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SYMMETRIC DUALITY IN LEXICOGRAPHIC PROBLEMS OF LINEAR OPTIMIZATION

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Abstract: The paper studies linear lexicographic problems and suggests a scheme under which the second dual recovers the primal. Lexicographic interpretation of duality in improper linear programming problems is also considered.

Keywords: Linear lexicographic problems, Symmetric duality, Pareto optimality.

1. INTRODUCTION

A construction of linear lexicographic problem (*lex*-problem) is proposed that makes it possible to formulate a dual *lex*-problem satisfying the conditions: let L_{lex} be the original *lex*-problem and let $\otimes$ be the scheme of forming its dual, then $(L_{lex}^{\otimes})^{\otimes} = L_{lex}$. A version of *lex*-duality theorem is presented, which is distinct from that given by Theorem 1 in author's paper [1].

By lexicographic (successive) programming problem with a feasible set $M \subset R^n$ and an ordered (according to their "importance") system of functions $f_k(x),...,f_0(x)$ defined on $M \subset R^n$ is meant the series of problems:

$$max\ \{f_k(x) \mid x \in M\}, \tag{1.k}$$
$$max\ \{f_{k-1}(x) \mid x \in Arg\ (1.k)\}, \tag{1.k-1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$max\ \{f_0(x) \mid x \in Arg\ (1.1)\}. \tag{1.0}$$

where the k-th objective is the most important one and the 0-th is the least important.

For an arbitrary ordering $p = (i_k,...,i_0)$ of indices $\{i\}_0^k$ the problem formulated above will be written down as

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$$max_p \{F(x) \mid x \in M\}. \tag{1}$$

For the sake of definiteness we assume that $p=(k, k\text{-}1,...,0)$ so that

$$F(x)=[f_k(x),...,f_0(x)]^T$$

2. CONDITIONS FOR SYMMETRIC DUALITY

Consider a dual pair of linear lexicographic optimization problems in the following setting:

$$L_{lex} : \max_p \left\{ \begin{bmatrix} C^T x \\ c_0^T x \end{bmatrix} \middle| \; Ax \le b_0 + Br, \quad x \ge 0 \right\}, \tag{2}$$

$$L_{lex}^{\otimes} : \min_q \left\{ \begin{bmatrix} B^T u \\ b_0^T u \end{bmatrix} \middle| \; A^T u \ge c_0 + CR, \quad u \ge 0 \right\}. \tag{3}$$

Here $C = [\, c_1,...,c_k \,]$, $c_i \in R^n$ (c_i is a column vector); $B = [\, b_1,...,b_l]$, $b_j \in R^m$ (b_j is a column vector); $R^l \ni r \ge 0$ and $R^k \ni R \ge 0$ are vector parameters, p and q are the orderings of indices defining the vectors $\begin{bmatrix} C^T x \\ c_0^T x \end{bmatrix}, \begin{bmatrix} B^T u \\ b_0^T u \end{bmatrix}$.

Corresponding scalarized problems assume the form:

$$L_{scal}: \max \{ (c_0, x) + (C^T x, R) \mid Ax \le b_0 + Br, \ x \ge 0\}, \tag{4}$$

$$L_{scal}^{*}: \min \{ (b_0, u) + (B^T u, r) \mid A^T u \ge c_0 + CR, u \ge 0\}. \tag{5}$$

The $\otimes$-duality theorem linking problems (2) and (3) consists in the realization of the scheme:

$$\begin{array}{ccc} L_{lex} & \underset{\rightarrow}{\overline{Arg}} & L_{scal} \\ \otimes \downarrow\uparrow & & (*) \ \downarrow\uparrow \\ L_{lex}^{\otimes} & \underset{\rightarrow}{\overline{Arg}} & L_{scal}^{*} \end{array}$$

Scheme 1

Here by $\overset{Arg}{\rightarrow}$ is denoted the transition to a new problem with retention of the optimal set, i.e. $Arg\ L_{lex} = Arg\ L_{scal}$, $Arg\ L_{lex}^{\otimes} = Arg\ L_{scal}^{*}$. Problems L_{scal} and L_{scal}^{*} are related by classical duality.

For problem (2) we introduce the following condition: there exists a sequence of indices $O, i_1, \ldots, i_k$ such that

$$c_{i_s} \in cone\,\{ -c_{i_{s+1}}, \ldots, -c_{i_k}; a_1^T, \ldots, a_{n+m}^T \} \tag{6}$$

where $s = k, \ldots, 0$ and $I_0 = 0$, $\{ a_j \}_{j=1}^{m}$ denotes the rows of matrix A and $\{ -a_{m+i} \}_{i=1}^{n}$ denotes the unit co-ordinate vectors of R^n.

Lemma 1. *Problem* (2) *with a consistent system of constraints* (*that is* $M(r) = \{ x \geq 0 \mid Ax \leq b_0 + Br \} \neq \varnothing$) *is solvable for some ordering of functions* $\{(c_i, x)\}_0^k$ *if and only if the condition* (6) *holds; for all that* $(i_k, \ldots, i_1, 0)$ *can play the role of ordering* p.

P r o o f . The proof consists of sequential applications of the well-known condition of solvability of linear programming problem (LP):
the problem $\max \{(c, x) \mid Ax \leq b,\ x \geq 0 \}$ with a consistent system of contraints is solvable if and only if the system $A^T u \geq c$, $u \geq 0$ is consistent too, or (in the other form) if and only if

$$c \in cone\,\{ a_j \}_1^{m+n} \tag{7}$$

Indeed, since problem (2) is solvable for some $p = (i_k, \ldots, i_1, 0)$, the following LP problems are solvable

$$\max \{(c_{i_k}, x) \mid x \in M(r) \} \ (=: \alpha_k),$$
$$\max \{(c_{i_{k-1}}, x) \mid x \in M(r), -(c_{i_k}, x) \leq -\alpha_k \} \ (=: \alpha_{k-1}),$$

and so on. Therefore according to (7) we have

$$c_{i_k} \in cone\,\{ a_j \}_1^{m+n}\ ;$$
$$c_{i_{k-1}} \in cone\,\{ -c_{i_k}, a_1, \ldots, a_{m+n} \}$$

and so on. After all, solvability of the final LP problem

$$\max \{(c_0, x) \mid x \in M(r), -(c_{i_s}, x) \leq -\alpha_s, \quad s = k, \ldots, 1 \}$$

gives us the relation (6).

Clearly, an inverse chain of reasonings is valid too.
Proof of Lemma 1 is completed.

R e m a r k. For problem (3), the condition which is analogous to (6) will consist of the following inclusions which must be valid

$$b_{j_s} \in cone\{ -b_{j_{s+1}}, \ldots, -b_{j_l};\ h_1, \ldots, h_{n+m} \}, \tag{8}$$

where $s = 0,\ldots,l$ and $s_0 = 0$; $\{ h_i \}_1^n$ denotes the columns of matrix A and $\{-h_{n+j}\}_{j=1}^m$ denotes the unit co-ordinate vectors of R^m. To be sure that this is true it is sufficient to transform problem (3) into the form of problem (2) by means of elementary transformations.

Lemma 2. *Let problem* (2) *be solvable for some* p *and* $r \geq 0$. *Then there exists a nonempty range of values* $R \geq 0$ *such that problem* (4) *is solvable. Moreover, the choice of corresponding* R *does not depend on the* realization of the right-hand-side vector b_0+Br, in particular, it does not depend on r (see [2, Lemma 3, p. 767]).

The extended version of this statement can be formulated as follows.

Lemma 3. *Let* $r \geq 0$ *and* $R \geq 0$ *be such that the constrained systems in problems* (2) *and* (3) *are consistent. Then there exists a nonempty range of values* $\bar{r} \geq 0$ *and* $\bar{R} \geq 0$ *such that for the scalarlized problems*

$$\max\ \{(c_0, x) + (C\bar{R}, x)\ \mid\ Ax \leq b_0 + Br,\quad x \geq 0\}, \tag{9}$$

$$\min\ \{(b_0, u) + (B\bar{r}, u)\ \mid\ A^T u \geq c_0 + CR,\quad u \geq 0\ \} \tag{10}$$

the following relations hold:

$$Arg\ L_{lex} = Arg(9) \neq \varnothing, \qquad Arg\ L_{lex}^{\otimes} = Arg(10) \neq \varnothing.$$

As it has already been mentioned, duality theorem in *lex*-optimization consists in realization of Scheme 1 under some assumptions, which can be of different nature.

Let us introduce the following condition:

$$\exists\, r_0 > 0:\quad Br_0 > 0. \tag{11}$$

Theorem 1. *Let conditions* (6), (8) *and* (11) *hold. Then there exist orderings* p *and* q *of systems of functions* $\{c_i^T x\}_0^k$ *and* $\{b_j^T u\}_0^l$ *and parameter vectors* $R>0$ *and* $r>0$ ($R \in R^k$, $r \in R^l$) *such that Scheme 1 is realized.*

P r o o f. First of all, we note that condition (11) implies consistency of system $Ax \leq b_0 + Br$ for some $r = t\bar{r}$, $t > 0$. This follows from the fact that the inequality $b_0 + t_0 \bar{r} \geq A\bar{x}$ holds for each $\bar{x} \geq 0$ and for a sufficiently large $t_0 > 0$.

Now, if $M(r) \neq \varnothing$ and condition (6) holds, according to Lemma 1, there exists an ordering of indices $\{j\}_0^k$ such that problem L_{lex} is solvable. Assume that

$p=(k,...,0)$. Then, by Lemma 2, there exists $\overline{R}>0$ such that $Arg\ L_{lex} = Arg\ L_{scal}$ and this property (i.e. the upper line of Scheme 1) does not depend on r. Moreover, if this property holds for $\overline{R}$, then it holds for any $t\overline{R}$, $t>0$ (see the proof of Lemma 3 in [2]).

Solvability of problem L_{scal} implies solvability of the dual (in a classical sense) problem

$$L^{*}_{scal}\colon \min \{(b_0,u)+(Br,u) \mid A^T u \ge c_0 + C\overline{R},\quad u \ge 0\}.$$

Then the situation considered above occurs. Since condition (8) holds, there exists an ordering q of indices $\{\ i\ \}_0^l$ (assume that $q = (\ l,......,0)$) and $\bar{r}>0$ such that the problem

$$\min{}_q \left\{ \begin{bmatrix} B^T u \\ b_0^T u \end{bmatrix} \middle| \ A^T u \ge c_0 + C\overline{R},\quad u \ge 0 \right\} \tag{12}$$

is solvable and

$$Arg(12) = Arg\ L_{scal}\big|_{\ r=\bar{r}}$$

Thus a system of transitions

$$r \to p \to \overline{R}$$
$$\overline{R} \to q \to \bar{r}$$

has been contructed. Since, as it has been mentioned earlier, the choice of $\overline{R}$ does not depend on r, and the choice of $\bar{r}$ does not depend on $\overline{R}$, we can equate r and $\bar{r}$ (it may be necessary to substitute $\bar{r}$ by $t\bar{r}$ in order to provide that $M(\bar{r}) \ne \varnothing$).

With $\bar{r}>0$ and $\overline{R}>0$ thus constructed, the proof of Theorem 1 is completed.

R e m a r k. Conditions of Theorem 1 may be chosen in such a way that realization of Scheme 1 will become possible for any orderings p and q. E.g. conditions providing the consistency of the inequality systems

$$A_j x \le b_j,\quad j=0,1,\ldots,l,\qquad B_i^T u \ge c_i,\quad i=0,1,\ldots,k \text{ are valid.}$$

3. APPLICATIONS TO IMPROPER LINEAR PROGRAMMING PROBLEMS

Now we consider lexicographic interpretation of duality in improper linear programming problems [3].

A pair of dual LP problems

$$L\colon \max\,(c,x),\quad Ax \le b,\quad x \ge 0 \qquad\qquad L^{*}\colon \min\,(b,u),\quad A^T u \ge c,\quad u \ge 0$$

which are not supposed to be solvable, can be put into correspodence with the pair of problems:

$$P\colon \max\,\{(c,x) - \sum_{j=1}^{m_0} R_j \| (A_j x - b^j)^+ \|_j \mid A_0 x \le b^0,\quad x \ge 0,\ \|x^i\|_i \le r_i, i = 1,\dots,n_0\},$$

$$P^{\#}\colon \min\,\{(b,u) + \sum_{i=1}^{n_0} r_i \|(c^i - B_i^T u)^+\|_i^* \mid B_0^T u \ge c^0,\quad u \ge 0, \|u^j\|_j^* \le R_j,\quad j = 1,\dots,m_0\}.$$

Problems P and $P^{\#}$ are related by the duality theorem in the classical formulation [3].

Here $\{b^j\}$ and $\{u^j\}$, $\{c^i\}$ and $\{x^i\}$ are partitions of vectors b and u, c and x that correspond to the partitions of matrix A into horizontal and vertical submatrices $\{A_j\}$ and $\{B_i\}$: $A = \begin{bmatrix} A_0 \\ \cdots \\ A_{m_0} \end{bmatrix} = [B_0 \dots B_{n_0}]$; $\{\|\cdot\|_j\}$ and $\{\|\cdot\|_i\}$ are arbitrary norms in the corresponding spaces, $\|\cdot\|_j^*$, $\|\cdot\|_i^*$ are norms dual to them; $\{R_j\}$ and $\{r_i\}$ are systems of nonnegative parameters; $(\cdot)^+$ denotes a vector obtained from vector $(\cdot)$ by substituting zeros for its negative components.

Functions $f_j(x) = \|(A_j x - b^j)^+\|_j$ and $g_i(u) = \|(c^i - B_i^T u)^+\|_i^*$ represent the residuals of subsystems $A_j x \le b^j$ and $B_i^T u \ge c^i$, $j = 1,\dots,m_0$, $i = 1,\dots,n_0$ and may be ordered according to the permutations $p = (m_0,\dots,1)$ and $q = (n_0,\dots,1)$, which can be interpreted as an ordering (according to their "importance") of subsystems of restrictions $\{A_j x \le b^j\}_0^{m_0}$ and $\{B_i^T u \ge c^i\}_0^{n_0}$. Subsystems $A_0 x \le b^0$, $x \ge 0$ and $B_0^T u \ge c^0$, $u \ge 0$ are interpreted as directive restrictions and in view of this fact they are supposed to be consistent. Based on the given interpretations we can formulate the following pair of lexicographic optimization problems:

$$P_{lex}\colon \max_p \{F(x) \mid x \in M(r)\},$$
$$P_{lex}^{\#}\colon \min_q \{G(u) \mid u \in M^{\#}(R)\},$$

where $F(x) = [-f_{m_0}(x),\dots,-f_1(x),(c,x)]^T$, $G(u) = [g_{n_0}(u),\dots,g_1(u),(b,u)]^T$; $M(r)$ and $M^{\#}(R)$ are feasible sets of problems P and $P^{\#}$. In lexicographic settings P_p and $P_q^{\#}$, functions (c, x) and (b, u) occupy (according to their "importance") the last places in the corresponding lists.

Theorem 2. *Let the norms* $\{\|\cdot\|_j\}$, $\{\|\cdot\|_i\}$ *be monotone (together with their duals) and piecewise linear. Then there exists a nonempty range of* $r \geq 0$ *and* $R \geq 0$, *which can be constructively determined, such that*

$$Arg\ P = Arg\ P_p \neq \varnothing, \qquad Arg\ P^{\#} = Arg\ P_q^{\#} \neq \varnothing, \qquad opt\ P = opt\ P^{\#}.$$

The content of Theorem 2 is pictured in Scheme 2:

$$\begin{array}{ccccccc} L & \to & P & \overset{\text{Arg}}{\to} & P_{lex} \\ (*)\downarrow\uparrow & & (\#)\downarrow\uparrow & & (lex)\downarrow\uparrow \\ L^* & \to & P^{\#} & \overset{\text{Arg}}{\to} & P_{lex}^{\#} \end{array}$$

Scheme 2

The p r o o f of Theorem 2, which is too cumbersome but essentially the same as that of Theorem 1, is omitted.

In the sequel we shall focus on a particular case of Theorem 2, which is of independent interest. Let $A_0 x \leq b^0$, $x \geq 0$ and $B_0^T u \geq c^0$, $u \geq 0$ be maximal consistent subsystems (MCS) of constraint systems of problems P and $P^{\#}$, and , for the sake of definiteness, let $(a_j, x) \leq b_j$, $j = 1, \ldots, k$, $(h_i, u) \geq c_i$, $i = 1, \ldots, l$ be the remainders of these systems. Note that the values of $(a_j, x) - b_j$ and $c_i - (h_i, u)$ are positive on the solution sets of the chosen MCS's. If we put $A_j = a_j$ (a row vector), $B_i = c_i$ (a column vector), P and $P^{\#}$ assume the form:

$$P_L: \max\ \{(c,x) - \sum_{j=1}^{k} R_j [(a_j, x) - b_j] \mid A_0 x \leq b^0, x \geq 0, x_i \leq r_i, i = 1, \ldots, l\};$$

$$P_L^{\#}: \min\ \{(b,u) + \sum_{i=1}^{l} r_i [c_i - (h_i, u)] \mid B_0^T u \geq c,\ u \geq 0,\ u_j \leq R_j,\ j = 1, \ldots, k\}.$$

We define matrices $\overline{A}$ and $\overline{B}$ by the equalities $A = \begin{bmatrix} \overline{A} \\ A_0 \end{bmatrix} = [\overline{B}\ \ B_0]$; let $p = (k, k-1, \ldots, 0)$ and $q = (l, l-1, \ldots, 0)$ be orderings which are used to relate problems P_L and $P_L^{\#}$ to the following *lex*-problems:

$$L_{lex}: \max{}_p \left\{ \begin{bmatrix} -\overline{A}x \\ (c,x) \end{bmatrix} \,\middle|\, x \in M(r) \right\},$$

$$L_{lex}^{\#}: \min{}_q \left\{ \begin{bmatrix} -\overline{B}^T u \\ (b,u) \end{bmatrix} \,\middle|\, u \in M^{\#}(R) \right\},$$

where $M(r) = \{x \geq 0 | A_0 x \leq b^0, x_i \leq r_i, i = 1, \ldots, l\}$,
$M^{\#}(R) = \{u \geq 0 | B_0^T u \geq c^0, u_j \leq R_j, j = 1, \ldots, k\}$.

Theorem 3. *There exists a nonempty range of parameters $r \geq 0$ and $R \geq 0$, which can be constructively determined, such that an analog of Scheme 2 is realized (where, in this situation, P_L and $P_L^{\#}$ are used in place of P and $P^{\#}$).*

R e m a r k. The idea of "schematizing" duality outside linear programming goes back to the author's and Astafiev's 1976 book [4]. Later their schemes have been used and extended by several authors (e.g. [5-6]).

4. CONCLUSIONS

This paper throws light upon some questions of symmetric lexicographical duality for optimization problems in linear setting. Models of such duality are not only of academic interest: e.g. the application to improper optimization problems gives an opportunity to analyse the contradictory problems, where both the criterial function and the functions defining constraints are arranged according to their significance for someone. The author believes that the dual scheme proposed above certainly admits a transposition onto some classes of nonlinear optimization problems.

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