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A SEMI-TOPOLOGICAL CLASSIFICATION OF LINE FIGURES

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Abstract: In this paper a subclass of line figures, which are finite unions of arcs, is considered. We call such a figure a form. To each form a numerical sequence related to the Euler characteristic of the underlying space of the form is attached. We prove that a form can be decomposed into simple ones. On the basis of this decomposition, to each form a matrix, as a way of arithmetization of the structure of the form, is attached. Finally, this leads to a semi-topological classification of forms. Then, invariant properties of this classification are taken as a basis for character recognition.

Keywords: Line figures, semi-topological classification, character recognition.

1. INTRODUCTION

Each problem of recognition supposes an understanding of the relationship of objects as equivalent to one another in certain ways. For instance, when we relate two figures in the plane according to their geometrical shape, we compare properties which are preserved under the action of the Euclidean group. And the set of properties upon which we find that two figures represent the same numeral may be quite different, much as two processes of comparison are different.

In interpersonal communication, a pattern is often transferred from a person to another by means of its graphic realization. For example, the standard patterns of letters and numerals, when freely realized (hand-written) may be more or less like the followed pattern. Recognition of such realizations is the process of relating them to their standard patterns. The process is based on structural properties of patterns and, when hindered, the difficulties are due to the presence of noise (those properties which are not structural).

Is the vast literature on the character recognition problems, structural properties of specific classes of characters are expressed in the form of different analytical, geometrical or topological features of line figures. And, as the development of mathematics reflects it, in the case of general, synthetic approaches, instead of selecting properties, an idea of equivalence has to be taken as a classification criterion.

Having in mind the realistic model of a page with a piece of writing, and being inspired by the current research (e.g. [1], [2], [3], [4], [5], [6], [7], [8], [14]), we synthesize such an equivalence for a class of line figures on the basis of topological ideas. Once we have a precise mathematical formulation of equivalence, we also have a ground upon which we select those properties shared by all equivalent objects as being structural ones.

Reflecting individual styles, handwritten characters may vary very much in shape and magnitude. This makes a unique metrical comparison with the followed standard patterns impossible. Thus, the researchers select some outstanding points and features of line figures and those which depend on topological and positional properties obtain a precise mathematical meaning in our approach. Being so semi-topological, this approach requires consideration of a plane supplied by a given coordinate system (and in such a plane we can define anthropomorphic "left", "right", "east", "south", etc.). Further, we select a subclass of line figures which are finite unions of arcs being either graphs of continuous functions or vertical segments. When connected, we call such a union a form. Each form determines a multi-valued function $t \mapsto F(t)$. A sequence τ_F is attached to a form F, registering the number of components $F(t)$ and discontinuity points of the corresponding function.

This sequence can be considered as a refinement of the idea of crossing counts, which is often present in the literature on character recognition problem [5], [8]. It is shown that this sequence is related to the Euler characteristic of space F. When the members of τ_F are equal to 1, the form is called simple. Then a family Λ of simple subforms of F is selected and the pair (F, Λ) is taken to be the structure upon which the graphical meaning of F is based. When a homeomorphism $h : F \to F'$ preserves the structure (maps simple forms of F upon simple ones of F', respecting "left", "right", "up", and "down"), we say that two forms F and F' are equivalent. Some invariants are treated, such as node points, the number of simple forms, equivalent matrices, etc. The whole conceptualization permits a precise, semi-topological definition of such symbols as numerals and letters of an alphabet. In order to make this paper self-contained, we repeat several ideas and results first exposed in [12], but it also includes a detailed treatment of the role and properties of node points.

Finally, we hope that our mathematically-oriented approach is interesting enough for specialists in the field of pattern analysis and we expect their appreciation of the involved topological ideas.

2. SELECTION OF A PROPER CLASS OF LINE FIGURES

A line figure is commonly understood to be a subset in the plane which consists of lines. For instance, the letters of the English alphabet, as well as the words written in that alphabet are line figures.

A Euclidean plane is a topological space and a line figure as its subset inherits the relative topology. Recall that a mapping $f : F_1 \to F_2$ of line figure F_1 to line figure F_2 is called a *homeomorphism* if f is bijective and both f and f^{-1} are continuous. Then we say that F_1 and F_2 are *homeomorphic* or *topologically equivalent*, and we write $F_1 \approx F_2$. A property of a figure which is unaltered by homeomorphisms is called a *topological property*.

If our purpose is to investigate the conventional symbols graphically - as being line figures possibly deformed out of their regular shape, but still with recognizable meaning, then topology may seem to be beyond this scope. The reason is seen in the fact that a homeomorphism may cause a big distortion of the shape of a figure. For example, an interval and a spiral are topologicaly equivalent. This situation entails a classification of figures which would be stricter than topological equvalence and still far from geometrical congruence.

Thus, we are led to consider a subclass of line figures which are constructed from arcs as basic building blocks. Roughly speaking, we cannot be haphazard about placing these blocks without respecting "left" and "right", "up" and "down". In order to express these positional properties formally, we also require that the plane is supplied with a coordinate system.

Since we will associate some numerical sequences and matrices with a line figure, we would not welcome a situation in which that figure in intersected by a straight line in infinitely many separated parts. This leads us to suppose that the number of building blocks is finite. So, we have gathered a clear motivation for the following formal steps.

Let E^2 be the Euclidean plane together with a given coordinate system. A realistic model is a page with its lower edge being x-axis and its left edge being y-axis.

First, we define building blocks of the line figures that we want to select.

A *stretching arc* in E^2 is the graph of a continuous function defined on a closed interval. A *vertical arc* in E^2 is a closed interval belonging to line $x = a$.

Now we define the subclass of line figures which will be convenient for our considerations and which we will call forms. When we say "arc", we mean that the arc is either stretching or vertical.

A *form* is a connected set in E^2 which is the union of a finite family of arcs which intersect only in end points. Note that the same form may be seen in many different ways as a family of arcs. For example, the forms represented in Fig. 1 are three families of five, three and two arcs, respectively, where the heavy dots represent the ends of arcs. In all three cases we have, in fact, the same form.

A stretching arc is the graph of a continuous function $f : [a.b] \to R$ and the point $(a, f(a))$ is called *left* and the point $(b, f(b))$ the *right* end of the arc. All its other points are called *interior points*.

Among all representations of a form (as the union of a family of arcs), there will be one having the fewest possible arcs. To see it, let us fix our attention on some particular points.

The point of a form F which is of any of the following types:

1. end point of exactly one arc

2. left or right end point of exactly two arcs
3. end point of three or more arcs
4. end point of exactly one vertical arc and possibly some stretching ones

we call a *node* of the form F. The drawings shown in Fig. 2a illustrate each of these cases.

Notice that nodes are never interior points of the arcs of a form. Therefore, in each possible representation of a form, a node must be the end of an arc.

On the other hand, if an end is not a node, then it is a point which is either

1. the left end of one arc and the right end of another one
2. the end of two vertical arcs

These two cases are illustrated in Fig. 2b. Notice that in both cases such an end is the interior point of the arc wich is the union of two arcs ending at that point. Replacing such two arcs by their union in a given representation of a form, we get another representation having less arcs and ends. If we keep replacing such pairs of arcs, then in a finite number of steps we shall reach the unique representation of a form with its nodes as the only end points. We call this representation *canonical*. For example, each of the forms represented in Fig. 3 has its canonical representation with respect to the page as a model of E^2 and the coordinate system with the origin at the lower-left corner. Loosely speaking, the concepts depend not only on the properties of a figure, but also on the position of an observer. In the model of this page, two congruent figures may have different canonical representations, as well as different sets of nodes and families of arcs. This is easily demonstrated with the forms represented in Fig. 4, with the reader as an observer.

The introduced concepts remain invariant under translation of the coordinate system. Thus, given a form F, we can translate the coordinate system to a unique position where F is a subset of the first quadrant and with the axes touching F. The coordinate system in that position will be called the *associated coordinate system* of the form F. Starting with this page as a model, the drawings shown in Fig. 5 represent two forms with their associated coordinate systems.

Projecting a form F to the axes of its associated system, two intervals $[0,a]$ and $[0,b]$ are obtained. We will call the interval $[0,a]$ (on x-axis) the *domain* of the form F and the rectangle $[0,a]\times[0,b]$ the *frame* of F.

3. CONTINUITY DISCRIMINANT OF A FORM

Suppose we have a form together with its associated coordinate system. Then, we can look on the form as being the graph of a multi-valued function and, in that way, we establish a correspondence between points of the domain and compact subsets of the set R of real numbers.

Before a formal exposition, we shall use an example to illustrate this idea. Let us consider two realizations of the numeral "8", one being standard and the other contaminated with noise (Fig. 6). Projecting nodes to the x-axis, the following subdivisions of domain are obtained:

$$(1): 0 < t_1 < a \qquad (2): 0 < t_1' < t_2' < t_3' < a'.$$

Let us denote these forms by F and F', respectively. Then to each $t \in [0,a]$, the set

$$F(t) = \{ y \mid (t,y) \in F \}$$

is attached. ($F(t)$ is the projection to the y-axis of the intersection of the form F and the straight line $x = t$). The attachement $t \mapsto F(t)$ is a multi-valued function (as is, similarly, the function $t \mapsto F'(t)$).

Let $t \in (0,t_1')$ and let $t \mapsto t_1'$. Then, $F'(t)$ is a two point set which "stays far from" the three point set $F'(t_1')$. This means that the multi-valued function $t \mapsto F'(t)$ is discontinuous on the left at t_1' and, as it is easy to see, also discontinuous on the right at t_3'. If the set $F(t)$ "comes close to" the set $F(t_0)$, when $t \to t_0$ we say that the function $t \mapsto F(t)$ is continuous at t_0. By inspection, we see that $t \mapsto F(t)$ is continuous at each point and $t \mapsto F'(t)$ is discontinuous only at t_1' and t_3'.

Now, let us attach to a form a sequence of numbers of components of $F(t)$, when t runs from 0 to the end of the domain. Such a number is the same for each t belonging to the interior of a subinterval and we use brackets "( )" to point it out. In the case of forms F and F' (considered above), the sequences are

$$\tau_F = 2(4)3(4)2, \quad \tau_{F'} = 1(2)_*3(4)3(4)3_*(2)1$$

where the asterisk denotes the place of discontinuity.

For the form represented in Fig. 7a, $F(0)$ and $F(a)$ are sets consisting of a point and an interval, so the number of components is two. The sequence is $\tau = 2_*(3)2(3)_*2$. When unbalanced, this form becomes the form represented in Fig. 7b, and the corresponding sequence is

$$\tau = 1(2)_*3_*(3)2(3)2_*(1)_*1$$

Remark that according to the classification of forms which we are going to define later, these two line figures will be of the same type (have the same meaning), just as they will be the same in the case of two considered realizations of the numeral "8". Thus, these sequences are not only dependent on structural properties of forms, but, being also affected by noise, they register its presence.

To give a precise meaning to the figurative "comes close to", we need a metric on the set of subsets of R. More generally, let (M,d) be a metric space and $\exp(M)$ the set of all non-empty compact subsets of M. For $A,B \in \exp(M)$, let

$$\rho(A,B) = \sup_{x \in A} \inf_{y \in B} d(x,y), \quad \rho(B,A) = \sup_{y \in B} \inf_{x \in A} d(x,y).$$

Two numbers, $\rho(A,B)$ and $\rho(B,A)$, may be different and the bigger of them is the distance between A and B. Write

$$d(A,B) = \max(\rho(A,B), \rho(B,A));$$

and then, this metric on $\exp(M)$ is called the *Hausdorff metric*.

The following useful proposition describes the relation $d(A,B) < \varepsilon$.

3.1. *$d(A,B) < \varepsilon$ if and only if for each $x \in A$ there exists an $y \in B$ such that $d(x,y) < \varepsilon$ and for each $y \in B$ there exists an $x \in A$ such that $d(x,y) < \varepsilon$.*

Now, it is easy to define the continuity of a multi-valued function. Let (M_1,d_1) and (M_2,d_2) be two metric spaces and $F: M_1 \to M_2$ a multi-valued function such that for each $x \in M_1$, $F(x)$ is a compact subset of M_2. The function F is continuous at $x_0 \in M_1$ if for each sequence (x_n) in M_1, $x_n \to x_0$ implies $F(x_n) \to F(x_o)$ in $\exp(M_2)$.

A thorough exposition of the material related to the Hausdorff metric and to multi-valued functions can be found in Kuratowski [9].

Given a form F, let $[0,a]$ be its domain. Correspond to each $x \in [0,a]$ the set

$$F(x) = \{\, y \mid (x,y) \in F \,\}.$$

Then, $x \mapsto F(x)$ is a multi-valued function and the set $F(x)$ is a finite union of isolated points and closed intervals which are the components of connectedness of $F(x)$.

Projecting all nodes of F to the x-axis a subdivision

$$0 = t_0 < t_1 < \ldots < t_k = a$$

is obtained. For $t \in (t_i,t_{i+1})$, $F(t)$ is a finite set of the same number n_t^i points. Let n_{t_i} denote the number of components of $F(t_i)$.

In order to register the points of discontinuity, let

$$\tilde{n}_{t_i} = \begin{cases} n_{t_i} & F \text{ is continuous at } t_i \\ \cdot n_{t_i} & F \text{ is continuous on the right and discontinuous on the left at } t_i \\ n_{t_i}\cdot & F \text{ is continuous on the left and discontinuous on the right at } t_i \\ \cdot n_{t_i}\cdot & F \text{ is discontinuous on both sides at } t_i \end{cases}$$

Following the order of points of the subdivision

$$t_0 < t_1 < \cdots < t_k,$$

we attach to the form F, in a unique way, the following sequence

$$\tau_F = \tilde{n}_{t_0}(n_t^0)\tilde{n}_{t_1}\cdots\tilde{n}_{t_i}(n_t^i)\tilde{n}_{t_{i+1}}\cdots\tilde{n}_{t_{k-1}}(n_t^{k-1})\tilde{n}_{t_k},$$

which will be called the *continuity discriminant* of F.

Notice that the sequence τ_F has an odd number $2k+1$ of members and that brackets could be omitted. If we look for a meaning, then notice that (n_t^i) is actually a constant function on (t_i, t_{i+1}) and τ_F could be considered as a step function on $[0, a]$. On the other hand, τ_F is related to the Euler characteristic of the form F.

Forms are finite cell complexes of dimension one and for a form F, its Euler characteristic is the difference $\chi(F) = \beta_0 - \beta_1$ of Betti numbers β_0, β_1. If the number of nodes of F is n and the number of arcs a, then $\chi(F) = n - a$ also holds. For general topological details, see for example, Maunder [10] or Munkres [11].

The following proposition, proved in [12], establishes the relation between τ_F and $\chi(F)$.

3.2. *Let F be a form, and* τ_F *its continuity discriminant, then*

$$n_{t_0} - n_t^0 + n_{t_1} \cdots + n_{t_i} - n_t^i + \cdots - n_t^{k-1} + n_{t_k} = \chi(F).$$

4. SUBFORMS AND SIMPLE FORMS

According to the definition, a form F is the union of a family $\{a_i \mid i \in \{1,\dots,n\}\}$ of arcs (which are either stretching or vertical and which intersect only in end points). Let J be a subset of $\{1,\dots,n\}$, then the union $\cup\{a_j \mid j \in J\}$, if connected, is called a *subform* of the form F.

For example, the forms F and F', represented in Fig. 8a, have for some of their subforms the forms represented in Fig. 8b, and $a_2 \cup a_4$ is not a subform of F (being not connected).

Notice that a subform is a subspace determined by the way we see a form as a union of arcs. Suppose the drawings in Fig. 8a represent the same form seen differently as a union of arcs. Then, no subform of F listed under Fig. 9b is a subform of F' and $a_1 \cup a_4$ is a subspace of F' which is a form in its own.

When we consider a form to be a graph of a multi-valued function, then its intersections with straight lines parallel to the y-axis play a significant role. Figurativaly speaking, then everything seems as we were looking at the form from a point in infinity, and in the direction of the vector $\vec{j}$. Thus, vertical arcs are seen as "big" points which contribute a "$_*1_*$" to the continuity discriminant.

A form having all members of its continuity discriminant equal to one will be called a *simple form*. Thus, one might view a simple form as being an arc having possibly "big" points.

As a convention, let us denote the discriminant of the form consisting of a single vertical line by $_*1_*$. All forms in Fig. 9 are simple ones. A subform of a form F which is a simple form will be called the *simple subform* of F. Of course, each arc is a simple subform.

The family of all simple subforms of a form F is finite and partially ordered by inclusion. Since, then, each chain has the maximal element, which will be called *maximal simple subform*, we derive the following proposition.

4.1. *Each arc of a form is contained in a maximal simple subform*

For illustration, consider the form represented in Fig. 10, where $a_1 \cup a_2 \cup a_3$ is a simple subform, but not maximal. Subforms

$$a_1 \cup a_2 \cup a_3 \cup a_4 \quad \text{and} \quad a_1 \cup a_2 \cup a_4 \cup a_5$$

are maximal and they both contain arcs a_1, a_2, a_4.

Let us remark that the concepts of subforms, related to the canonical representation of a form, are also canonical.

Let a be a stretching (vertical) arc. Denote $end_-(a)$ its left (lower) end and by $end_+(a)$ its right (upper) end.

Combinatorially, a form F is the set N of its nodes, the family A_1 of its stretching arcs and the family A_2 of its vertical arcs. We will use the ordered triple (N, A_1, A_2) to express this structure of a form.

Inspired by the general idea of isomorphic simplicial complexes, we say that two forms F and F' are *combinatorially isomorphic* if there exist bijections $N \to N'$, $A_1 \to A_1'$, and $A_2 \to A_2'$ such that whenever an arc a corresponds to the arc a', then $end_-(a)$ and $end_+(a)$ correspond to $end_-(a')$ and $end_+(a')$, respectively.

For example, the pairs of forms depicted in Fig. 11 are combinatorially isomorphic and they still have some positional properties which we would like to consider discriminating. Another classification, described in the next Section, will be more satisfactory, and then the structure of a form will be seen through a family of maximal simple subforms.

In the case of simple forms, this isomorphism is exactly what desirable classification should be. Namely, let F and F' be two combinatorially isomorphic simple forms. Then, there is a mapping $h : F \to F'$ which is a homeomorphism and $h(x,y) = (h^1(x), h_x^2(y))$, where h^1 is a function increasing in x and, for each x, h_x^2 a function increasing in y. (For all but a finite number of x's, the domain of h_x^2 is a one-point set). Let us call such a mapping h *homeomorphism of forms*.

On the other hand, as it is easy to see, when $h : F \to F'$ is a homeomorphism, then forms F and F' are combinatorially isomorphic.

The first two forms in Fig. 12 are homeomorphic (isomorphic) and the second two are not.

5. CONTINUITY MATRICES

As has been fixed by definition, a form is a connected space. Leaving out this assumption, then we have a finite union of arcs which intersect only in their end points. Such a figure is the disjoint union of a finite number of forms. Consideration of such objects is not motivated by our tendency for generality but by the procedure that we are going to describe.

Connectivity assumptions have not been used explicitely in the definitions of concepts related to a form, so that they have the same meaning when applied to a disjoint union of forms. The same remark holds for the validity of the already formulated propositions (3.2 and 4.1).

Let S be a disjoint union of a finite number of forms. A maximal simple subform L of S such that for each $(x,y) \in L$ and $(x,y') \in S \setminus L$, the inequality $y' > y$ holds, we call a *layer* of S.

Roughly speaking, the layers of S are the lowest maximal simple subforms of S.

Our definition of layers has a logical justification in the fact that they exist. And their existence is based on the following proposition (proved in [12]).

5.1. *Each disjoint union of a finite number of forms has at least one layer.*

We have already exhibited a tendency to see those components of the value $F(x)$ which are intervals as they were "big" points contributing a "$_*1_*(_*1.1_*)$" to the continuity discriminant. Preserving our tendency, we also see the ends of an arc as "big" points. Let us express it precisely.

Suppose a is a stretching arc of a form F (or of the union S of a finite family of forms). The left *end component* of a is the connected subset of F (or S) containing the point $end\_(a) = (u,v)$ and which lies on the line $x = u$. The right end component is similarly defined.

The layers of the forms represented in Fig. 13 are a_2 for (i), a_2 for (ii) and $a_0 \cup a_1 \cup a_3 \cup a_4 \cup a_5 \cup a_6 \cup a_7 \cup a_8 \cup a_{10}$ for (iii).

If we remove all stretching arcs of these layers and those arcs which belong to their end components whenever they do not belong to the end components of the remaining family of stretching arcs, we obtain the forms represented in Fig. 14.

As it is seen, in cases (ii) and (iii) the union of the remaining family of arcs is not connected. The layers of these unions of forms are:

$$\text{(i) } a_1; \quad \text{(ii) } a_1 \cup a_3; \quad \text{(iii) } a_0 \cup a_1 \cup a_2 \cup a_8 \cup a_9 \cup a_{10}.$$

Removing again, in cases (i) and (ii), we have the result represented in Fig. 15. Thus, we have illustrated a procedure which leads to the decomposition of forms in layers. In the examples we follow, these decompositions are the families in Fig. 16.

As it will be seen later, the family of layers in which a form is decomposed is exactly the structure which carries the essential properties of that form. We would mention also that when removing arcs of layers, an alternative way would be to remove all arcs. As a matter of preference, we consistently treat end components as "big" points and so we decompose a form as described.

In each step a sequence of layers is obtained and we attach to such a sequence a row of a matrix. These rows are taken to be the continuity discriminants of these sequences and they are written in order corresponding to the special position of the layers. The already written 1's corresponding to the end components are not repeated (and 0's are written instead). For example, in the case of the form under (ii), the matrix is

$$\begin{bmatrix} 1 & (1) & 1 & (1) & 0 & (0) & 0 & (1) & 1 & (1) & 1 \\ 0 & (0) & 0 & (1) & 1 & (0) & 1 & (1) & 0 & (0) & 0 \\ 0 & (0) & 1 & (1) & 1 & (1) & 1 & (1) & 1 & (0) & 0 \end{bmatrix}$$

$$\tau = 1(1)2(3)2(1)2(3)2(1)1$$

Notice that the sum of numbers in a column is equal to the corresponding member of the continuity discriminant τ of this form.

In the case of forms under (i) and (iii), their matrices are

$$\text{(i)} \begin{bmatrix} 1 & (1) & 0 \\ 0 & (1) & 1 \\ 1 & (1) & 1 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} 0 & (1) & 0 & (0) & 0 & (0) & 0 & (1) & 0 \\ 1_{\bullet} & (1) & 1 & (1) & {}_{\bullet}1_{\bullet} & (1) & 1 & (1) & {}_{\bullet}1 \end{bmatrix}$$

$$\tau = 2(3)2 \qquad\qquad \tau = 1_{\bullet}(2)1(1)_{\bullet}1_{\bullet}(1)1(2)_{\bullet}1$$

Examples (i), (ii) and (iii) have served as recipes for the described procedure. Now, we turn to the exact definitions in the case of arbitrarily given forms.

Let F be a canonically represented form and $[0,a]$ its domain. Let

$$0 = t_0 < t_1 < ... < t_k = a$$

be the corresponding subdivision of the interval $[0,a]$. Each layer of F is a simple form and let Λ_1 be the union of all of them. Then, τ_{Λ_1} is the sequence of $\tilde{n}_{t_i}$ and $(n_t^{(i)})$ as defined in Section 3, with $\tilde{n}_{t_i} = 0$ when the line $x = t_i$ does not intersect Λ_1 and $n_t^{(i)} = 0$ when there is no arc of Λ_1 stretching over $[\ t_i, t_{i+1}\ ]$.

Let us remove all stretching arcs of Λ_1 and all those arcs which belong to their end components whenever they are not end components of some remaining stretching arcs. Thus, the subset S_2 of F is obtained which is the union of forms. In addition, put formally $S_1 = F$. Let Λ_2 be the union of all layers of S_2 and let $\tilde{\tau}_{\Lambda_2}$ be τ_{Λ_2} with 0's in place of 1's corresponding to the end components already included in Λ_1. Put formally $\tilde{\tau}_{\Lambda_1} = \tau_{\Lambda_1}$.

To proceed by induction, assume the sequences

$$S_1, ..., S_{m+1};\ \ \Lambda_1, ..., \Lambda_m \ \text{ and } \ \tilde{\tau}_{\Lambda_1}, ..., \tilde{\tau}_{\Lambda_m}$$

have already been defined.

Let Λ_{m+1} be the union of all layers of S_{m+1}, $\tilde{\tau}_{\Lambda_{m+1}}$ be $\tau_{\Lambda_{m+1}}$ with 0's in place of 1's corresponding to the end components already included in $\Lambda_1 \cup ... \cup \Lambda_m$. Remove all stretching arcs of Λ_{m+1} and those which belong to the end components not being end components of the remaining family of stretching arcs. Denote by S_{m+2} the union of remaining arcs S_{m+1}. Thus, S_{m+2}, Λ_{m+1} and $\tilde{\tau}_{\Lambda_{m+1}}$ are defined.

Since each S_i contains at least one stretching arc of F, there exists a natural n such that $S_n \neq \emptyset$ and $S_{n+1} = \emptyset$, (when this procedure stops). Call the number n *height* of the form F. For example, simple forms have a height equal to 1.

The sequence

$$\Lambda = (\Lambda_1, ..., \Lambda_n)$$

will be called the *decomposition in layers* of the form F and the matrix

$$\begin{bmatrix} \tilde{\tau}_{\Lambda_n} \\ \cdots \\ \tilde{\tau}_{\Lambda_1} \end{bmatrix}$$

the *continuity matrix* of F.

6. CLASSIFICATION OF FORMS

Consider two realizations of the numeral "3", represented in Fig. 17a.

The continuity matrices of these forms are

$$\begin{bmatrix} 1 & (1) & 0 \\ 0 & (1) & 1 \\ 1 & (1) & 0 \\ 1 & (1) & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & (0) & 1 & (1) & 1 & (1) & 1 & (1) & 0 \\ 0 & (0) & 0 & (0) & 0 & (1) & 1 & (1) & 1 \\ 0 & (0) & 0 & (0) & 1 & (1) & 0 & (0) & 0 \\ 1 & (1) & 1 & (1) & 1 & (1) & 1 & (0) & 0 \end{bmatrix}$$

and their decompositions in layers are represented in Fig. 17b.

Comparing $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ and $\Lambda' = (\Lambda_1', \Lambda_2', \Lambda_3', \Lambda_4')$ we see that the layers Λ_i and Λ_i' are homeomorphic and they all are connected to each other in the same way. Equalization of these forms, on the grounds of such a comparison, leads to the idea of their eqiuvalence. Then, their different continuity matrices may be interpreted as a result of the presence of noise.

In general, each member Λ_i of the sequence Λ is the union of simple forms having disjoint domains. This motivates us to call the sequence $L_1, ..., L_n$ of simple forms having the domains $[a_i, b_i]$ for which

$$a_1 < b_1 < \cdots < a_n < b_n,$$

arranged and to extend slightly the definition of homeomorphism given in Section 4.

Let Λ and Λ' be the unions of two arranged sequences of simple forms. Let $h : \Lambda \to \Lambda'$ be a homeomorphism such that $h(x, y) = (h^1(x), h^2_x(y))$, where h^1 is increasing in x and, for each x, h^2_x increasing in y. Then, h will be called *homeomorphism of the unions of arranged sequences of simple forms*.

Now we fix the idea of equivalence in the form of a definition.

Let F and F' be two forms having the same height n and let

$$\Lambda = (\Lambda_1, \ldots, \Lambda_n) \text{ and } \Lambda' = (\Lambda'_1, \ldots, \Lambda'_n)$$

be their decomposition in layers. Then, F and F' are called *equivalent* if there exists a homeomorphism $h : F \to F'$ such that for each i the restrictions

$$h \big| \Lambda_i \div \Lambda_i \to \Lambda'_i$$

are homeomorphisms of the unions of arranged sequences of layers. Then, we write $F \approx F'$ and it is easily seen that "$\approx$" is an equivalence relation. Further, since for each $i : \Lambda''_i = h[\Lambda_i]$, we see that such homeomorphisms preserve the structure of the form understood as the pair (F, Λ). In order to be specific, we call the mapping h *homeomorphism of forms* (and the term homeomorphism is used with its standard meaning). We also say for two equivalent forms that they have the same type or that they are of the same type.

Classification of various concrete realizations of forms entails recognition of those having the same type and discrimination of those being of different types. In both cases, invariants of classification may be used. A *form invariant* is a property unaltered by homeomorphisms of forms. Thus, to show that two forms are not equivalent, it is enough to find an invariant being the property of one of them and not being the property of the other one. In general, to show that two forms are equivalent, we have to find a proper homeomorphism. But in the case of a selected set of possible types, invariants may be used as well.

The height of a form is evidently an invariant, as well as the number of stretching arcs and the number of vertical arcs in its canonical representation.

Observe that the types of nodes listed in Section 2, under 1. and 3. are preserved by any homeomorphism and those listed under 2. and 4. by homeomorphism of forms. Hence, the numbers of nodes of each type are also form invariants, as well as is the total number of nodes.

For example, numerals "2" and "3" have different types, because they have four and five nodes, respectively. On the other hand, each of the following forms (Fig. 18) is a realization of numeral "2" and they have the same number of nodes and all their invariant properties are the same. The numeral "1" has one vertical arc and the numeral "7" has no such arc. Thus, they are not equivalent.

If (F, Λ) and (F', Λ') are equivalent, then for each $k \le n$, the sets

$$\Lambda_1 \cup \ldots \cup \Lambda_k \quad \text{and} \quad \Lambda'_1 \cup \ldots \cup \Lambda'_k,$$

when connected, represent equivalent forms. On the opposite, if for some k, they are not equivalent, then forms F and F' are of different type. For instance, the numerals "6" and "9" are not of the same type, because $\Lambda_1 \cup \Lambda_2$ is not equivalent to $\Lambda_1' \cup \Lambda_2'$.

Let us call two continuity matrices equivalent if they correspond to equivalent forms. For example, the forms in Fig. 19 are equivalent and so are their continuity matrices

$$\begin{bmatrix} 0 & (0) & 1 & (1) & 0 \\ 0 & (1) & 1 & (1) & 1 \\ 1 & (1) & 1 & (1) & 1 \end{bmatrix}, \begin{bmatrix} 1 & (1) & 0 & (0) & 0 \\ 0 & (1) & 1 & (0) & 0 \\ 1 & (1) & 1 & (1) & 1 \end{bmatrix}$$

It is worth thinking of an algorithm by which any two equivalent matrices are transformed to the canonical one being uniquely attached to each type of forms. In the example above, two matrices are transformed into

$$\begin{bmatrix} 1 & (1) & 0 \\ 0 & (1) & 1 \\ 1 & (1) & 1 \end{bmatrix},$$

which corresponds to the type of numeral "2". Canonical matrices would also be form invariant. We leave this matter with the remark that these matrices could be important invariants, because they register a lot of the structure of a form.

7. IMPLEMENTATION

The above considerations, although theoretically oriented, have been mainly motivated by the idea to design a robust character recognition system, able to cope with problems of practical applications to different character sets (numerals, isolated handwritten characters, cursive text). Some of the corresponding results have been described in [12], [13]; a complete presentation will appear elsewhere. In this section we shall give only some ideas of how to apply the developed methodology in more realistic environments.

7.1. Extraction of Line Forms

Real inputs to machine character recognition systems are not in the form of ideal lines, but appear as pixel patterns, i.e. matrices with binary elements. Any practical application of the developed concept is basically concerned with the problem of correspondence between real pixel patterns and line forms.

One of the possibilities to connect line forms directly with given patterns is to construct a polygonal line form, a sort of simplified skeleton. Fig. 20 depicts two realistic, "heavy" patterns, representing numerals "2" and "3". The main idea is to introduce "nodes" of the patterns, similarly as the nodes have been defined in the case of line forms. Notice that, when a form is regular, the nodes are touching points of the form with its frame (see Fig. 21) (include also the touching points with the horizontal lines of the frame). Consequently, in the case of a pixel pattern, touching points with a frame will have the role of "nodes". Expecting, in general, unbalanced forms, touching points can be obtained by "moving" parts of the frame to the touching positions (see Fig. 20). The obtained set of points is the first "approximation" of the form. The next step is to connect these points by admissible arcs. These arcs have the meaning given above by the corresponding definition (see Section 2), and, in addition, they have to belong to the body of the pattern. In such a way we correspond a line form to a pattern. Defining the closeness of an interval and the body of the pattern, a polygonal form with straight lines can also be constructed. The line forms obtained in such a way can be efficiently subjected to the procedure based on the conceptualization presented in the paper.

Notice that the balancing of the line forms is now eaiser, and consists of fixing possible discontinuity lines and of erasing the outer parts of the forms. The idea is seggested in Fig. 22.

Details related to the machine realization of the described procedures are out of the scope of the paper.

7.2 Decomposition of Forms

A form is simpler when its height, defined in the preceding section, is smaller. Looking alternatively in the direction of the vector $\bar{i}$ to the numerals represented in Fig. 23, we conclude that all have smaller height, compared with the look in direction $\bar{j}$. Therefore, the $\bar{i}$ look makes these forms simpler. Observe that all Λ_t (with respect to $\bar{j}$) are one member families (and speaking descriptively then such forms are "even"), and with respect to $\bar{i}$ some are not. Decomposing the forms along the discontinuity lines (which are structural), we obtain "sums" of more "even" forms (which are more regular, and so more easily subjected to recognition). The "seen" structural discontinuities are essential properties of forms here.

The above concepts can be related to pixel patterns in real recognition procedures. The way is to consider a pixel pattern as a form drawn by heavy lines, A column of the pixel matrix stays in the role of the line $x = t$. Their intersections consist now of connectivity components, which are seen as points, and counted as 1's. As a result, a sequence of groups of numbers is obtained. If the form is not "too heavy", short groups correspond to odd positions in τ, and long sequences to even positions, i.e.

$$\overbrace{\alpha\ldots\alpha}^{short}\ \overbrace{\beta\ldots\beta}^{long}\ \overbrace{\gamma\ldots\gamma}^{short}\ldots\mapsto\alpha(\beta)\gamma\ldots$$

We also have to respect the structure of the sequence; when $\alpha = \beta$ the correspondence is

$$\overbrace{\alpha\ldots\alpha}^{long}\ \overbrace{\gamma\ldots\gamma}^{short}\ldots\mapsto\alpha(\alpha)\gamma\ldots$$

and so on. The practical way of relating τ to a pixel pattern is faced with difficulties which have to be treated separately. Some ideas, such as the estimation of line width, can be utilized.

Continuity of line forms, when related to pixel patterns, takes the aspect, say, of two nonapproaching components followed by one component, or of a sequence of one component sets followed by a vertical element (Fig. 24). Remark that our approach "sees" the vertical elements as "big points". This supposes specific methods for their selection, which are not included in the above general considerations. Detection of the discontinuities leads to decomposition of pixel patterns into "simpler" parts.

The above mentioned relations between ideal line forms and realistic pixel patterns have been combined to make a computer program for the recognition of numerals. The length of such a consideration requires a separate paper which is under preparation by the authors.

8. CONCLUSION

In this paper the conceptualization of a semi-topological classification of line figures has been presented.

The above-considered continuity sequences and matrices are attached to each realization of a form, and not to its equivalence class. In this way they are sensitive to the presence of noise existing in the realizations. It is quite natural to tend to find algorithms leading to a canonical representative of a class (a well-balanced form) and to obtain the corresponding sequences and matrices as being invariants of the class. It would be also interesting to define invariantly a decomposition of forms into more regular ones. Both aspects could be important for implementation purposes.

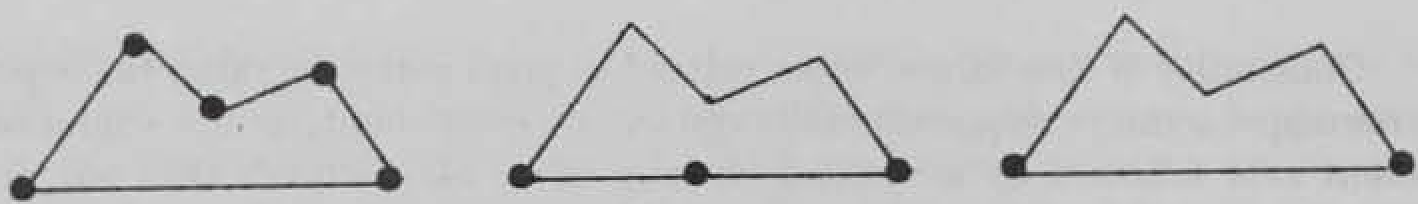


Fig. 1

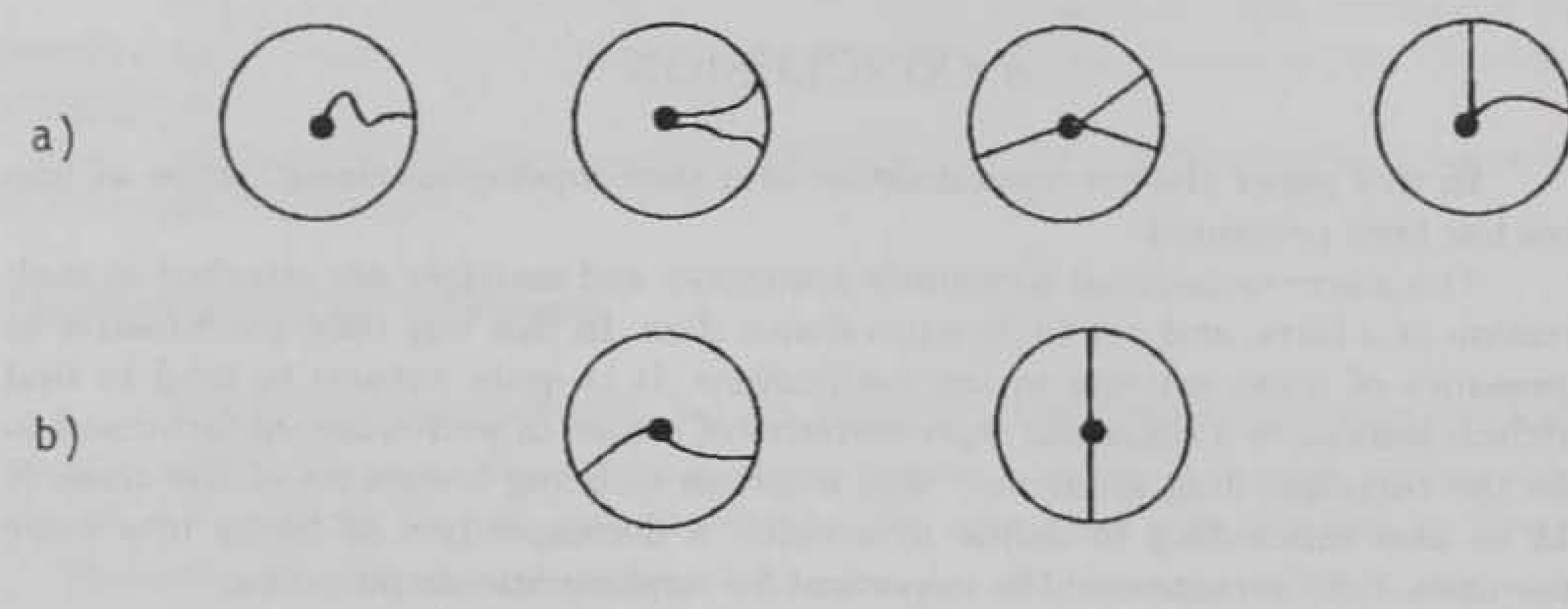


Fig. 2

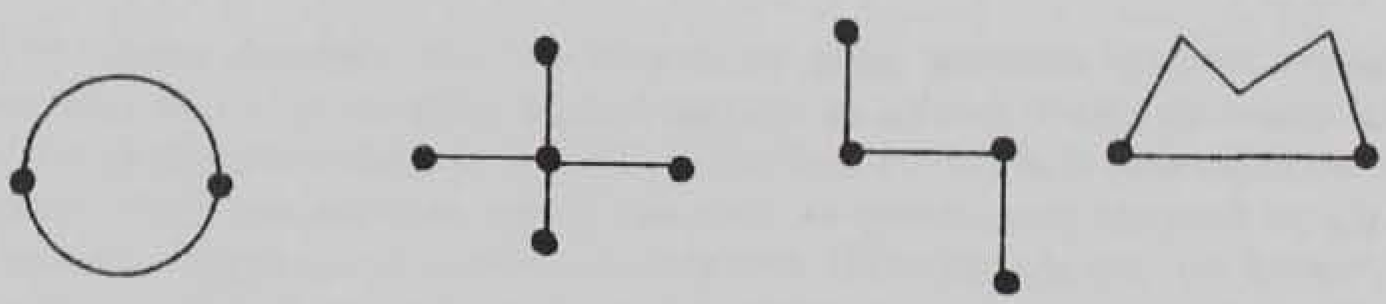


Fig. 3

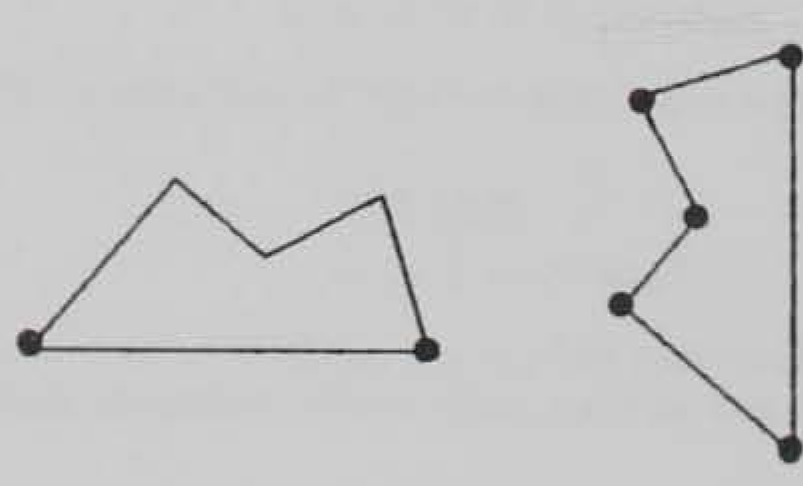


Fig. 4

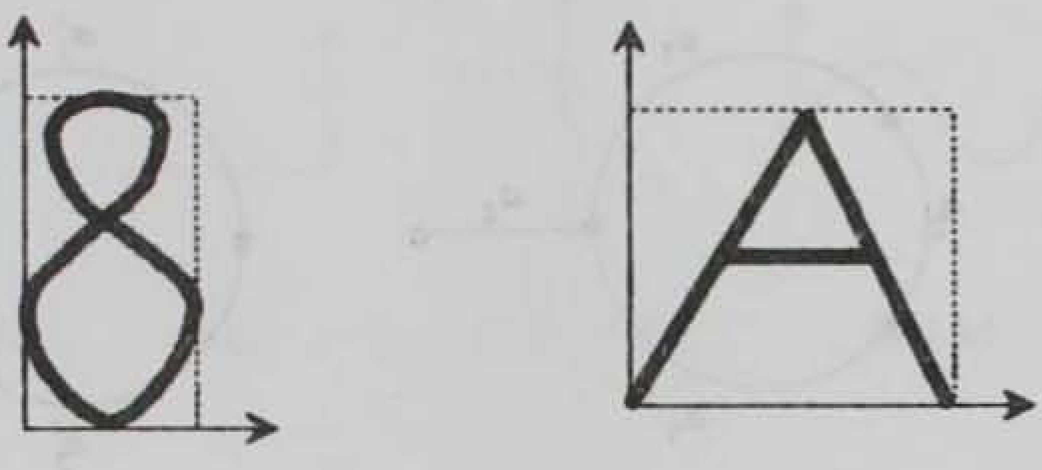


Fig. 5

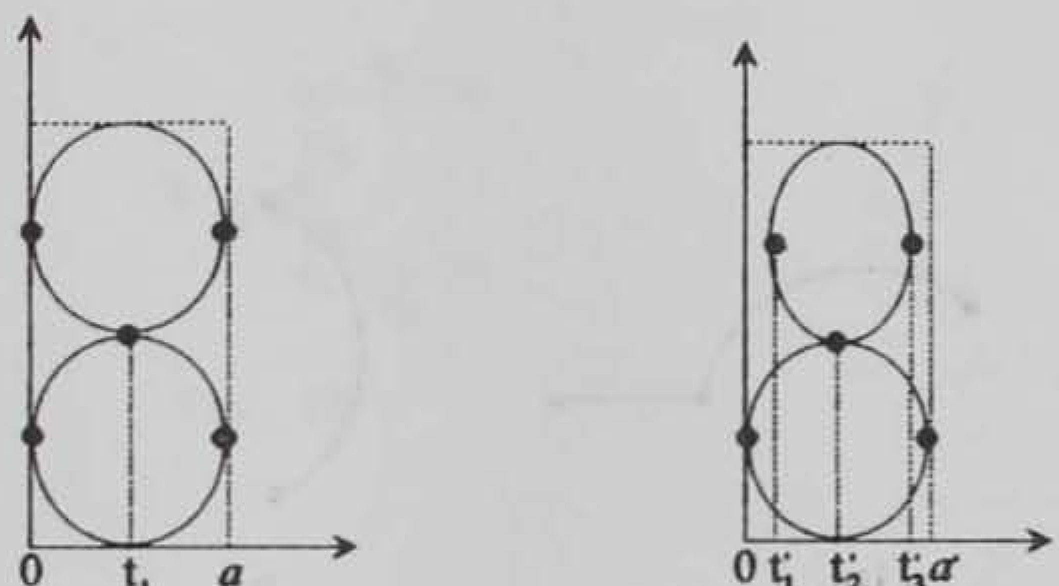


Fig. 6

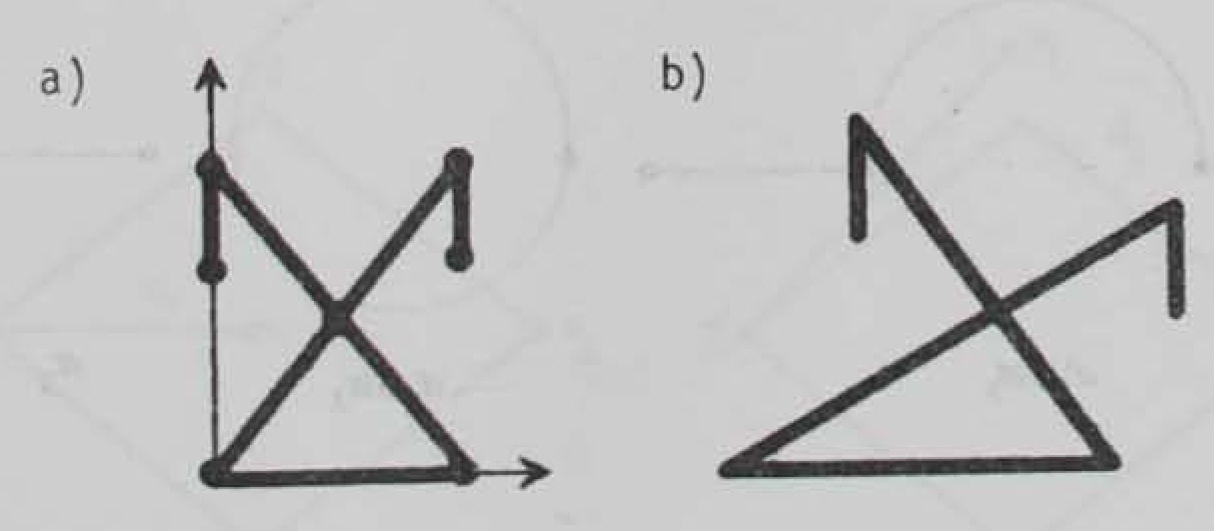


Fig. 7

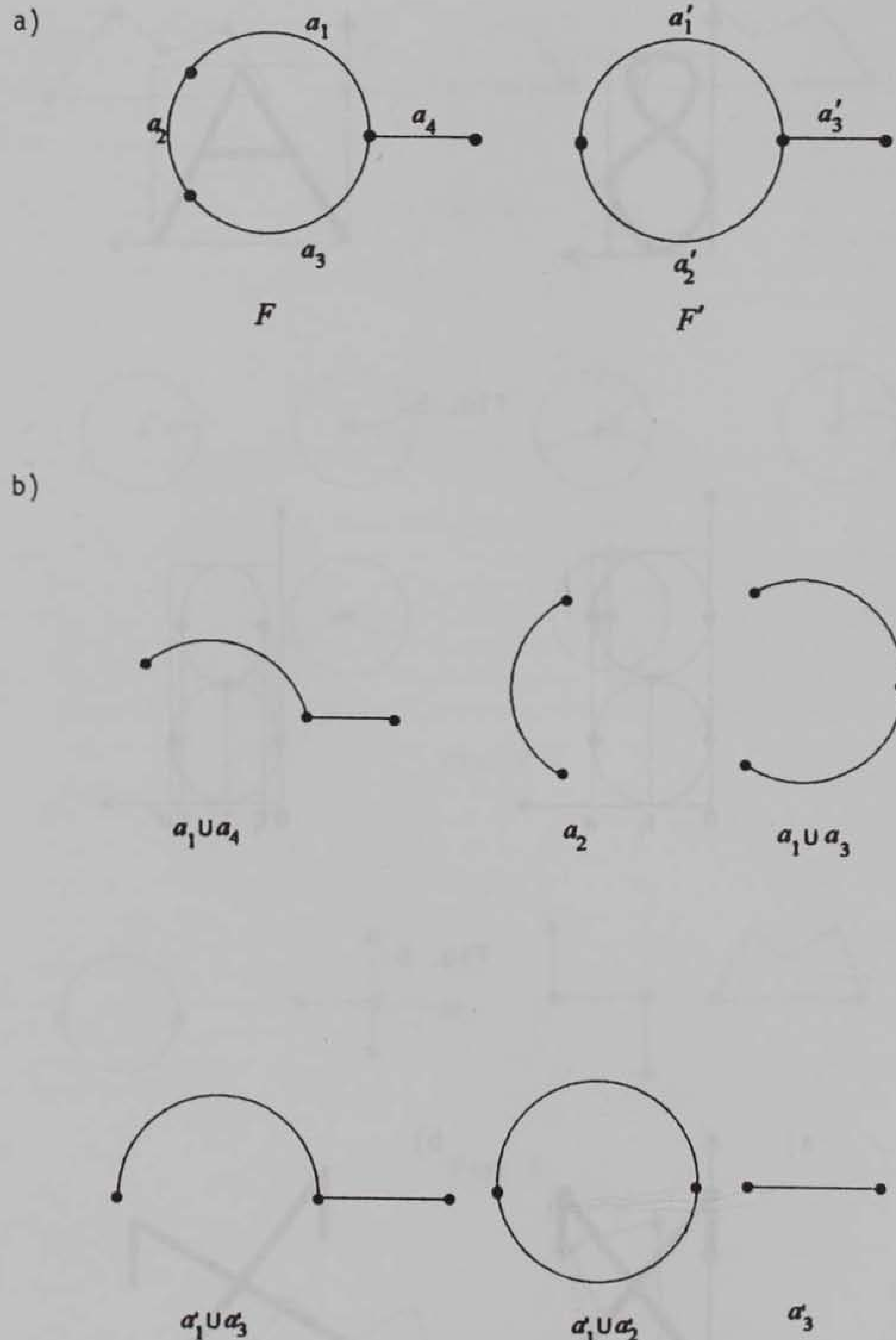

a)
a_1
a_2
a_4
a_3
F
a'_1
a'_3
a'_2
F'
b)
$a_1 \cup a_4$
a_2
$a_1 \cup a_3$
$a'_1 \cup a'_3$
$a'_1 \cup a'_2$
a'_3

Fig. 8

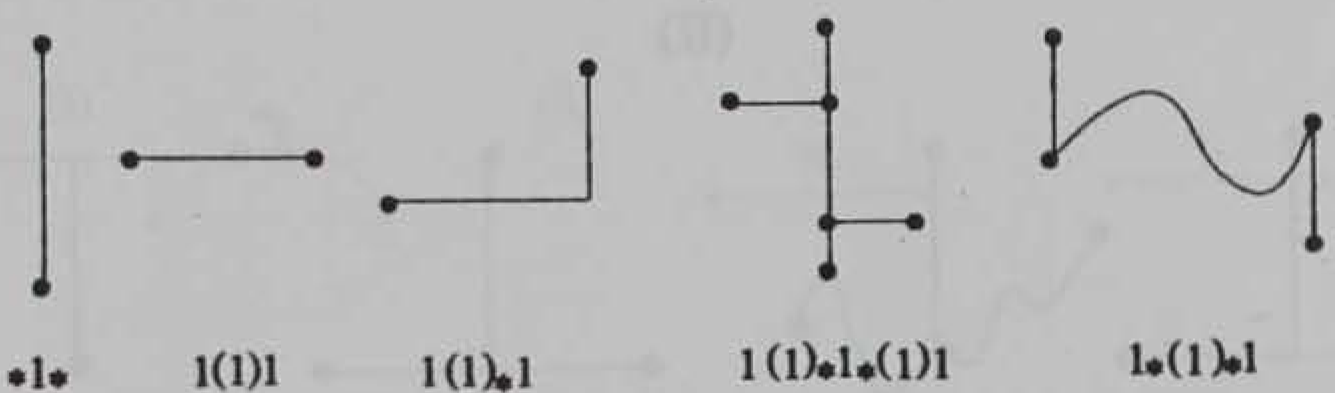


Fig. 9

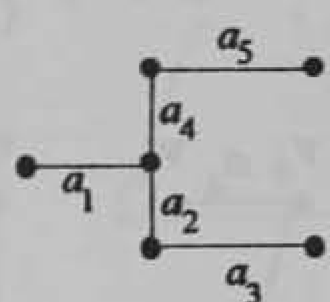


Fig. 10

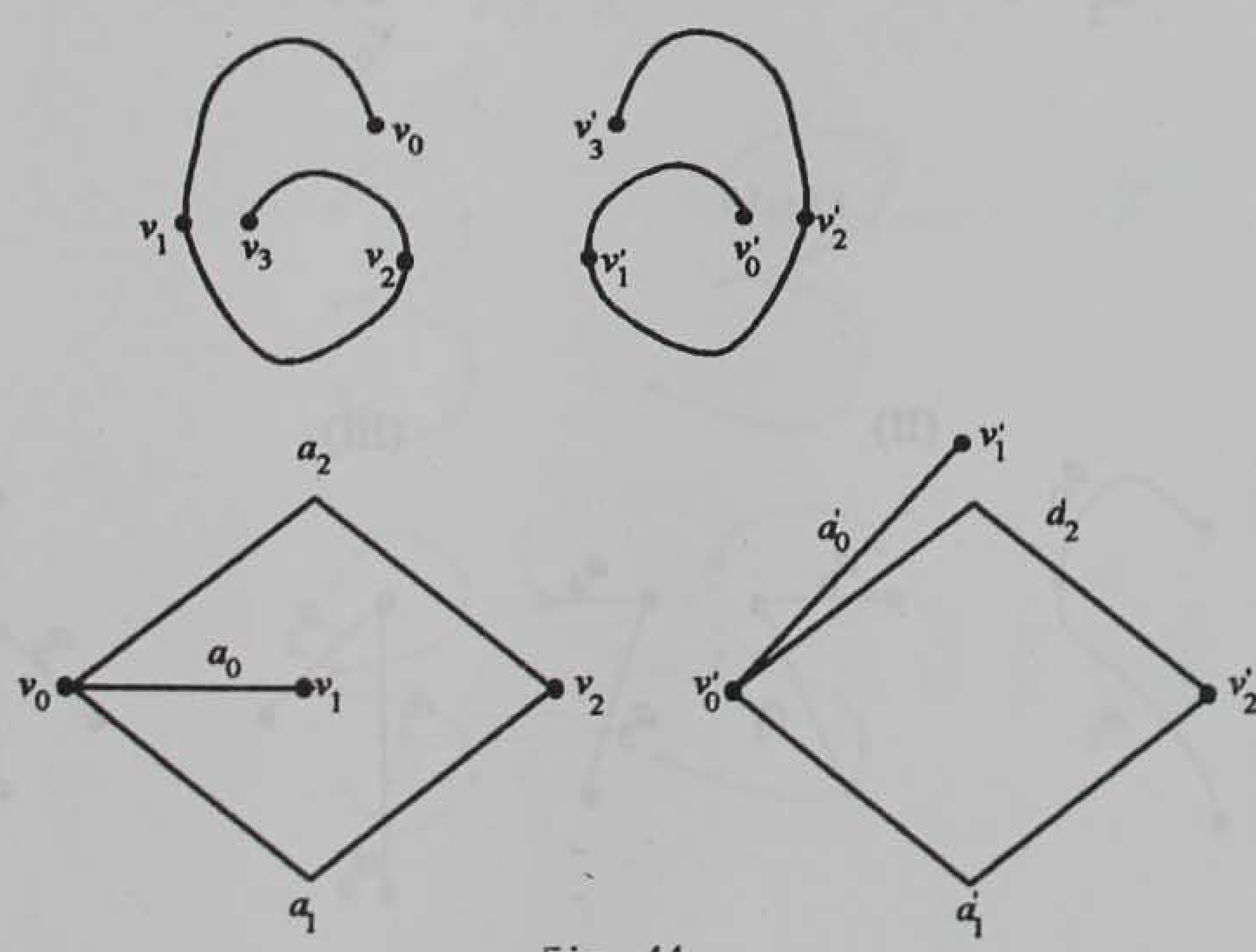


Fig. 11

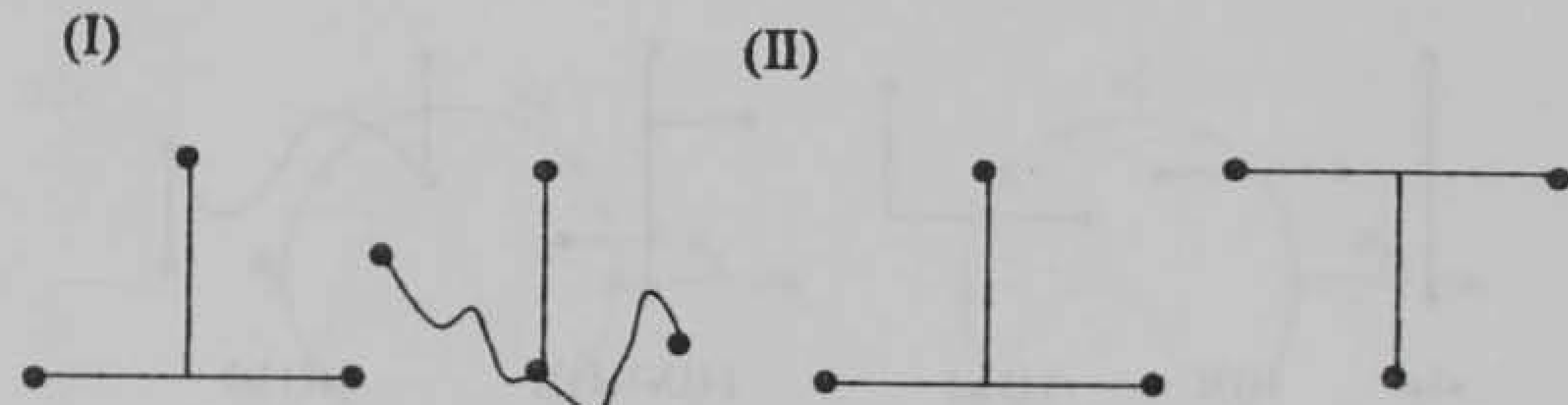


Fig. 12

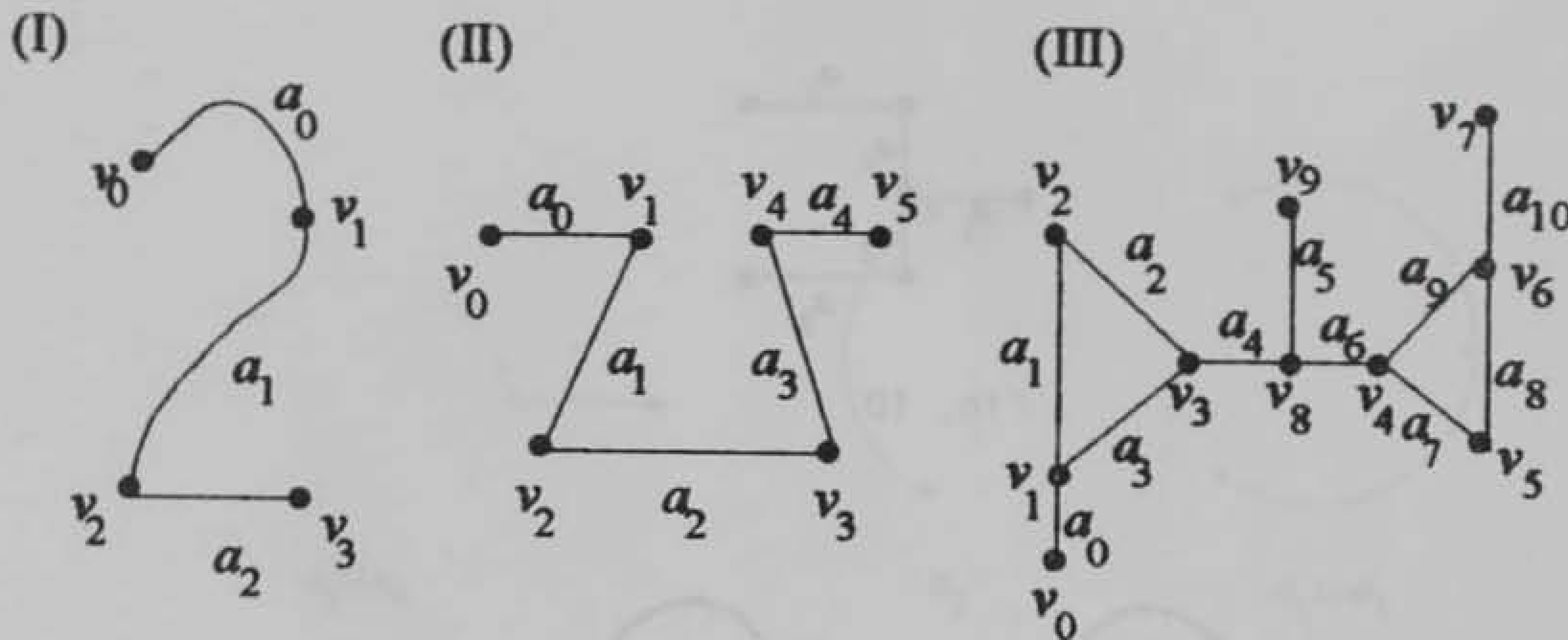


Fig.13

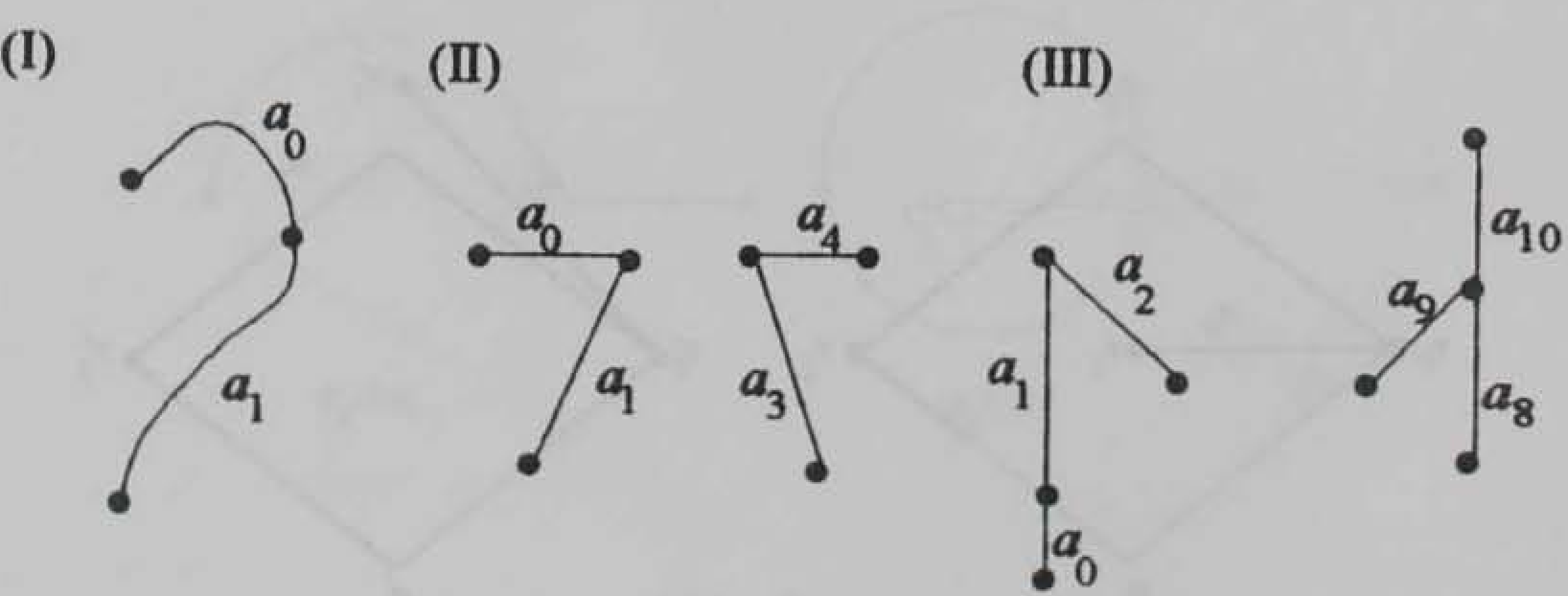


Fig. 14

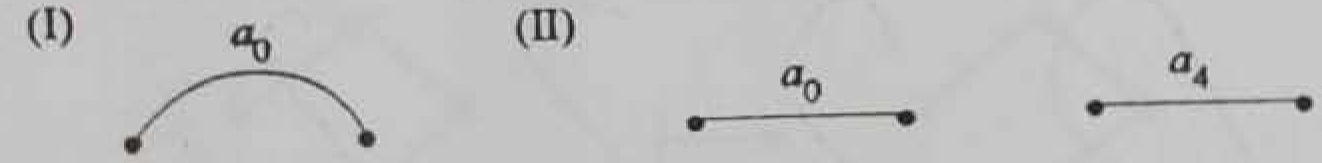


Fig. 15

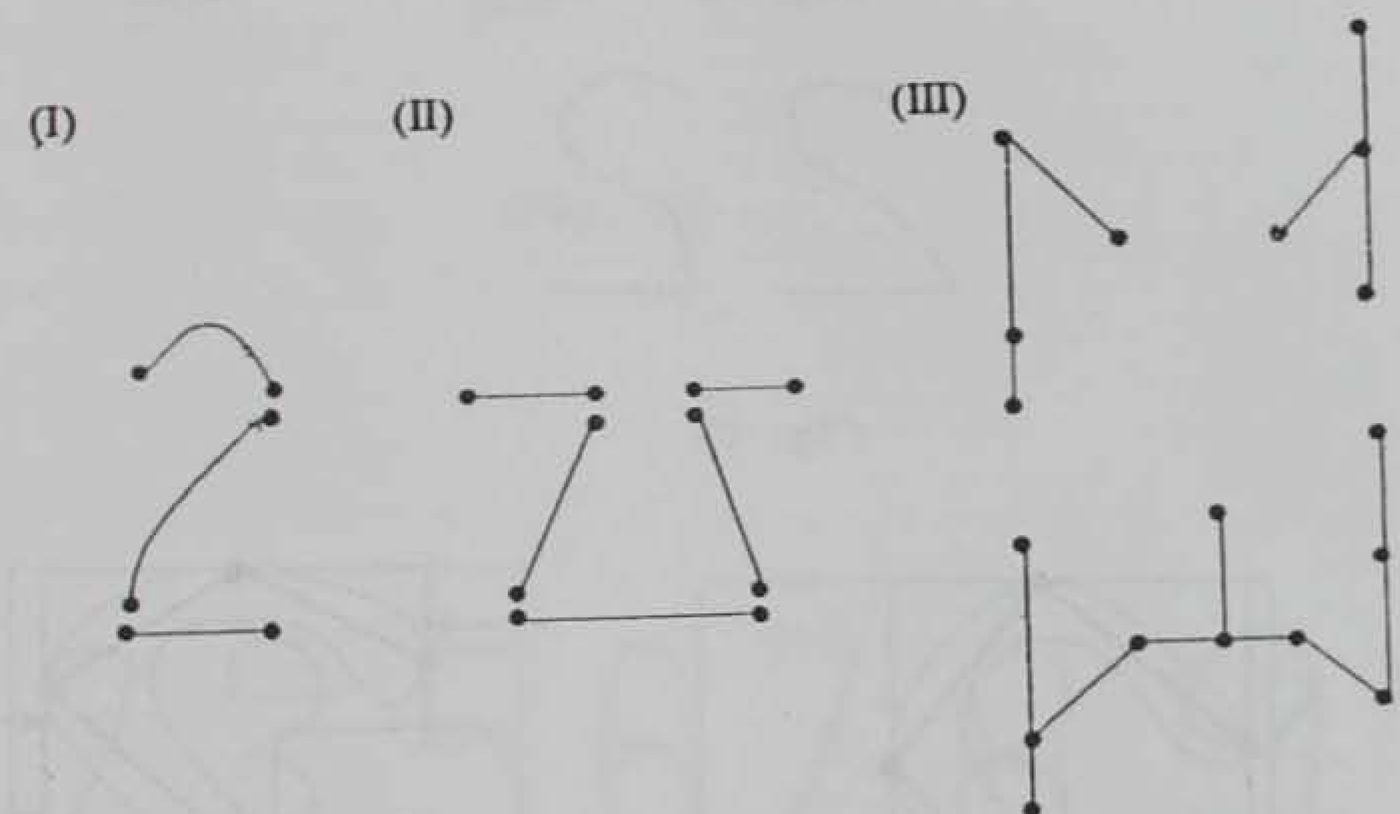


Fig. 16

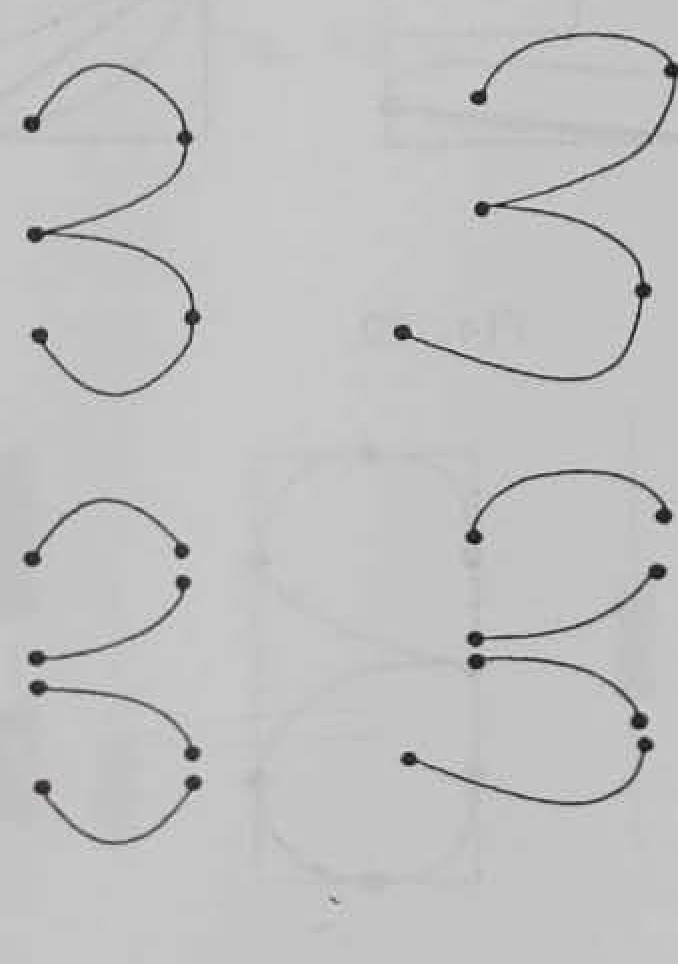


Fig. 17

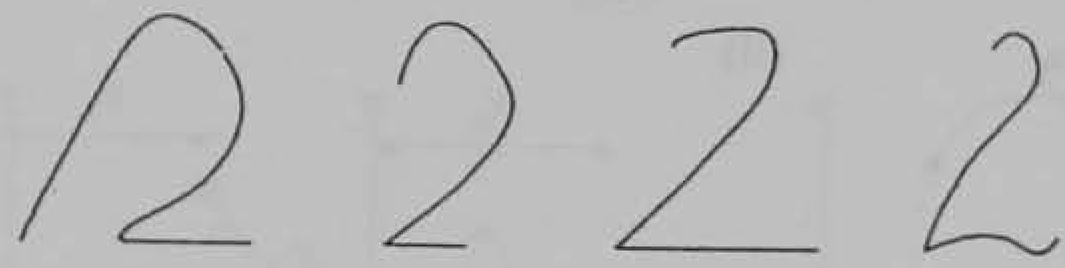


Fig. 18

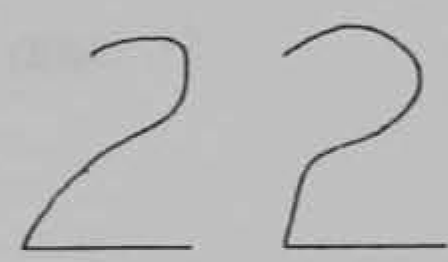
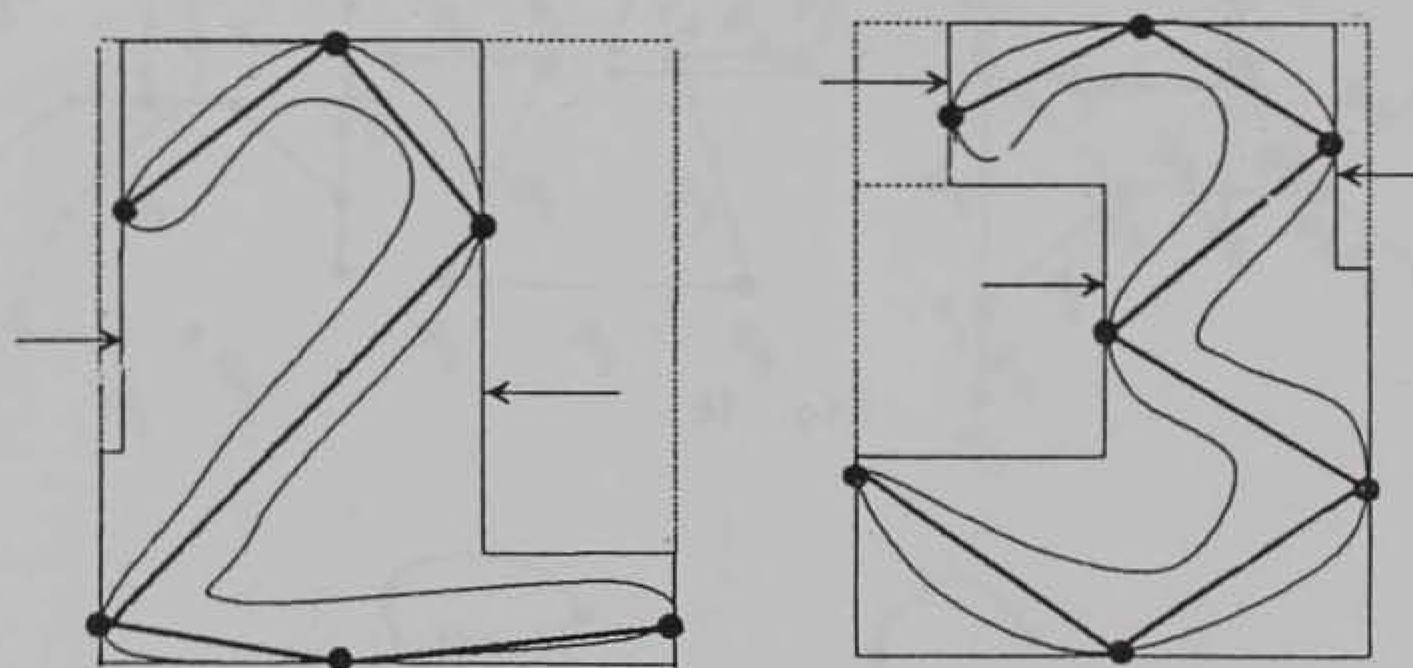
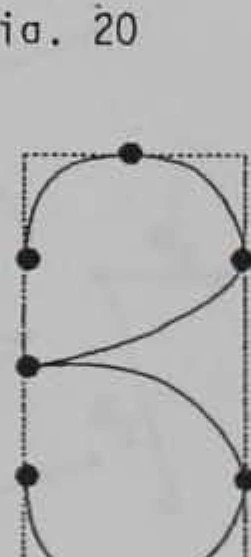


Fig. 19



Fia. 20



Fia. 21

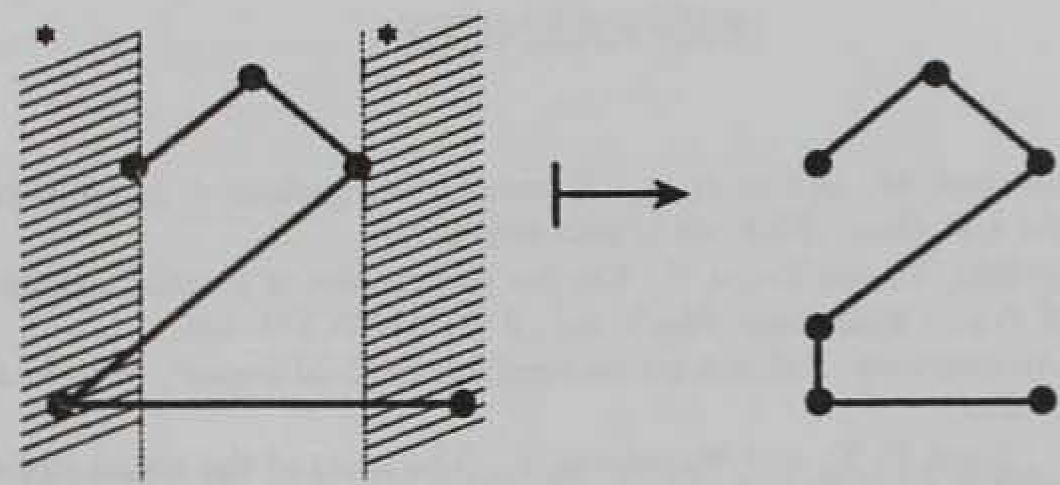


Fig. 22

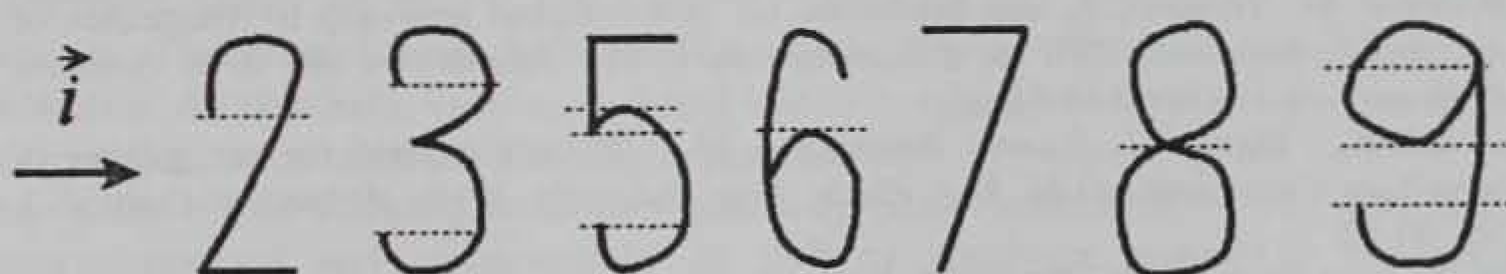


Fig. 23

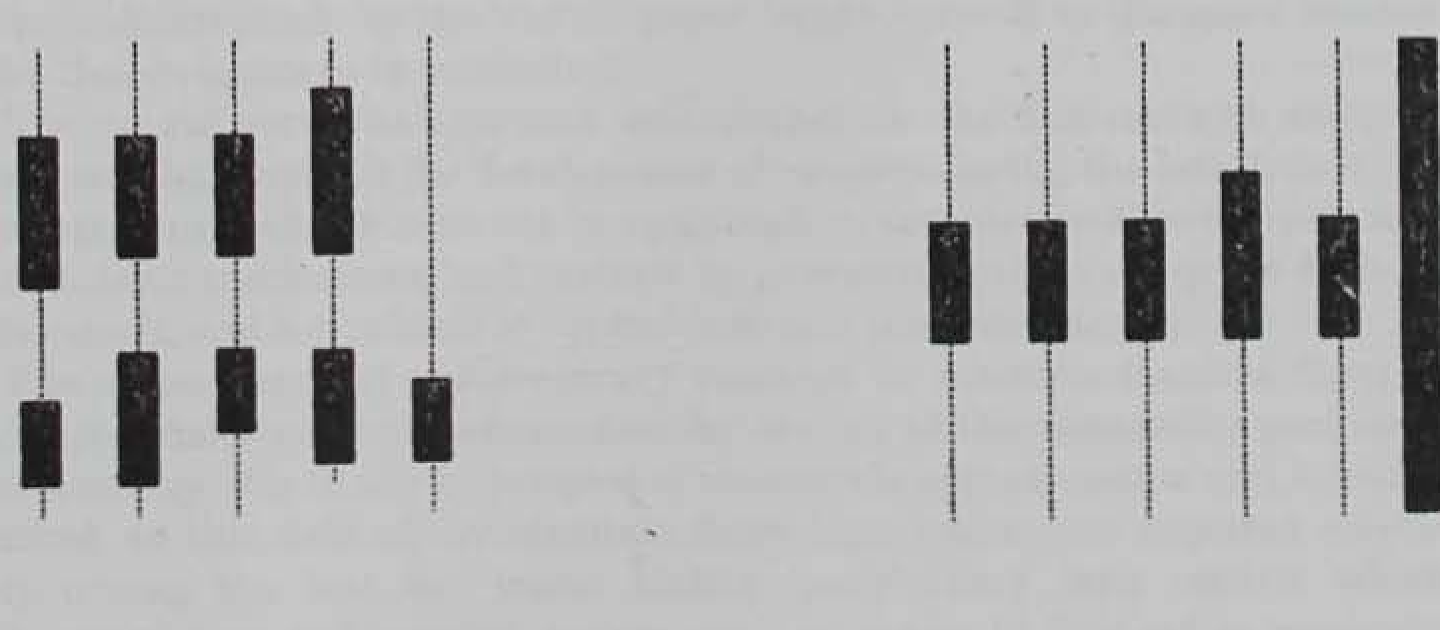


Fig. 24

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