{7}$$

2) the approximations $g_i(u,t), J'(u,t), g_i'(u,t)$ of the $g_i(u), J'(u), g_i'(u)$ are continuous in u for all $t\ge 0$, measurable in t for all $u\in H$ and

$$\begin{aligned}&\max_{1\le i\le m}|g_i(u,t)-g_i(u)|\le\delta(t)(1+\|u\|^2),\quad u\in H,\\ &\max\{\|J'(u,t)-J'(u)\|;\ \max_{1\le i\le m}\|g_i'(u,t)-g_i'(u)\|\}\le\\ &\le\delta(t)(1+\|u\|),\quad u\in H,\quad t\ge 0;\end{aligned} \tag{8}$$

3) the parameters $\alpha(t), \beta_4(t), \beta_3(t), \beta_2(t), \gamma(t), \delta(t)$, of the method (3)-(5) are such that

$$\begin{aligned}&\alpha(t),\ \beta_4(t),\ \gamma(t)\in C^4[0,+\infty),\quad \beta_3(t)\in C^3[0,+\infty),\\ &\beta_2(t)\in C^2[0,+\infty),\quad \delta(t),\quad \theta(t)\in C[0,+\infty),\\ &\alpha(t)>0,\quad \gamma(t)>0,\quad \delta(t)\ge 0,\quad \theta(t)\ge 0,\quad t\ge 0,\\ &\alpha'(t)\le 0,\quad \beta_i'(t)\le 0,\quad i=2,3,4,\quad \gamma'(t)\le 0,\\ &\alpha''(t)\ge 0,\quad \beta_i''(t)\ge 0,\quad i=2,3,4,\quad \gamma''(t)\ge 0,\\ &\alpha'''(t)\le 0,\quad \beta_i'''(t)\le 0,\quad i=3,4,\quad \gamma'''(t)\le 0,\\ &\alpha^{iv}(t)\ge 0,\quad \beta_4^{iv}(t)\ge 0,\quad \gamma^{iv}(t)\ge 0,\quad t\ge 0;\end{aligned} \tag{9}$$

$$\lim_{t\to\infty}\left(\alpha(t)+\gamma(t)+\delta(t)+\theta(t)+\frac{\delta(t)}{\theta(t)}\right)=0,$$

$$\lim_{t\to\infty}\left(\frac{\theta(t)}{\alpha^4(t)\gamma^3(t)}+\frac{|\alpha'(t)|}{[\alpha^7(t)\gamma^5(t)]^{1/2}}+\frac{|\gamma'(t)|}{\alpha(t)\gamma^2(t)}\right)=0, \tag{10}$$

$$\lim_{t\to\infty}\left(\frac{\alpha''(t)}{\alpha(t)}+\frac{\gamma''(t)}{\gamma(t)}+\frac{|\alpha'''(t)|}{\alpha(t)}+\frac{|\gamma'''(t)|}{\gamma(t)}\right)=0,$$

$$\begin{aligned}
&\lim_{t\to\infty}\beta_i(t)=\beta_{i_\infty}>0,\quad i=2,3,4,\quad 1-\beta_{2\infty}>0,\\
&\beta_{2\infty}^2+\beta_{4\infty}-2\,\beta_{3\infty}>0,\qquad \beta_{3\infty}^2-2\,\beta_{2\infty}\,\beta_{4\infty}>0,\\
&\beta_{2\infty}-\frac{3}{2}\beta_{3\infty}>0,\qquad \beta_{2\infty}\,\beta_{3\infty}-3\,\beta_{4\infty}>0\,.\\
&\delta(t)\max\{3+3\|u_c\|;\ \ 5+\|u_c\|\}\le 2\theta(t),\quad t\ge 0\,.
\end{aligned} \tag{11}$$

Then

$$\lim_{t\to\infty}\left(\sum_{i=1}^{4}\ \|u^{(i)}(t)\|+\|u(t)-u_*\|\right)=0, \tag{12}$$

where $u_*\in U_*$, $\|u_*\|=\inf_{u\in U_*}\|u\|$ is the normal solution to problem (1). The convergence in (12) is uniform with respect to the choice of approximations $g_i(u,t), J'(u,t), g_i'(u,t)$ in (8).

First we will note that such parameters satisfying (9)-(11) can be chosen, for example, in the following way

$$\alpha(t)=(1+t)^{-\alpha},\ \ \beta_i(t)=\beta_{i\infty}+[1+(1+t)^{-1}],\ \ i=2,3,4,$$
$$\gamma(t)=(1+t)^{-\gamma},\ \ \theta(t)=(1+t)^{-\theta},\ \ \delta(t)=a(1+t)^{-\delta},\ \ t\ge 0\,,$$

where $a,\alpha,\gamma,\delta,\theta,\beta_{i\infty}$ are positive real numbers, such that $4\alpha+3\gamma<\theta<\delta$, $\alpha+\gamma<\frac{2}{5}$, a is small enough so that

$$a\max\{3+3\|u_c\|;\ \ 5+\|u_c\|\}\le 2,$$

and $\beta_{i\infty}$, i=2,3,4 satisfy (11).

Proof. Let us show that $U(u,t)\neq\varnothing$ for all $t\geq 0$, $u\in H$. As we know [8], for every convex differentiable function $g(u)$ on H, it is true that

$$g(v)+<g_i'(v),w-v>\leq g(w),\quad w,v\in H. \tag{13}$$

Taking into account (7), (8), (11) and inequality (13), we have

$$\begin{aligned}&g_i(u,t)+<g_i'(u,t),u_c-u>\leq g_i(u)+<g_i'(u),u_c-u>+\\&+\delta(t)\,(1+\|u\|^2)+\delta(t)\,(1+\|u\|)\|u_c-u\|\leq\\&\leq g_i(u_c)+\delta(t)\,[(1+\|u\|^2+\|u_c\|+\tfrac{1}{2}(1+\|u_c\|)\times\\&\times(1+\|u\|^2)+\|u\|^2]<\tfrac{1}{2}\delta(t)\,[(3+3\|u_c\|)+\\&+(5+\|u_c\|)\,\|u\|^2]\leq\theta(t)\,(1+\|u\|^2)\,.\end{aligned}$$

This means $u_c\in U(u,t)$, so $\forall t\geq 0$, $\forall u\in H$, $U(u,t)\neq\varnothing$. Besides that, the set $U(u,t)$ is convex and closed, therefore for every $t\geq 0$ the right hand side of (3) is uniqually defined, and under the established assumptions, satisfies the Caratheodory conditions [6]. Thus the differential equation (3), (4) has the solution u(t) defined for small values of t and for any choice of the initial points u_0, u_1, u_2, $u_3\in H$ [5]. We will assume the existence of the solution $u(t)$ on the half-line $[0,+\infty)$.

Let $v_\tau\in U$ be the solution to the minimization problem

$$Q(u,\tau)\to inf,\quad u\in U=\{u\in U_0: g_i(u)\leq 0, i=1,\dots,m\}. \tag{14}$$

The Tichonov function $Q(u,\tau)$ is strongly convex on H, so that for every $\tau\geq 0$, there is a unique point $v_\tau\in U$, such that (see [7]):

$$\sup_{\tau\geq 0}\|v_\tau\|\leq\|u_*\|,\qquad \lim_{\tau\to\infty}\|v_\tau-u_*\|=0. \tag{15}$$

Besides that, under the given assumptions, the Kuhn-Tucker theorem can be applied to problem (14) for every $\tau\geq 0$. Therefore, there exist

$$\lambda_1(\tau)\geq 0,\dots,\lambda_m(\tau)\geq 0, \tag{16}$$

such that

$$<Q'(v_\tau,\tau)+\sum_{i=1}^{m}\lambda_i(\tau)g_i'(v_\tau),\ \ v-v_\tau>\geq 0,\quad v\in U_0,\quad \tau\geq 0, \tag{17}$$

$$\lambda_i(\tau)g_i(v_\tau)=0,\quad g_i(v_\tau)\leq 0,\quad i=1,\dots,m;\quad \tau\geq 0. \tag{18}$$

Furthermore, the Kuhn-Tucker theorem can also be applied to the problem

$$\Phi(z,t) = \| z - [u(t) - \gamma(t)\, T'(u(t),t)\,] \|^2 \to inf\,,$$
$$z \in U(u(t),t), \quad t \ge 0,$$

which is equivalent to the problem of projecting the point $u(t) - \gamma(t) \times T'(u(t),t)$ on the set $U(u(t),t)$. Consequently, there exist

$$\upsilon_1(t) \ge 0, \ldots, \upsilon_m(t) \ge 0, \quad t \ge 0, \tag{19}$$

such that

$$\begin{aligned} &< \beta_4(t)\, u^{iv}(t) + \beta_3(t) u'''(t) + \beta_2(t) u''(t) + u'(t) + \\ &+\gamma(t) T'(u(t),t) + \sum_{i=1}^{m} \upsilon_i(t)\, g_i'(u(t),t),\, w - [\, \beta_4(t) u^{iv}(t) + \\ &+\beta_3(t) u'''(t) + \beta_2(t) u''(t) + u'(t) + u(t)\,] > \ge 0, \quad \forall\, w \in U_o, \end{aligned} \tag{20}$$

$$\begin{aligned} &\upsilon_i(t)\, [\, g_i(u(t),t) + < g_i'(u(t),t),\, \beta_4(t) u^{iv}(t) + \\ &+\beta_3(t) u'''(t) + \beta_2(t) u''(t) + u'(t) > - \\ &-\theta(t)(1 + \| u(t) \|^2)\,] = 0, \quad i = 1,\ldots,m; \quad \forall t \ge 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &g_i(u(t),t) + < g_i'(u(t),t), \beta_4(t)\, u^{iv}(t) + \beta_3(t) u'''(t) + \\ &+\beta_2(t) u''(t) + u'(t) > - \theta(t)\, (1 + \| u(t) \|^2) > \ \le 0, \end{aligned} \tag{22}$$

for $i = 1,\ldots,m$, $\forall t \ge 0$. Setting $v = \beta_4(t)\, u^{iv}(t) + \beta_3(t) u'''(t) + \beta_2(t)\, u''(t) + u'(t) + u(t) \in U(u(t),t) \subseteq U_0$ in (17), $w = v_\tau \in U \subseteq U_0$ in (20), multiplying the obtained inequalities, respetively, by $(-\gamma(t))$ and (-1), and summing them, we have

$$\begin{aligned} &< \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u',\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > + \\ &+\gamma < T'(u,t) - Q'(v_\tau,\tau),\, \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > + \\ &+\sum_{i=1}^{m} \upsilon_i < g_i'(u,t),\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > - \\ &-\gamma \sum_{i=1}^{m} \lambda_i(\tau) < g_i'(v_\tau),\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > \ \le 0, \end{aligned} \tag{23}$$

for all $t,\ \tau \geq 0$. Here and further, we will often omit the argument t of the functions $\alpha, \beta_i, \gamma, \delta, \theta, u, u^{(i)}$, in order to shorten the expressions. Let

$$r = r(t) = \| u'''(t) \|^2, \quad q = q(t) = \| u''(t) \|^2,$$
$$p = p(t) = \| u'(t) \|^2, \quad x = x(t,\tau) = \tfrac{1}{2} \| u(t) - v_\tau \|^2. \tag{24}$$

Then

$$\begin{aligned}
&2 < u^{iv}, u''' >= r', \quad 2 < u''', u'' >= q', \quad 2 < u'', u' >= p', \\
&2 < u^{iv}, u'' >= q'' - 2r, \quad 2 < u''', u' >= p'' - 2q, \\
&2 < u^{iv}, u' >= p''' - 3q', \quad < u', u - v_\tau >= x', \\
&< u'', u - v_\tau >= x'' - p, \quad < u''', u - v_\tau >= x''' - \tfrac{3}{2} p', \\
&< u^{iv}, u - v_\tau >= x^{iv} - 2p'' + q,
\end{aligned} \tag{25}$$

where $x^{(i)} = \dfrac{d^i}{dt^i} x(t,\tau), \quad t \geq 0, \quad \tau \geq 0$. Using (24) and (25) the first term in (23) can be written as

$$\begin{aligned}
&< \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u',\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau >= \\
&= \beta_4^2 \| u^{iv} \|^2 + [\beta_3^2 - 2\beta_4\beta_2\,] r + \beta_3\beta_4 r' + [\beta_2^2 + \beta_4 - 2\beta_3\,] q + \\
&+ [\beta_3\beta_2 - 3\beta_4\,] q' + \beta_4\beta_2 q'' + [1 - \beta_2\,] p + [\beta_2 - \tfrac{3}{2}\beta_3\,] p' + \\
&+ [\beta_3 - 2\beta_4\,] p'' + \beta_4 p''' + \beta_4 x^{iv} + \beta_3 x''' + \beta_2 x'' + x', \quad t, \tau \geq 0,
\end{aligned} \tag{26}$$

According to [8], p.175, for any convex differentiable function $g(u)$ on H, whose gradient satisfies the Lipschitz condition (6), the following inequality is true

$$< J'(u) - J'(v), v - w > \leq \tfrac{L}{4} \| u - w \|^2, \quad u, v, w \in H. \tag{27}$$

Now we will estimate the second term in (23). Using (8), (15), (27) and $2|ab| \leq \varepsilon a^2 + \varepsilon^{-1} b^2, \varepsilon > 0$, we get

$$\begin{aligned}
&< T'(u,t) - Q'(v_t,\tau),\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau >= \\
&=< J'(u,t) - J'(u),\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > + \\
&+ < J'(u) - J'(v_\tau),\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > + \\
&+ < \alpha(t) u - \alpha(\tau) v_\tau,\ \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > \geq
\end{aligned} \tag{28}$$

$$
\begin{aligned}
&\geq -\delta(1+\|u-v_\tau\|+\|v_\tau\|)(\|\beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'\|^2+\\
&+\|u-v_\tau\|-\tfrac{L}{4}\|\beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'\|^2+\\
&+\alpha < u-v_\tau,\ \beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'+u-v_\tau >+\\
&+(\alpha(t)-\alpha(\tau)) < v_\tau,\ \beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'+u-v_\tau > \geq\\
&\geq -\delta[2(1+\|v_\tau\|^2)+\tfrac{1}{2}\|\beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'\|^2+\\
&+\tfrac{9}{4}\|u-v_\tau\|^2]-\tfrac{L}{4}\|\beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'\|^2+\\
&+\alpha\{\beta_4(x^{iv}-2p''+q)+\beta_3(x'''-\tfrac{3}{2}p')+\beta_2(x''-p)+\\
&+x'+2x\}-|\alpha(t)-\alpha(\tau)|^2\|v_\tau\|^2-\tfrac{1}{4}\|\beta_4 u^{iv}+\beta_3 u'''+\\
&+\beta_2 u''+u'\|^2-\tfrac{1}{\alpha(t)}\|\alpha(t)-\alpha(\tau)\|^2\|v_\tau\|^2-\\
&-\tfrac{\alpha(t)}{4}\|u-v_\tau\|^2 \geq -[1+L+2\delta_{\max}][\beta_4^2\|u^{iv}\|^2+\beta_3^2 r+\\
&+\beta_2^2 q+p]+\alpha\{\beta_4(x^{iv}-2p''+q)+\beta_3(x'''-\tfrac{3}{2}p')+\\
&+\beta_2(x''-p)+x'+\tfrac{3}{2}(1-3\tfrac{\delta}{\alpha})x\}-C_0[\delta(t)+\\
&+\tfrac{1}{\alpha(t)}|\alpha(t)-\alpha(\tau)|^2],\quad \forall\, t,\tau\geq 0.
\end{aligned}
\tag{28}
$$

where $\delta_{\max}=\sup\limits_{t\geq 0}\delta(t)$, $C_0=max\,\{2(1+\|u_*\|^2);(1+\alpha_{\max})\|u_*\|^2\}$, $\alpha_{\max}=\sup\limits_{t\geq 0}\delta(t)= =\alpha(0)$.
Let us show that the third term in (23) is nonegative for all $t\geq t_0$ is a large enough number. From (8), (13), (15), (19) and (21), we have

$$
\begin{aligned}
&\nu_i < g_i'(u,t),\ \beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'+u-v_\tau >=\\
&=\nu_i[-g_i(u,t)+ < g_i'(u,t),\ u-v_\tau > +\theta\,(1+\|u\|^2)]\geq\\
&\geq \nu_i[-g_i(u)- < g_i'(u),\ v_\tau-u > +\theta\,(1+\|u\|^2)]-\\
&-\delta\,(1+\|u\|^2)-\delta\,(1+\|u\|)\,\|u-v_\tau\|\geq\\
&\geq -\nu_i\, g_i(v_\tau)+\nu_i(1+\|u\|^2)\,\tfrac{\theta}{2}[2-\tfrac{\delta}{\theta}(5+\\
&+2\,\|u_*\|)]\geq 0,\quad \forall\, t\geq t_0,\quad i=1,\dots,m,
\end{aligned}
\tag{29}
$$

because $\lim\limits_{t\to\infty}\frac{\delta(t)}{\theta(t)}=0$, according to (10). Finally, we will estimate the last term in (23). Using the inequality (see [8], p.93, Lemma 1)

$$| \; g(u) - g(v) - < g'(v), u - v > \; | \le \frac{L}{2} \| u - v \|^2, \quad u, v \in H,$$

and (13), (18), (19), (22), we get

$$\begin{aligned} &\lambda_i(\tau) < g_i'(v_\tau), \; \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau >= \\ &= \lambda_i(\tau) [g_i(v_\tau) + < g_i'(v_\tau), \; \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + \\ &+ u - v_\tau >] \le \lambda_i(\tau) g_i(\beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u) \le \\ &\le \lambda_i(\tau) \frac{L}{2} \| \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' \|^2 + \lambda_i(\tau) [g_i(u) + \\ &+ < g_i'(u), \; \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' >] \le \lambda_i(\tau) \frac{L}{2} \| \beta_4 u^{iv} + \\ &+ \beta_3 u''' + \beta_2 u'' + u' \|^2 + \lambda_i(\tau) [g_i(u,t) + < g_i'(u,t), \; \beta_4 u^{iv} + \\ &+ \beta_3 u''' + \beta_2 u'' + u' > - \theta (1 + \|u\|^2)] + \lambda_i(\tau) [\delta (1 + \|u\|^2) + \\ &+ \delta (1 + \|u\|) \| \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' \| + \theta (1 + \|u\|^2)]. \end{aligned} \tag{30}$$

Since $1 + \| u \|^2 \le 1 + 2 \| \; u - v_\tau \; \|^2 + 2 \| \; v_\tau \; \|^2 \le 1 + 4x + 2 \| \; u_* \; \|^2, (1 + \| \; u \; \|) \| \; \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' \; \| \le (1 + \| \; v_\tau \; \| + \| \; u - v_\tau \; \|) \| \; \beta_4 u^{iv+} + \beta_3 u''' + + \beta_2 u'' + u' \; \| \le (1 + \; \| \; u_* \; \|^2) + \frac{1}{2} \| \beta_4 u^{iv} + + \beta_3 u''' + + \beta_2 u'' + u' \; \|^2 + 2x(t)$ from (30), it follows that

$$\begin{aligned} &\lambda_i(\tau) < g_i'(v_\tau), \; \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' + u - v_\tau > \le \\ &\le \lambda_i(\tau) [(L + \delta) \frac{1}{2} \| \beta_4 u^{iv} + \beta_3 u''' + \beta_2 u'' + u' \|^2 + (\delta + \theta) \times \\ &\times (1 + 2 \; \|u_*\|^2 + 4x) + \delta (1 + \|u_*\|^2) + 2x)] \le \\ &\le \lambda_i(\tau) [2 \, (L + \delta_{\max}) \, (\beta_4^2 \|u^{iv}\|^2 + \beta_3^2 r + \beta_2^2 q + p) + \\ &+ 6(\delta + \theta) x + C_1 (\delta + \theta)], \quad \forall \; t, \tau \ge 0, \quad i = 1, \dots, m. \end{aligned} \tag{31}$$

$C_1 = max \{ 1 + 2 \|u_*\|^2 ; \;\; (1 + \|u_*\|)^2 \}$. Let us prove that

$$0 \le \sum_{i=1}^{m} \lambda_i(\tau) \le C_2 = const, \quad \forall \; \tau \ge 0. \tag{32}$$

From (17), where $v = u_c$, and (13) we have

$$J(v_\tau)+\tfrac{1}{2}\alpha(\tau)\| v_\tau \|^2+\sum_{i=1}^{m}\lambda_i(\tau)g_i(v_\tau)\le J(u_c)+$$

$$+\tfrac{1}{2}\alpha(\tau)\| u_c \|^2+\sum_{i=1}^{m}\lambda_i(\tau)g_i(u_c).$$

Using the last inequality and (7), (13), (15), (16), (18), we get

$$\min_{1\le i\le m} | g_i(u_c) | \sum_{i=1}^{m}\lambda_i(\tau)\le -\sum_{i=1}^{m}\lambda_i(\tau)g_i(u_c)\le J(u_c)-J(v_\tau)+$$
$$+\tfrac{1}{2}\alpha(\tau)\| u_c \|^2\le < J'(u_c),\ u_c-v_\tau>+\tfrac{1}{2}\alpha(\tau)\| u_c \|^2\le$$
$$\le\| J'(u_c)\| (\| u_c \|+\| u_* \|)+\tfrac{1}{2}\alpha_{\max}\| u_c \|^2,$$

which proves (32). From (31), (32) follows the necessary estimation for the last term in (23):

$$-\gamma(t)\sum_{i=1}^{m}\lambda_i(\tau)< g_i'(v_\tau), \beta_4 u^{iv}+\beta_3 u'''+\beta_2 u''+u'+$$
$$+u-v_\tau>\ge -v(t)C_2\,\{2(L+\delta_{\max})[\beta_4^2\|u^{iv}\|^2+\beta_3^2 r+ \tag{33}$$
$$+\beta_2^2 q+p]+6(\delta+\theta)x+C_1(\delta+\theta)\},\quad \forall t,\tau\ge 0.$$

Putting the estimations (26), (28), (29), (33) in (23), we find

$$\beta_4^2[1-\gamma(t)C_3]+\|u^{iv}(t)\|^2+[(\beta_3^2(1-\gamma(t)C_3)-2\beta_4\beta_2)]r+$$
$$+\beta_3\beta_4 r'+[(\beta_2^2(1-\gamma(t)C_3)+\beta_4(1+\alpha\gamma)-2\beta_3]q+[\beta_3\beta_2-$$
$$-3\beta_4]q'+\beta_4\beta_2 q''+[(1-\gamma(t)C_3)-\beta_2(1+\alpha\gamma)]p+[\beta_2-$$
$$-\tfrac{3}{2}\beta_3(1+\alpha\gamma)]p'+[\beta_3-2\beta_4(1+\alpha\gamma)]p''+\beta_4 p'''+\alpha\gamma\,(\tfrac{3}{2}- \tag{34}$$
$$-C_4\tfrac{\delta+\theta}{\alpha})x+(1+\alpha\gamma)[x'+\beta_2 x''+\beta_3 x'''+\beta_4 x^{iv}]\le C_5[\tfrac{\gamma(t)}{\alpha(t)}|\alpha(t)-$$
$$-\alpha(\tau)|^2+\gamma(t)\,(\delta(t)+\theta(t))],\quad \forall t\ge 0,\quad \forall\tau\ge 0,$$

where $C_3=L+1+2\delta_{\max}+2C_2(L+\delta_{\max})$, $C_4=6C_2+\frac{9}{2}$, $C_5=\max\{C_0;\frac{3}{2}C_1C_2\}$. The inequality (34) has the same structure as the inequality (22) in [10]. Therefore the equality

$$\lim_{t\to\infty} \{\| u(t)-v_* \|+\| u'(t) \|+\| u''(t) \|+\| u'''(t) \|+\| u^{iv}(t) \| \}=0,$$

which follows from (34), can be obtained in the same way as the equalities (34), (35) in [10].

3. THE STOPPING CRITERION

In practice the error of the initial data is usually greater than a certain fixed positive number. A more realistic condition than (8), with $\lim_{t\to\infty} \delta(t)=0$, for considered problem (1) is as follows: for each fixed $u \in H$, instead of computing the exact values of $g_i(u), J'(u), g_i'(u)$, it is possible to compute their approximations $g_{i\delta}(u)$, $J_\delta'(u), g_{i\delta}'(u)$ such that

$$\begin{aligned} &\max_{1\le i\le m} | g_{i\delta}(u)-g_i(u) | \le \delta(1+\| u \|^2), \quad u \in H, \\ &\max \{\| J_\delta'(u)-J' \|; \max_{1\le i\le m} \| g_{i\delta}'(u)-g_i'(u) \| \}\le \\ &\le \delta\,(1+\| u \|), \quad u \in H, \end{aligned} \tag{35}$$

where $\delta > 0$ is a known positive number. Then, instead of process (3) - (5), we have the process

$$\begin{aligned} &\beta_4(t)w^{iv}(t)+\beta_3(t)w'''(t)+\beta_2(t)w''(t)+w'(t)+w(t)= \\ &= P_{U(w(t),\delta)}[w(t)-\gamma(t)\,(J_\delta'(w(t))+\alpha(t)w(t))]. \end{aligned} \tag{36}$$

$$w(0)=u_0, \quad w'(0)=u_1, \quad w''(0)=u_2, \quad w'''(0)=u_3, \tag{37}$$

$$\begin{aligned} &U(w,\delta)=\{z \in U_0 : g_{i\delta}(w)+< g_{i\delta}'(w), \quad z-w > \le \\ &\le \theta(t) \quad (1+\|w\|^2), \quad i=1,\dots,m\}. \end{aligned} \tag{38}$$

which will be continued up to the moment $t(\delta)$ defined by the following condition

$$t(\delta)=sup\{t:\delta(s)>\delta, \quad 0\le s\le t\}. \tag{39}$$

Since $\delta(t)\to 0$ when $t\to+\infty$, $\delta(0)>\delta$, the required moment of time $t(\delta)$ can be found with certainty . The parameters $\alpha(t), \beta_i(t), \gamma(t), \theta(t), t\ge 0$, in (36) - (39) satisfy the conditions (9) - (11), and do not depend on the number δ from (35).

The justification for using the criterion (39) to stop the process (36) - (38) is given by the following theorem.

Theorem 2. Let all the conditions of Theorem 1 be satisfied, apart from (8), and let the approximations $g_{i\delta}(u)$, $J'_\delta(u), g'_{i\delta}(u)$ of $g_i(u), J'(u), g'_i(u)$ satisfy condition (35). Suppose that the trajectory $w(t)$, $0 \le t \le t(\delta)$ has been obtained by the method (36) - (38), where the moment $t(\delta)$ is determined in accordance with the stopping criterion (39). Then

$$\lim_{\delta \to 0} \| w(t(\delta)) - u_* \| = 0.$$

This theorem can be proved in the same way as Theorem 2 in [10]. From Theorem 2, it follows that the operator R_δ which sets the point $w(t(\delta))$, defined by the method (36) - (39), in correspondence with $(g_{i\delta}(u),\ J'_\delta(u), g'_{i\delta}(u), \delta)$ from (35), is a regularizing operator in the sense [1], [5].

4. CONCLUSION

The paper shows that the continuous linearization method of the fourth order proposed in [4] can be regularized and applied to minimization problems with inaccurately specified objective function and set. The proposed regularized method (3) - (5) for $m = 0$ in (5) reduces to the regularized continuous projection-gradient method of the fourth order treated in [10], while for $m = 0, \beta_4(t) = \beta_5(t) = 0$ it reduces to the regularized continuous projection-gradient method of the second order presented in [9]. As pointed out in [3], the advantages and the importance of higher order continuous methods stem from their higher order of convergence and from the fact that continuous methods give a large choice of numerical integration methods to solve the corresponding differential equations.

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