Yugoslav Journal of Operations Research
7 (1997), Number 1, 39-47

THE CONTINUOUS LINEARIZATION METHOD OF THE FOURTH ORDER

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Abstract: This paper considers the continuous linearization method of the fourth order for solving the convex programming problem Euclidean space. The sufficient conditions for convergence are established.

Keywords: Linearization method, convex programming problem.

1. INTRODUCTION

Consider the following minimization problem

$$J(u) \to inf, \quad u \in U = \{ z \in U_0 : g_i(u) \le 0, \quad i = 1, \dots, m \}, \tag{1}$$

where U_o is a closed, convex subset of real Euclidean space $\mathbf{E}^n$, functions $J(U)$, $g_i(u)$ are defined, continuously differentiable and convex on $\mathbf{E}^n$. The scalar product of two elements $u, v \in \mathbf{E}^n$ will be denoted by: $< u, v >$; $\|u\| = < u, u >^{1/2}$ is the norm of an element $u \in \mathbf{E}^n$.
Suppose that

$$J_* = \inf_{u \in U} J(u) > -\infty, \quad U_* = \{ u \in U : J(u) = J_* \} \ne \varnothing. \tag{2}$$

In order to solve the problem (1), when $m = 0$ (i.e. $U = U_o$), the continuous projection-gradient method of the fourth order

$$\beta_4(t) u^{iv}(t) + \beta_3(t) u'''(t) + \beta_2(t) u''(t) + u'(t) + u(t) = P_U[u(t) - \alpha(t) J'(u(t))], \; t \ge 0,$$

$$u(0)=u_0,\quad u'(0)=u_1,\quad u''(0)=u_2,\quad u'''(0)=u_3,$$

has been proposed and investigated in [2]. When the structure of set U is too complicated for the projection operation, it is more convenient, instead of projecting on U, to project on its linear approximation $U(u)=\{z\in U_0: g_i(u)+ + <g_i'(u), z-u>\leq 0, i=1,\dots,m\}$. This linearization idea is treated in this work.

2. THE METHOD

In order to solve problem (1) we will use the continuous linearization method described by the following differential equation of the fourth order:

$$\beta_4(t)u^{iv}+\beta_3(t)u'''+\beta_2(t)u''+u'+u= \\ = P_{U(u(t))}\left[u-\alpha(t)J'(u)\right],\quad t\geq 0, \tag{3}$$

$$u(0)=u_0,\quad u'(0)=u_1,\quad u''(0)=u_2,\quad u'''(0)=u_3, \tag{4}$$

where

$$U(u)=\{z\in U_0: g_i(u)+<g_i'(u), z-u>\leq 0, i=1,\dots,m\}; \tag{5}$$

$P_{U(u(t))}(z)$ denotes the projection of point z on the set $U(u(t))$; $u_i\in \mathbf{E}^n$, $i=0,1,2,3$ are given initial points; $\alpha(t), \beta_i(t)$, $i=2,3,4$ are the parameters of the method (3) - (5); $u\in u(t)$, $u^{(i)}(t)=\dfrac{d^i u(t)}{dt^i}$, $t\geq 0$, $i=1,2,3,4$.

We can remark that when $m=0$ the method (3)-(5) turns into the projection-gradient method (see [2]). If U_0 is a polyhedral set, and $U_0=\mathbf{E}^n$ or $U_0=\mathbf{E}^n_+=\{u\in \mathbf{E}^n: u^i\geq 0,\ i=1,\dots,n\}$, projection problem (3) is a standard quadratic programming problem. For $\beta_4(t)\equiv\beta_3(t)\equiv 0$, the method (3)-(5) becomes the continuous linearization method investigated in [1].

Suppose that the Slater condition is satisfied, i.e.

$$\exists u_c\in U_0,\quad g_i(u_c)<0,\quad i=1,\dots,m. \tag{6}$$

It is obvious that $U(u(t))$ is a nonempty, closed, convex set at any fixed moment $t\geq 0$, and the projection operation in (3) is correct. From the definition of the projection (see [3]), equation (3) can be replaced by the following minimization problem:

$$\tfrac{1}{2}\|z-[u(t)-\alpha(t)J'(u(t))]\|^2\to inf,\quad z\in U(u(t)), \tag{7}$$

for every $t\geq 0$. Under the given assumptions the Lagrangean functions for problems (1) and (7) have the saddle points (u_*,λ^*), $(\beta_4(t)\times u^{iv}(t)+\beta_3(t)u'''(t)+\beta_2(t)u''(t)+u'(t)+u(t),$

$\nu(t)) \in U_0 \times \Lambda_0$, where $\Lambda_0 = \{\lambda \in \mathbf{E}^m : \lambda_i \ge 0, \ i = 1,...m\}$. Then according [3], p. 237, Lemma 2, they satisfy:

$$\lambda_1^* \ge 0,...,\lambda_m^* \ge 0, \quad u_* \in U_*, \tag{8}$$

$$< J'(u_*) + \sum_{i=1}^{m} \lambda_i^* g_i'(u_*), \quad v - u_* > \ge 0, \quad v \in U_0, \tag{9}$$

$$\lambda_i^* g_i(u_*) = 0, \quad i = 1,...,m. \tag{10}$$

$$\nu_1(t) \ge 0,...,\nu_m(t) \ge 0, \quad t \ge 0, \tag{11}$$

$$\begin{aligned} &< \beta_4(t) u^{iv} + \beta_3(t) u''' + \beta_2(t) u'' + u' + \alpha J'(u) + \\ &+ \sum_{i=1}^{m} \nu_i(t) g_i'(u), \quad w - (\beta_4(t) u^{iv} + \beta_3(t) u''' + \beta_2(t) u'' \\ &+ u' + u) > \ge 0, \quad w \in U_0, \quad t \ge 0. \end{aligned} \tag{12}$$

$$\begin{aligned} &\nu_i(t) \ [g_i(u) + < g_i'(u), \beta_4(t) u^{iv} + \beta_3(t) u''' + \\ &+ \beta_2(t) u'' + u' + u >] = 0, \quad t \ge 0, \quad i = 1,...,m, \end{aligned} \tag{13}$$

$$\begin{aligned} &g_i(u) + < g_i'(u), \beta_4(t) u^{iv} + \beta_3(t) u''' + \beta_2(t) u' > \le 0, \\ &t \ge 0, \quad i = 1,...,m, \end{aligned} \tag{14}$$

3. THE CONDITIONS FOR CONVERGENCE

Here we will establish the sufficient conditions for the convergence of the method (3)-(5).

Theorem. Suppose that

1) U_0 is a convex closed set in Euclidean space $\mathbf{E}^n$; the function $J(u)$, $g_i(u)$are convex and differentiable on $\mathbf{E}^n$; the gradients $J'(u)$, $g_i'(u)$ satisfy the Lipschitz condition

$$max \ \{\|J'(u) - J'(v)\|; \ \max_{1 \le i \le m} \|g_i'(u) - g_i'(v)\|\} \le L \ \|u - v\|, \tag{15}$$

for all $u, \quad v \in \mathbf{E}^n$; conditions (2) and (6) are satisfied.

2) parameters $\alpha(t), \beta_i(t), i = 2,3,4$ are such that

$\alpha(t) \in \mathbf{C}[0,+\infty), \quad 0 < \alpha_0 \le \alpha(t) \le \alpha_1, \quad t \ge 0;$

$$\beta_2(t)\in \mathbf{C}^2[0,+\infty);\quad \beta_3(t)\in \mathbf{C}^3[0,+\infty);\quad \beta_4(t)\in \mathbf{C}^4[0,+\infty);$$
$$\beta_i'(t)\le 0,\quad \beta_i''(t)\ge 0,\quad i=2,3,4,\quad t\ge 0;$$
$$\beta_i'''(t)\le 0,\quad i=3,4;\quad \beta_4^{iv}(t)\ge 0,\quad t\ge 0;$$
$$\lim_{t\to\infty}\beta_i(t)=\beta_{i\infty}>0,\quad i=2,3,4;$$
$$1-\alpha_1L_0-\beta_{2\infty}>0,\quad \beta_{2\infty}^2(1-\alpha_1L_0)+\beta_{4\infty}-2\,\beta_{3\infty}>0,$$
$$\beta_{3\infty}^2(1-\alpha_1L_0)+2\,\beta_{2\infty}\,\beta_{4\infty}>0,\quad \beta_{2\infty}-\tfrac{3}{2}\beta_{3\infty}-\beta_{3\infty}\beta_{4\infty}>0,$$
$$\beta_{2\infty}\,\beta_{3\infty}-3\,\beta_{4\infty}-\beta_{2\infty}\beta_{4\infty}^2>0.$$

where $L_0=L\left(1+2\sum_{i=1}^{m}\lambda_i^*\right)$ and L is defined in (15). Then for every initial points $u_0,u_1,u_2,u_3\in\mathbf{E}^n$, there is a point $u_\infty\in U_*$ such that

$$\lim_{t\to\infty}\left\{\sum_{i=1}^{4}\|\,u^{(i)}(t)\,\|+\|\,u(t)-u_\infty\,\|\right\}=0,$$
$$\int_0^{+\infty}\left\{\sum_{i=1}^{4}\|\,u^{(i)}(s)\,\|^2+f(s)\|\,u(s)-u_\infty\,\|^2\right\}ds<+\infty,$$

where $f(s)=\beta_2''(s)-\beta_3'''+\beta_4^{iv}(s)$, for all $s\ge 0$.

P r o o f. As we konw [3], for any convex differentiable function $g(u)$ on $\mathbf{E}^n$, the following is true

$$g(v)+<g'(v),\ w-v>\le g(w),\quad v,w\in\mathbf{E}^n, \tag{16}$$

From (5) and (16), it follows $U\subseteq U(u(t))\subseteq U_0$, t [illegible]. Setting $v=\beta_4(t)u^{iv}+\beta_3(t)u'''+$ $+\beta_2(t)u''+u'=u$ in (9), $w=u_*$ in (12), and multiplying the obtained inequalities, respectively, by $-(\alpha(t))$ and (-1), and summing them up we have

$$<\beta_4(t)u^{iv}+\beta_3(t)u'''+\beta_2(t)u''+u'+$$
$$+\sum_{i=1}^{m}\nu_i(t)\,g_i'(u),\quad \beta_4(t)u^{iv}+\beta_3(t)u'''+\beta_2(t)u''+$$
$$+u'+u-u_*>\le\alpha(t)<\sum_{i=1}^{m}\lambda_i^*\,g_i'(u_*),\quad \beta_4(t)u^{iv}+\beta_3(t)u'''+ \tag{17}$$
$$+\beta_2(t)u''+u'+u-u_*>+\alpha(t)\quad <J'(u)-J'(u_*),\,u_*-$$
$$-(\beta_4(t)u^{iv}+\beta_3(t)u'''+\beta_2(t)u''+u'+u)>,\quad t\ge 0,\quad u_*\in U_*.$$

Further, we will prove that

$$< \sum_{i=1}^{m} \nu_i(t) g_i'(u), \quad \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + \\ + u' + u - u_*) >, \quad t \ge 0, \quad u_* \in U_*. \tag{18}$$

From (11), (13), (16) and $u_* \in U_* \subseteq U$, it follows that

$$\nu_i(t) < g_i'(u), \quad \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + u - u_* > =$$

$$= \nu_i(t)\big[< g_i'(u), \quad \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' > + g_i(u) \ \big] -$$

$$- \nu_i(t)\big[< g_i'(u), u_* - u > + g_i(u) \ \big] \ge - \nu_i(t) g_i'(u_*) =$$

$$= \nu_i(t) \big| \ g_i'(u_*) \ \big| \ge 0, \quad t \ge 0, \quad i = 1, \dots, m.$$

From here, it is obvious that (18) is true. The first term on the right-hand side of (17) can be estimated in the following way

$$\sum_{i=1}^{m} \lambda_i^* < g_i'(u_*), \quad \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + \\ + u - u_* > \le \tfrac{L}{2} \sum_{i-1}^{m} \lambda_i^* \| \ \beta_4(t)u^{iv} + \beta_3(t)u''' + \\ + \beta_2(t)u'' + u' \ \|^2, \quad t \ge 0, \quad u_* \in U_*. \tag{19}$$

Let us prove it. From (10) and (16) we have

$$\lambda_i^* < g_i'(u_*), \quad \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + u - u_* > \le \\ \le \lambda_i^* \big[\ g_i(\beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + u) - g_i(u_*) \ \big] = \\ = \lambda_i^* g_i(\beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + u). \tag{20}$$

For a convex function $g(u)$, with gradient $g'(u)$ satisfying the Lipschitz condition (15), it holds that (see [3], p. 93, Lemma 1)

$$g(w) \le g(v) + < g'(w), w - v > + \tfrac{L}{2} \| \ w - v \ \|^2, \quad w, v \in \mathbf{E}^n. \tag{21}$$

Combining (8), (14) and (21) we get

$$\lambda_i^* g_i(\beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + u = \lambda_i^* \big[g_i(\beta_4(t)u^{iv} +$$
$$\beta_3(t)u''' + \beta_2(t)u'' + u' + u) - g_i(u) - < g_i'(u), \beta_4(t)u^{iv} +$$
$$+\beta_3(t)u''' + \beta_2(t)u'' + u' > \big] + \lambda_i^* \big[g_i(u) + < g_i'(u), \beta_4(t)u^{iv} +$$
$$+\beta_3(t)u''' + \beta_2(t)u'' + u' > \big] \le \lambda_i^* \tfrac{L}{2} \| \beta_4(t)u^{iv} + \beta_3(t)u''' +$$
$$+\beta_2(t)u'' + u' \|^2, \quad i = 1, \dots, m, \quad t \ge 0.$$

Relation (20) and the last inequality imply (19). Finally, we can estimate the second term on the right-hand side of (17) using the inequality (see [3], p. 175, Theorem 16)

$$< J'(u) - J'(v), \quad v - w > \le \tfrac{L}{4} \| u - w \|^2, \quad u, v, w \in \mathbf{E}^n. \tag{22}$$

Taking into account estimations (18), (19) and (22), from (17) we get

$$< \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u', \quad \beta_4(t)u^{iv} + \beta_3(t)u''' +$$
$$+\beta_2(t)u'' + u' + u - u_* > \le \alpha(t) \tfrac{L}{2} \left(\tfrac{1}{2} + \sum_{i=1}^{m} \lambda_i^* \right) \| \beta_4(t)u^{iv} +$$
$$+\beta_3(t)u''' + \beta_2(t)u'' + u' \|^2 \le \alpha(t) L_0 \big[\beta_4^2(t) \| u^{iv} \|^2 +$$
$$+\beta_3^2(t) \| u''' \|^2 + \beta_2^2(t) \| u'' \|^2 + \| u' \|^2 \big], \quad t \ge 0, \quad u_* \in U_*$$

where $L_0 = L\left(1 + 2\sum_{i=1}^{m} \lambda_i^*\right)$. The obtained inequality can be written in the form:

$$\begin{aligned} &\big[1 - \alpha(t) L_0 \big] \{ \beta_4^2(t) \| u^{iv} \|^2 + \beta_3^2(t) \| u''' \|^2 + \beta_2^2(t) \| u'' \|^2 + \\ &+ \| u' \|^2 \} + 2\beta_4(t) \{ \beta_3(t) < u^{iv}, u''' > + \beta_2(t) < u^{iv}, u'' > + \\ &+ < u^{iv}, u' > \} + 2\beta_3(t) \{ \beta_2(t) < u''', u'' > + < u''', u' > \} + \\ &+ 2\beta_2(t) < u'', u' > + \beta_4(t) < u^{iv}, u - u_* > + \beta_3(t) < u''', u - u_* > + \\ &+ \beta_2(t) < u'', u - u_* > + < u', u - u_* > \le 0, \quad t \ge 0, \quad u_* \in U_*. \end{aligned} \tag{23}$$

Inequality (23) has the same form as (8) in [2]. Thus, the relations

$$\overline{\lim_{t \to \infty}} \| u(t) - u_* \|^2 \le b_0 C_0(\xi, u_*), \quad \xi \in \bar{t}, \quad u_* \in U_*, \tag{24}$$

$$\overline{\lim_{t\to\infty}}\| u^{(i)}(t) \|^2 \le b_i C_0(\xi,u_*),\quad \xi \ge \bar{t},\quad u_* \in U_*,\quad i=1,2,3,4 \tag{25}$$

$$\int_0^\infty \left\{ \sum_{i=1}^4 \| u^{(i)}(s) \|^2 + f(s)\| u(s)-u_* \|^2 \right\} ds \le +\infty,\quad u_* \in U_*, \tag{26}$$

$$\begin{aligned} C_0(\xi,u_*) &= \beta_4(\xi)\beta_3(\xi)\| u'''(\xi) \|^2 + 2\beta_4(\xi)\{ \beta_2(\xi) < u'''(\xi),u''(\xi) > + \\ &+ < u''(\xi),u'(\xi) > \} + [\beta_3(\xi)\beta_2(\xi) - \beta_4(\xi) - (\beta_2(\xi)\beta_2(\xi))']\| u''(\xi) \|^2 + \\ &+[2\beta_3(\xi) - \beta_4(\xi) - 2\beta_4'(\xi)] < u''(\xi),u'(\xi) > + [\beta_2(\xi) - \tfrac{1}{2}\beta_3(\xi) + \\ &+\beta_4'(\xi) - \beta_3'(\xi) + \beta_4''(\xi)]\| u'(\xi) \|^2 + \beta_4(\xi) < u'''(\xi),u(\xi)-u_* > + \\ &+[\beta_3(\xi) - \beta_4'(\xi)] < u''(\xi),u(\xi)-u_* > + [\beta_2(\xi) - \beta_3'(\xi) + \beta_4''(\xi)]\times \\ &\times < u'(\xi),u(\xi)-u_* > + \tfrac{1}{2}[1 - \beta_2'(\xi) + \beta_3''(\xi) - \beta_4'''(\xi)]\| u(\xi)-u_* \|^2, \end{aligned} \tag{27}$$

for $\xi \ge \bar{t},\ u_* \in U_*$, which are consequences of (23), can be proved in the same way as (16), (18)-(21), (25) in [2]. Here the moment $\bar{t}$ is large enough, $b_1 = const.$

Now we will show that

$$\forall\, i \in \{1,\dots,m\},\qquad \sup_{t\ge 0} \nu_i(t) \le C < +\infty. \tag{28}$$

Setting $w = u_c$ in (12), where u_c is taken from (6), we get

$$\begin{aligned} &< \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + \alpha(t)J'(u)),\quad u_c - (\beta_4(t)u^{iv} + \\ &\beta_3(t)u''' + \beta_2(t)u'' + u' + u > \ge \sum_{i=1}^m \nu_i(t) < g_i'(u),\quad \beta_4(t)u^{iv} + \\ &+\beta_3(t)u''' + \beta_2(t)u'' + u' + u - u_c >,\quad t \ge 0. \end{aligned}$$

From Condition 2) of the theorem, and (24), 25), it is obvious that the left-hand side of the last inequality is bounded. Using (6), (11), (13) and (16), it can be shown that

$$\begin{aligned} C_1 &\ge \sum_{i=1}^m \nu_i(t) < g_i'(u),\quad \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' + u - u_c >= \\ &= \sum_{i=1}^m \nu_i(t)[g_i(u) + < g_i'(u),\quad \beta_4(t)u^{iv} + \beta_3(t)u''' + \beta_2(t)u'' + u' >] - \end{aligned}$$

$$-\sum_{i=1}^{m} v_i(t)\left[\ g_i(u) + < g_i'(u),\ \ u_c - u >\ \right] \ge -\sum_{i=1}^{m} v_i(t) g_i(u_c) =$$

$$= \sum_{i=1}^{m} v_i(t) \left|\ g_i(u_c)\ \right| \ge 0.$$

Thus

$$0 \le \sum_{i=1}^{m} v_i(t) \le C_1 \left[\ \min_{1 \le i \le m} \left|\ g_i(u_c)\ \right|\ \right]^{-1}.$$

This proves (28). From relation (26) the following can be derived

$$\underline{\lim}_{s \to \infty} \left\{ \left\| u^{iv}(s) \right\| + \left\| u'''(s) \right\| + \left\| u''(s) \right\| + \left\| u'(s) \right\| \right\} = 0.$$

Let $\{ s_j \} \subseteq [0, +\infty)$ be a sequence such that

$$\underline{\lim}_{s \to \infty} \left\{ \left\| u^{iv}(s) \right\| + \left\| u'''(s) \right\| + \left\| u''(s) \right\| + \left\| u'(s) \right\| \right\} =$$

$$= \lim_{j \to \infty} \left\{ \left\| u^{iv}(s_j) \right\| + \left\| u'''(s_j) \right\| + \left\| u''(s_j) \right\| + \left\| u'(s_j) \right\| \right\} = 0.$$

From (24) and (28) it is obvious that trajectory $u(t)$ and Lagrangean multipliers $v_i(t)$, $i = 1,\dots,m$, are bounded. Since $\alpha(t)$ is also bounded, there exist $u_\infty \in \mathbf{E}^n$, $\alpha_\infty > 0$, $v_i \ge 0$, $i = 1,\dots,m$ and the subsequence $\{ s_k \} \subseteq \{ s_j \}$ such that

$$\lim_{k \to \infty} \left\| u(s_k) - u_\infty \right\| = 0.$$

$$\lim_{k \to \infty} \left\| u^{(i)}(s_k) \right\| = 0, \quad i = 1,2,3,4, \tag{29}$$

$$\lim_{k \to \infty} \alpha(s_k) = \alpha_\infty > 0, \quad \lim_{k \to \infty} v_1(s_k) - v_i^* \ge 0, \quad i = 1,\dots,m.$$

Setting $t = s_k$ in (12)-(14), when $k \to \infty$ we have

$$\alpha_\infty < J'(u_\infty) + \sum_{i=1}^{m} v_i^* g_i'(u_\infty),\ \ w - u_\infty > \ge 0, \quad w \in U_0,$$

$$v_i^* g_i(u_\infty) = 0, \quad g_i(u_\infty) \le 0, \quad i = 1,\dots,m.$$

According to [3], p. 237, Lemma 2, the obtained inequalities give $u_\infty \in U_*$. From (27), where $u_* = u_\infty$, $\xi = s_k$, $k \ge k_0$ (k_0 is such that $s_{k_0} \ge \bar{t}$), and (29), we get

$$\lim_{k\to\infty} C_0(s_k, u_\infty) = 0.$$

The first statement of the theorem follows from here and (24), (25). Putting $u_* = u_\infty$ in (26) we get the second statement of the theorem.

4. CONCLUSION

As peinted out in [2], the importance of higher order projection-gradient methods stems from their higher order of convergence in comparison with first order methods of that type and from the fact that continuous methods give a large choice of numerical integration methods to solve the corresponding differential equations. When the structure of the feasible set U is too complicated it is convenient, instead of projecting the gradient on U to project it on the appropriate linear approximation of U. This paper shows that under suitable assumptions the method based on the linearization idea has the same convergence properties as the continuous projection-gradient method of the fourth order proposed in [2].

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