Yugoslav Journal of Operations Research
1 (1991), Number 1, 45–50

# NP-HARD PROBLEMS AND TEST PROBLEMS FOR GLOBAL CONCAVE MINIMIZATION METHODS

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**Abstract.** This paper presents a new method of obtaining hard test problems for global concave minimization problems. The method starts from a three-satisfiability type of problem and it transforms it into a minimization problem over the unit cube with quadratic or cubic concave objective function. An alternative proof of the $NP$-hardness of such problems is also given.

*Key words and phrases*: $NP$-hard, concave, global optimization, complexity

## 1. INTRODUCTION

In this paper we present a new way of obtaining hard test problems for global concave minimization problems. The same issue was addressed by several authors; see for example [**2**, **4**, **6**, **7**]. The approach taken here is diferent: we show how to transform 3-SAT problems (known to be $NP$-complete) to concave minimization problems whith quadratic or cubic objective functions. This paper was inspired by a recent paper by P. Gritizmann and V. Klee [**5**] and it also contains an alternative proof of the main result of [**5**], i.e. the proof that the problems CUBEMAX and POSDEF-0-1-MAX are $NP$-hard. The transformation used is very simple and the time required is bounded above by a constant times the size of the input. Details are given in Section 2.

We now decribe the classes of problems and the notation to be used in this paper. First of all, we will consider *minimization* rather than *maximization* of functions; however the notation CUBEMAX and POSDEF-0-1-MAX will be retained (as in [**5**]) to avoid confusion and introduction of too many names and symbols. As usual, $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{R}$ will denote the integers, the positive integers and the reals respectively.

The cubic concave minimization problem over $[0,1]^n$ may be desribed in decision form as follows:

CUBICONC.

Instance: $n \in \mathbf{N}$; a cubic polynomial $P \in \mathbf{Z}[x_1, \dots, x_n]$ strictly concave on $[0,1]^n$; an integer $\lambda$.

Question: Does there exist a vector $x \in [0,1]^n$ such that $P(x) \leq \lambda$?

Similarly, by the quadratic concave minimization problem we mean:

QUADCONC.

Instance: $n \in \mathbf{N}$; a quadratic polynomial $P \in \mathbf{Z}[x_1, \dots, x_n]$ strictly concave on $\mathbf{R}^n$; an integer $\lambda$.

Question: Does there exist a vector $x \in [0,1]^n$ such that $P(x) \leq \lambda$?

Notice that we require the cubic polynomials to be concave only on the cube $[0,1]^n$. The reason is simple: If a polynomial $P$ is concave on $\mathbf{R}^n$ and if $\deg(P) \leq 3$, then $\deg(P) \leq 2$. On the other hand, if a quadratic function is concave on a (convex) set with nonempty interior then it is concave everywhere.

If in addition we require our quadratic polynomial to be homogeneous, we obtain the problem CUBEMAX (compare with [5]):

CUBEMAX.

Instance: $n \in \mathbf{N}$; a negative definite symmetric integer $n \times n$ matrix $B$; an integer $\lambda$.

Question: Does there exist a vector $x \in [0,1]^n$ such that $x^{\mathrm{T}}Bx \leq \lambda$?

Since $B$ is negative definite the function $x^{\mathrm{T}}Bx$ is concave and it follows that CUBEMAX is equal (see [5]) to the following 0-1 optimization problem:

POSDEF-0-1-MAX.

Instance: $n \in \mathbf{N}$; a negative definite symmetric integer $n \times n$ matrix $B$; an integer $\lambda$.

Question: Does there exist a vector $x \in \{0,1\}^n$ such that $x^{\mathrm{T}}Bx \leq \lambda$?

In the following Section we describe how a three-satisfiability problem can be converted into a global optimization problem with a quadratic or cubic objective function. Although the transformation to CUBEMAX yields the strongest result (the $NP$-hardness of CUBEMAX), the transformations to QUADCONC and CUBICONC require fewer variables and are thus interesting in their own right as sources of hard global optimization problems with not too many variables.

The problem 3-SAT is mentioned in virtually every book that deals with $NP$-completeness and related topics. The following formulation will be used in what follows (cf. [3]):

3-SAT.

Instance: $m \in \mathbf{N}$; $n \in \mathbf{N}$; a set $S = \{D_1, \dots, D_m\}$ of expressions of the form $\hat{p} \vee \hat{q} \vee \hat{r}$ where $p, q, r \in \{p_1, \dots, p_n\}$ and $\hat{p}$ is either $p$ or $\bar{p}$.

Question: Does there exist a truth value assignment for the variables $p_1, \dots, p_n$ which makes all expressions in $S$ take on the value *'true'*?

The notation is somewhat simplified if *'true'* and *'false'* are replaced by '1' and '0' respectively. The logical connectives $\vee$ and $\bar{}$ then have the following "algebraic" properties: $p \vee q = p + q - pq$, $\bar{p} = 1 - p$.

Without loss of generality we will assume that each expression $D_j \in S$ contains three distinct variables. There are then eight possible truth value assignments for the variables in $D_j$ and $D_j$ will take on the value 1 for seven of them. The values in the remaining (eighth) assignment will be called *the bad values* for $D_j$. For example, $p_2 = 1$, $p_4 = 0$ and $p_7 = 1$ are the bad values for $D_9 = \bar{p}_2 \vee p_4 \vee \bar{p}_7$. These values will play an important rôle in Section 2.

## 2. THE TRANSFORMATIONS

Our first goal in this Section is to show how 3-SAT can be recast as a cubic concave minimization problem over $[0,1]^n$. The obtained problem will have $n$ variables, i.e. the same number as 3-SAT. We will use eight auxiliary functions which will serve as building blocks in the construction of the objective function of the resulting CUBICONC problem. To begin with, let us consider the polynomial $Q$ defined by

$$Q(x, y, z) = -xyz - 2(x^2 + y^2 + z^2) + xy + xz + yz + x + y + z + 1.$$

It is easy to check that $Q$ is strictly concave on the unit cube. Its minimum value on $[0,1]^3$ is zero and is attained at all vertices of the cube except at $(0,0,0)$. Consider now the problem 3-SAT. Using the same notation as in Section 1, for each $D_j \in S$ we construct a cubic polynomial $Q_j$ of three variables according to the following rules:

**a.** If $D_j$ contains the variables $p_i$, $p_k$ and $p_l$ the polynomial $Q_j$ will contain the variables $x_i$, $x_k$ and $x_l$.

**b.** Use the following table to choose the polynomial $Q_j$:

| *Type of $D_j$* | *Use of the polynomial:* |
|---|---|
| $p \vee q \vee r$ | $Q(x, y, z)$ |
| $p \vee q \vee \bar{r}$ | $Q(x, y, 1 - z)$ |
| $p \vee \bar{q} \vee r$ | $Q(x, 1 - y, z)$ |
| $p \vee \bar{q} \vee \bar{r}$ | $Q(x, 1 - y, 1 - z)$ |
| $\bar{p} \vee q \vee r$ | $Q(1 - x, y, z)$ |
| $\bar{p} \vee q \vee \bar{r}$ | $Q(1 - x, y, 1 - z)$ |
| $\bar{p} \vee \bar{q} \vee r$ | $Q(1 - x, 1 - y, z)$ |
| $\bar{p} \vee \bar{q} \vee \bar{r}$ | $Q(1 - x, 1 - y, 1 - z)$ |

In other words rule **a** changes the name of the variable (from $p$ to $x$) but leaves the subscripts unchanged. Rule **b** prescribes the coice of the polynomial depending on the position of negation signs. For example, starting from the expression $\bar{p}_2 \vee p_4 \vee \bar{p}_7$ we would obtain the polynomial $Q(1 - x_2, x_4, 1 - x_7)$.

The last step in the construction is to form the sum of the obtained polynomials; the resulting problem is:

$$\text{(2.1)} \qquad \begin{aligned} &\text{Minimize } P(x) = \sum_{j=1}^{m} Q_j \\ &\text{subject to } x \in [0,1]^n. \end{aligned}$$

The objective function is clearly a cubic polynomial strictly concave and nonnegative on $[0,1]^n$. A closer look at the table in rule **b** reveals that each polynomial in the right hand side column of the table attains the value 0 at seven out of eight vertices of its (three-dimensional) cube; the only exception is the vertex whose coordinates are the bad values for the corresponding expression in the left hand side column. The objective function in (2.1) will thus attain the value zero if and only if there exists a vertex of $[0,1]^n$ such that its coordinates do not give bad values to any expression in $S$. In other words, the question: Does there exist a vector $x \in [0,1]^n$ such that $P(x) \leq 0$? and the question in 3-SAT have the same answer. Moreover, any global minimizer of (2.1) will automatically yield a truth value assignment for which all expressions in $S$ will take on the value 1, provided of course that the minimum objective function value of (2.1) is zero. We have thus proved:

THEOREM 2.1. *The problem* CUBICONC *is NP-hard.* ■

We will use the same idea to show how 3-SAT can be reformulated as a quadratic concave minimization problem. This time, however, we need an extra variable for each expression in $S$; the total number of variables in the obtained problem will be $n+m$. Consider first the polynomial $R$ defined by:

$$R(x,y,z,u) = -2(x^2+y^2+z^2+u^2) + xy + xz + yz + (x+y+z)u + 3.$$

Again, it is easy to check that $R$ is a concave function (on $\mathbf{R}^4$). The minimum value of $R$ on $[0,1]^4$ is zero and for each $(x,y,z) \in \{0,1\}^3$, $(x,y,z) \neq (0,0,0)$ it is possible to chose $u \in \{0,1\}$ such that $R(x,y,z,u) = 0$. It is clear that such a choice of $u$ is impossible if $(x,y,z) = (0,0,0)$ because $R(0,0,0,u)$ is positive for $u \in [0,1]$.

Consider now the problem 3-SAT. For each $D_j \in S$ we construct a quadratic polynomial $R_j$ of four variables according to the following two rules:

**a′**. If $D_j$ contains the variables $p_i$, $p_k$ and $p_l$ the polynomial $R_j$ will contain the variables $x_i$, $x_k$, $x_l$ and $u_j$.

**b′**. Treating $u_j$ as a constant, use the table from the rule **b** above to choose the polynomial $R_j$.

For example, starting from the expression $D_9 = \bar{p}_2 \vee p_4 \vee \bar{p}_7$ we would obtain the polynomial $R(1-x_2, x_4, 1-x_7, u_9)$.

It is a straightforward, albeit somewhat tedious task to verify that if the $x$-variables of $R_j$ are from $\{0,1\}$ and are not the bad values for the corresponding expression in the left hand side column of the table then it is possible to choose $u_j \in \{0,1\}$ so that the value of $R_j$ is zero. Such a choice is impossible if the values are bad. The last step in the construction is to form the sum of the obtained polynomials; the resulting problem is:

$$\text{(2.2)} \qquad \begin{aligned} &\text{Minimize } \; P(x_1, \dots, x_n, u_1, \dots, u_m) = \sum_{j=1}^{m} R_j \\ &\text{subject to } \; x \in [0,1]^n, \; u \in [0,1]^m. \end{aligned}$$

The remaining part of the reasoning is the same as above. We have thus proved

THEOREM 2.2. *The problem* QUADCONC *is* $NP$*-hard.* ■

Our proof that CUBEMAX is also $NP$-hard is very similar to the proofs above. We will therefore give only a sketch of the proof, giving the details only where the proof is different. We need a convenient homogeneous quadratic polynomial as a starting point; a possible choice is:

$$T(x,y,z,u,t) = -2(x^2+y^2+z^2+u^2) + xy + xz + yz + (x+y+z)u - 16t^2.$$

The minimum value of $T$ on $[0,1]^5$ is $-19$ and can be attained only at the vertices that satisfy the following conditions: (i) the value of the $t$-coordinate is 1; (ii) the $x-$, $y-$ and $z-$ coordinates are not all equal to zero. In other words, if $(x,y,z) \neq (0,0,0)$ then a suitable choice of $u \in \{0,1\}$ and $t = 1$ will make $T$ attain its minimum value $-19$; this is not possible if $(x,y,z) = (0,0,0)$.

The first rule is now:

**a″**. If $D_j$ contains the variables $p_i$, $p_k$ and $p_l$, the polynomial $T_j$ will contain the variables $x_i$, $x_k$, $x_l$, $u_j$ and $t$.

Notice that we are using the same variable $t$ for all expressions in $S$. This is possible because the same value of $t$ ($t = 1$) is intended in all cases. The only real role of $t$ is to make the auxiliary polynomials homogeneous.

The second rule is much like **b′** above; the only significant difference is that $x$, $y$ and $z$ are to be replaced, where indicated, by $t-x$, $t-y$ and $t-z$ rather than by $1-x$, $1-y$ and $1-z$. This change makes sure that the auxiliary functions are all homogeneous. For example, starting from the expression $D_9 = \bar{p}_2 \vee p_4 \vee \bar{p}_7$ we would obtain the polynomial $R(t-x_2, x_4, t-x_7, u_9, t)$. The rest of the proof is the same as before (the optimal objective function value is now $-19m$, provided that $S$ is satisfiable). The number of variables in the CUBEMAX problem will be equal to $m+n+1$. Finally, the obtained objective function should be multiplied by two, in order to make the symmetric matrix $B$ of the quadratic form an integer matrix. The conclusion is:

THEOREM 2.3. *The problem* CUBEMAX *is* $NP$*-hard.* ■

The transformations described above replace one expression at a time by a polynomial (of 3, 4 or 5 variables). The time required for one such transformation is clearly bounded above by a constant times the size of the expression and it immediately follows that the total time needed is bounded by a constant times the total size of the input. Since the transformations are so simple it is clear that they can be used to construct hard test problems for global concave minimization methods. We are currently planning to test our method [1] and similar methods on a number of randomly generated problems using the approach from this paper.

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