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A NOTE ON FUZZY GROUPS

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Abstract: Motivated by various applications of finite algebraic structures, we give general solutions to the following problems, implicitly stated in [1] and [2]: for which groups is every fuzzy subgroupoid a fuzzy subgroup ? and: for which groups are all fuzzy subgroups of finite order ? The solutions are given for lattice valued fuzzy subgroups, and particularly for interval valued ones ([0,1] fuzzy subgroups).

Keywords: Fuzzy subgroup, fuzzy subgroupoid.

1. INTRODUCTION

Fuzzy algebras, particularly fuzzy groups, have been extensively studied recently. Apart from purely algebraic reasons, it turned out that these structures can be applied in various fields: informatics, coding theory (see [5, 6, 10] and references there in), biology [12], etc. Importance of finite fuzzy structures lies in the fact that they uniquely represent ordered collections of crisp structures (levels), which can be used for representations of various collections of objects: binary block-codes, particular correspondences between species (in biology) etc.

The present note was motivated by the foregoing applications. Its aim is to give some answers to problems concerning the finitness of fuzzy groups.

In paper [2], an example of a fuzzy subgroupoid which is not a fuzzy subgroup of the corresponding group has been given (as the correction to a theorem in [1]). Here we give the necessary and sufficient conditions for a group under which all its fuzzy subgroupoids are fuzzy subgroups. The solution is given for lattice valued subgroups, however the answer is also valid for interval valued ones (i.e. for [0,1] fuzzy subgroups). It turns out that the group satisfying these conditions has to be a torsion one.

In the cited papers some particular problems concerning fuzzy subgroups of finite order have been solved; here in we give a general answer to the question: for which groups are all fuzzy subgroups of a finite order ? In this case, the answer is different for [0,1]-valued and for lattice valued fuzzy subgroups. Namely, in order to have lattice valued subgroups of finite order only, a group should be finite, and in the case of interval valued subgroups, it suffices that a group satisfies the ascending and descending chain conditions on subgroups.

We use the technique developed in [7, 8], considering fuzzy subgroups as collections of ordinary subgroups satisfying some conditions. Further applications of these methods and of finite fuzzy structures can be found in other papers listed in the References.

2. PRELIMINARIES

Here we give some basic notions on fuzzy groupoids and groups. In the following, $\mathcal{G}=(G,*)$ will be a group and $(L,\wedge,\vee)$ a complete lattice, with the corresponding ordering $\leq$, and the top and the bottom elements 1 and 0.

(I) The mapping $\mu\colon G \to L$ is an L-fuzzy groupoid of $\mathcal{G}$ if for all $x \in G$

$$\mu(x*y) \geq \mu(x) \wedge \mu(y).$$

(II) An L-fuzzy subgroupoid of $\mathcal{G}$ is an L-fuzzy subgroup of $\mathcal{G}$ for all $x \in G$,

$$\mu(x^{-1}) \geq \mu(x).$$

(III) Since [0,1] is a complete lattice, definitions (I) and (II) are also valid for [0,1] valued fuzzy sets. We will call the corresponding notions fuzzy subgroupoids and fuzzy groups.

(IV) [1] If $\mu\colon G \to L$ is an L-fuzzy set, then order of μ is the cardinality of the set

$$\{\ \mu(x) \mid x \in G\ \}.$$

3. RESULTS

The following propositions have been proved in [7] for arbitrary algebras; here we state them for groupoids and groups.

Proposition 1. *Fuzzy set* $\mu\colon G \to L$ *is a fuzzy groupoid if and only if all its level sets are subgroupoids of* $\mathcal{G}$.

Proposition 2. *Fuzzy set* $\mu\colon G \to L$ *is a fuzzy group if and only if all its level sets are subgroups of* $\mathcal{G}$.

Proposition 3. Let $\mathcal{F} = \{B_i, i \in I\}$*be a family of subgroups of group* $\mathcal{G} = (G,*)$ *closed under intersection and containing* $\mathcal{G}$. *Let also* $(\mathcal{F},\leq)$ *be the lattice dual to* $(\mathcal{F},\subseteq)$. *Then* $\mu{:}G \to \mathcal{F}$, *defined with: for* $x \in G$

$$\mu(x) = \bigcap(\mathcal{B} \in \mathcal{F} \mid x \in \mathcal{B}) \tag{1}$$

is an L-valued subgroup of $\mathcal{G}$. *The collection of level subgroups of* μ *is* $\mathcal{F}$.

Recall that a torsion group is one for which all elements are of finite order.

Theorem 1. $\mathcal{G}$ is a torsion group if and only if every fuzzy subgroupoid of $\mathcal{G}$ is a fuzzy subgroup.

Proof. Suppose that $\mathcal{G}$ is a torsion group and that $\mu{:}G \to L$ is a fuzzy subgroupoid on $\mathcal{G}$. Consider a family $\mathcal{F}= \{B_i, i \in I\}$of level subgroupoids. For each i, $\mathcal{G}_i$ is a subgroupoid of $\mathcal{G}$. Suppose that $x \in G_i$. Then, since $\mathcal{G}$ is a torsion group, $x^n = e$ (n is a natural number, and e is a neutral element of ($\mathcal{G}$), and obviously $e \in G_i$. Hence, $x^{-1} = x^{n-1} \in G_i$, and G_i is a group. Since all level subsets are subgroups of $\mathcal{G}$, by Proposition 2, μ is a fuzzy group.

We prove the converse by contraposition. Suppose that $\mathcal{G}$ is not a torsion group. Then, there is an element of infinite order $x \in G$. We consider a subgroup $\mathcal{H}$ of $\mathcal{G}$ generated by x. This subgroup is isomorphic with $(Z,+)$, and has a subgroupoid $\mathcal{K}$ isomorphic with $(N,+)$. We denote the corresponding elements of H with $x_i, i \in Z$, if x_i corresponds to i under the mentioned isomorphism. Now, we define a fuzzy subset G in the following way:

for $\alpha \in L, \alpha \neq 0$:

$$\mu(x) = \begin{cases} \alpha, \text{ if } i \in N \\ 0, \text{ elsewhere.} \end{cases}$$

Level subsets of μ are $\mathcal{K}$ and $\mathcal{G}$ itself, and $\mathcal{K}$ is a subgroupoid of $\mathcal{H}$, and hence of $\mathcal{G}$ (it is isomorphic with $(N,+)$, while $\mathcal{H}$ is isomorphic with $(Z,+)$), μ is a fuzzy subgroupoid of $\mathcal{G}$ (Proposition 1). Hence, it is not its fuzzy subgroup (by Proposition 2). This proves that there is a fuzzy subgroupoid which is not a fuzzy group.

Obviously, since $[0,1]$ is a complete lattice, Theorem 1 is valid for $[0,1]$ -valued fuzzy groups as well.

Recall that a group satisfies the Descending Chain Condition (DCC) on subgroups if every strictly descending chain of its subgroups is finite. The dual notion is the Ascending Chain Condition (ACC).

Theorem 2. Every fuzzy subgroup of a group $\mathcal{G}$ is of finite order if and only if $\mathcal{G}$ satisfies DCC and ACC on subgroups.

P r o o f. We shall prove both implications by contrapositions. First, if there is a fuzzy subgroup $\mu: G \to [0,1]$ which is of infinite order then for all values of the function μ, the corresponding level subgroups are different. If $\mu(x)=\alpha_1$ and $\mu(y)=\alpha_2$ then G_{α_1} and G_{α_2} differ at least in x or y. This means that there are infinitely many different level subgroups, i.e., in the subgroup lattice $\mathcal{G}$ there is an infinite chain, which proves that $\mathcal{G}$ does not satisfy ACC nor DCC.

Conversely, if $\mathcal{G}$ does not satisfy ACC nor DCC, then there is an infinite chain of subgroups. By Theorem 6 in [1] we can construct a fuzzy subgroup of $\mathcal{G}$ of infinite order.

The previous theorem is valid only for [0,1]-valued fuzzy subgroups, because for the lattice valued ones it is possible that a group satisfies DCC and ACC and that it is infinite, and hence could have an L-fuzzy subgroup of infinite order, which is shown by the following example.

Example.

Let $\mathcal{G}$ be an infinite group all of whose proper subgroups are of prime order (the construction of such a group is due to A. Olshanskii, see [3]; the number of primes occuring as orders of subgroups in such a group is also infinite). The lattice $Sub\mathcal{G}$ of subgroups of such group is given in Fig. 1 (proper subgroups are denoted by H_{p_i}).

We shall construct an L-fuzzy subgroup of $\mathcal{G}$, with L being the above lattice of subgroups of $\mathcal{G}$. Let $\mu: G \to L$ be such that for $x \in G$

$$\mu(x) = H_{p_i}, \text{ provided that } x \in H_{p_i}, x \neq e, \text{ and}$$

$$\mu(e) = G.$$

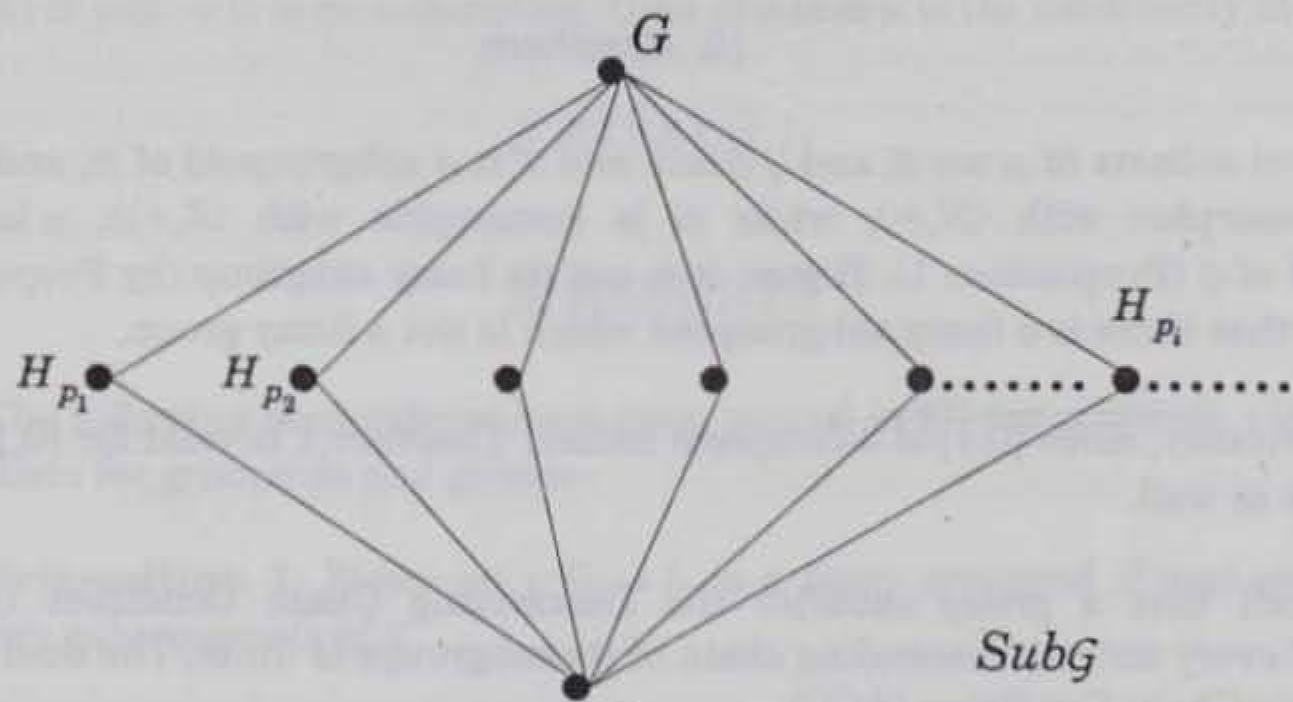

Figure 1.

μ is well defined, since the intersection of any two different proper subgroups of $\mathcal{G}$ contains only the neutral element e. The proof that μ is a fuzzy subgroup of $\mathcal{G}$ is straightforward.

The order of μ is infinite, however $\mathcal{G}$ obviously satisfies both DCC and ACC on subgroups.

The next theorem proves that a necessary and sufficient condition for all fuzzy subgroups of a group $\mathcal{G}$ to have finite order is that this group is finite.

Theorem 3. Every L-fuzzy subgroup of a group $\mathcal{G}$ is of finite order if and only if $\mathcal{G}$ is finite.

P r o o f. If $\mathcal{G}$ is a finite group, then obviously it is of finite order.

Conversely, if $\mathcal{G}$ is infinite, then it has infinitely many subgroups. Indeed, if it contains a subgroup isomorphic with an infinite cyclic group, then this subgroup has infinitely many subgroups which are subgroups of $\mathcal{G}$ as well. If it does not contain an infinite cyclic subgroup then it contain infinitely many cyclic groups of finite order. In any case, the lattice of subgroups of the group $\mathcal{G}$ is infinite. Now, we can construct a fuzzy subgroup of infinite order $\mathcal{G}$ in the following way:

We take $Sub\mathcal{G}$ to be a set of values, and define $\mu : G \to Sub\mathcal{G}$ in the following way:

$$\mu(x) := \bigcap_{H<G \mid x \in H} H .$$

By Proposition 3 μ is an L-fuzzy subgroup of $\mathcal{G}$ the set of levels of which is $Sub\mathcal{G}$. Since $\mathcal{G}$ is infinite it has infinitely many cyclic subgroups, and hence μ has infinitely many different values.

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