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SOME PROPERTIES OF TRANSPORTATION MANIFOLD*

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Abstract. In this paper it is shown that the set of equality constraints of transportation problem can be represented as an intersection of two orthogonal affine sets. Projection point onto transportation manifold then could be achieved by consecutive projections onto these sets using simple formulas. This result can be used in solving a linear transportation problem (in variant of Karmarkar projective algorithm), as well as in a nonlinear case (by gradient projection method).

Key words and phrases: transportation problem, point projection

1. INTRODUCTION

A well known classical transportation problem is as follows: minimize transportation costs from m sources to n destinations, where holding capacities of both sources (a_i) and destinations (b_j) are known. Mathematically, the problem is to find

$$(\min)\ f(x) \tag{1}$$

s.t.

$$A_1 x = a \tag{2}$$

$$A_2 x = b \tag{3}$$

$$x \geq 0 \tag{4}$$

where $x \in R^{mn}$, $a \in R^m$, $b \in R^n$,

$$A_1 = \begin{bmatrix} 1\dots1 & & & \\ & 1\dots1 & 0 & \\ 0 & & \dots & \\ & & & 1\dots1 \end{bmatrix}$$

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$$A_2 = \begin{bmatrix} 1 & \dots & 1 & \dots & 1 \\ 1 & \dots & 1 & \dots & 1 \\ 0 & \dots & \dots & 0 & \dots \\ 1 & \dots & 1 & \dots & 1 \end{bmatrix}$$

Let us denote by

$$\begin{aligned} V_1 &= \{x \in R^{mn} \mid A_1 x = a\} \\ V_2 &= \{x \in R^{mn} \mid A_2 x = b\} \\ V &= V_1 \cap V_2 \\ X &= \{x \in R^{mn} \mid x_{ij} \geq 0\}. \end{aligned}$$

Then, the feasible set is

$$D = V \cap X.$$

In this paper it is shown that projection point onto V could be reached in 2 steps after consecutive projections onto V_1 and V_2, using simple formulas (with complexities $O(m)$ and $O(n)$).

2. PROJECTION OF A POINT ONTO A TRANSPORTATION MANIFOLD

In this section we shall consider the problem

$$(P_V) \qquad \inf \tfrac{1}{2}\|x - y\|^2, \qquad x \in V,$$

i.e.

$$(P_{V_j}) \qquad \inf \tfrac{1}{2}\|x - y\|^2, \qquad x \in V_j,\ j = 1, 2.$$

where y is any point in R^{mn}. Optimal solution of (P_V) we shall denote by $p_V(y)$. It is well known that the number of linear independent equations is $m + n - 1$ and that the following holds

$$V \neq \emptyset \implies \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j. \tag{5}$$

At first, we shall find $p_{V_1}(y)$ and $p_{V_2}(y)$ explicitly. The solution of (P_{V_1}) is explicitly written as (Lawsdon, Hanson [74])

$$p_{V_1}(y) = y - A_1^T(A_1 A_1^T)^{-1}(A_1 y - a) \tag{6}$$

and if we introduce notation

$$\mu^* = (A_1 A_1^T)^{-1}(A_1 y - a)$$

we obtain $y - P_{V_1}(y) = A_1^T \mu^*$; μ^* is a vector of Lagrangian multipliers.

We denote by

$$\begin{aligned} A_1^+ &= A_1^T(A_1 A_1^T)^{-1} \\ A_2^+ &= A_2^T(A_2 A_2^T)^{-1} \end{aligned} \tag{7}$$

the Moore-Penrose generalized inverse A and by

$$\begin{aligned} Q_1 &= I - A_1^+ A_1 = I - P_1 \\ Q_2 &= I - A_2^+ A_2 = I - P_2 \end{aligned} \tag{8}$$

the projector onto the manifold defined by (2) and (3).

PROPOSITION 1.

$$\begin{aligned} A_1^+ &= (1/n)A_1^{\mathrm{T}} \\ A_2^+ &= (1/m)A_2^{\mathrm{T}}. \end{aligned}$$

Proof. It is easy to check that the following holds:

$$\begin{aligned} A_1 A_1^{\mathrm{T}} &= nI, \\ A_2 A_2^{\mathrm{T}} &= mI \end{aligned}$$

so we get from (7), $A_1^+ = (1/n)A_1^{\mathrm{T}}$ and $A_2^+ = (1/m)A_2^{\mathrm{T}}$.

Using Proposition 1, we get

$$\begin{aligned} P_{V_1}(y) &= y - (1/n)A_1^{\mathrm{T}}(A_1 y - a) \\ P_{V_2}(y) &= y - (1/m)A_2^{\mathrm{T}}(A_2 y - b) \end{aligned}$$

i.e.

$$[p_{V_1}(y)]_{ij} = y_{ij} - (1/n)(\sum y_{iq} - a_i) = y_{ij} - (u_i - a_i)/n \tag{9}$$

$$[p_{V_2}(y)]_{ij} = y_{ij} - (1/m)(\sum y_{pj} - b_j) = y_{ij} - (v_j - b_j)/m, \tag{10}$$

where $u_i = \sum_q y_{iq}$, $v_j = \sum_p y_{pj}$.

PROPOSITION 2. *If we denote by $1_{p\times q}$ the matrix with p rows and q columns and with all elements equal to 1, then the following holds:*

1° $A_2 A_1^{\mathrm{T}} = 1_{n\times m}$

2° $A_1 A_2^{\mathrm{T}} = 1_{m\times n}$

3° $(A_1 A_2^{\mathrm{T}})A_2 = 1_{m\times(mn)}$

4° $(A_2 A_1^{\mathrm{T}})A_1 = 1_{n\times(mn)}$

5° $A_2^{\mathrm{T}}(A_2 A_1^{\mathrm{T}}) = 1_{(mn)\times m}$

6° $A_1^{\mathrm{T}}(A_1 A_2^{\mathrm{T}}) = 1_{(mn)\times n}$

7° $A_2^{\mathrm{T}}(A_2 A_1^{\mathrm{T}})A_1 = A_1^{\mathrm{T}}(A_1 A_2^{\mathrm{T}})A_2 = 1_{(mn)\times(mn)}$.

Proof. Equalities 1° and 2° are obvious. Other equalities can be obtained from 1° and 2°.

PROPOSITION 3. $Q_2 Q_1 = Q_1 Q_2$.

Proof. After dividing 7° by mn, we get

$$(1/m)A_2^{\mathrm{T}}(A_2(1/n)A_1^{\mathrm{T}})A_1 = ((1/n)A_1^{\mathrm{T}})A_1((1/m)A_2^{\mathrm{T}})A_2.$$

From Proposition 1 and (8), we obtain

$$(A_2^+ A_2)(A_1^+ A_1) = (A_1^+ A_1)(A_2^+ A_2),$$

i.e.,

$$P_2 P_1 = P_1 P_2.$$

Now, we have ($Q_1 = I - P_1$ and $Q_2 = I - P_2$)

$$\begin{aligned} Q_2 Q_1 &= (I - P_2)(I - P_1) = I - P_1 - P_2 + P_2 P_1 = I - P_2 - P_1 + P_1 P_2 \\ &= (I - P_1)(I - P_2) - Q_1 Q_2. \end{aligned}$$

PROPOSITION 4. *If $V \neq 0$, then the following holds*

$$A_2^{\mathrm{T}} A_2 A_1^{\mathrm{T}} a = A_1^{\mathrm{T}} A_1 A_2^{\mathrm{T}} b. \tag{11}$$

Proof. Substituting 5° and 6° (from Proposition 2) in (11), we get

$$(1_{(mn)m})a = (1_{(mn)n})b,$$

which is equivalent with condition (5).

PROPOSITION 5. *For every $y \in R^{mn}$ the following holds*

$$p_{V_1}(p_{V_2}(y)) = p_{V_2}(p_{V_1}(y)).$$

Proof. From (6), (7) and Proposition 1, we obtain

$$\begin{aligned} p_{V_1}(p_{V_2}(y)) &= Q_1(p_{V_2}(y)) + A_1^+ a = Q_1(Q_2 y + A_2^+ b) A_1^+ a \qquad (12) \\ &= Q_1 Q_2 y + (I - P_1) A_2^+ b + A_1^+ a = Q_1 Q_2 y + A_2^+ b + A_1^+ a - (1/mn) A_1^{\mathrm{T}} A_1 A_2^{\mathrm{T}} b. \end{aligned}$$

In the same way we have

$$p_{V_2}(p_{V_1}(y)) = Q_2 Q_1 y + A_2^+ b + A_1^+ a - (1/mn) A_2^{\mathrm{T}} A_2 A_1^{\mathrm{T}} a. \tag{13}$$

Because of Propositions 3 and 4, right-hand sides of (12) and (13) are equal, so Proposition 5 is proved.

Now, the projection point onto a manifold V could be achieved by consecutive projections onto V_1 and V_2 using simple formulas (9) and (10).

ALGORITHM

Step 1. There are given $y \in R^{mn}$, $a \in R^m$, $b \in R^n$

Step 2. Find $p_{V_1}(y)$ as $u_i = \sum_q y_{iq}$,

$$z_{ij} = y_{ij} - (1/n)(u_i - a_i), \qquad i = 1, \dots, m;\ j = 1, \dots, n.$$

Step 3. Find $p_V(y)$ as $v_j = \sum z_{pj}$

$$x_{ij} = z_{ij} - (1/m)(v_j - b_j), \qquad i = 1, \dots, n;\ j = 1, \dots, n.$$

3. CONCLUSION

Projection of a point onto a manifold needs $O(n^3)$ computations in general (in our case $O((m+n-1)^3)$). In this paper it is shown that this number is $O(m+n)$, for transportation manifold. For that purpose, the manifold is divided into two affine sets that have a very simple structure.

This result could be applied in various ways. For instance, in solving a linear transportation problem by "modified Karmarkar strategy" (Cavalier, Soyster [1985]), most time consuming activity is in finding projection of Dc onto the null space of AD ($D = \mathrm{diag}(x_{11}, \ldots, x_{mn})$, x — feasible point). Unfortunatuly, halfspaces $V_1' = \{(A_1 D)_x = 0\}$ and $V_2' = \{(A_2 D)_x = 0\}$ are not orthogonal, but an idea of consecutive projections onto these sets in order to find projections is still valid.

REFERENCES

Cavalier T. M. and Soyster A. L. [1985] *Some computational experience and a modification of the Karmarkar algorithm*, ISME working paper 85/105, The Pennsylvania State University.

Karmarkar N. [1984] *A new polynomial time algorithm for linear programming*, Combinatorics **4**, 373–395.

Lawson L. and Hanson J. [1974] *Solving Least Squares problems*, Prentice Hall, New Jersey.