Yugoslav Journal of Operations Research
1 (1991), Number 1, 3–14

CHARACTERIZING OPTIMALITY IN NONCONVEX OPTIMIZATION[1]

Sanjo ZLOBEC

McGill University, Department of Mathematics and Statistics, Burnside Hall, 805 Sherbrooke Street West, Montreal, Quebec, Canada H3A 2K6

Abstract. A saddle-point condition, that is both necessary and sufficient for a feasible point to be globally optimal, is given for a large class of nonconvex programming problems. The condition assumes lower semicontinuity of the feasible set point-to-set mapping.

AMS Subject Classification: 90 C 25, 90 C 30, 90 C 31

Key words and phrases: nonconvex program, convex program, saddle point, point-to-set mapping, optimality conditions

1. INTRODUCTION

It is well known that meaningful complete characterizations of optimality exist only for convex programs and their convex-like generalizations, e.g., **[2, 6]**. In this paper we consider general (possibly nonconvex) programs

$$(P) \qquad \operatorname*{Min}_{s.t.} f^0(z)$$
$$f^i(z) \le 0, \qquad i \in \mathcal{P} = \{1, \dots, m\}$$

where all functions $f^0, f^i : R^N \to R$, $i \in \mathcal{P}$ are assumed continuous. We study those programs (P) with the property that, for some arbitrary splitting of the variable z into $z = (x, \theta)$, the functions $f^0(\cdot, \theta), f^i(\cdot, \theta) : R^n \to R$, $i \in \mathcal{P}$ are convex for every $\theta \in R^p$, where $N = n + p$. Many highly nonconvex programs share this property, for example the following two:

1.1. EXAMPLE.

$$\begin{aligned} \operatorname*{Max}_{s.t.} \ & \frac{z_1}{z_3} \\ & z_1 + z_2 \le 1 \\ & z_1 + z_2 z_3 \ge 1 \\ & z_1 \ge 0, \quad z_2 \ge 0, \quad z_3 \ge 1. \end{aligned} \tag{1.1}$$

---

[1] Research partly supported by NSERC.

The feasible set of this program has a peculiar property: Along $z_3 > 1$ it is a threedimensional convex body, at $z_3 = 1$ it is a line segment, and at $0 < z_3 < 1$ it is a point.

1.2. EXAMPLE.

$$\begin{aligned} &\operatorname*{Min}_{s.t.} \ (z_1 - 2)^2 + z_2^2 \\ &(1 - z_1)^3 - z_2 \geq 0 \\ &z_1 \geq 0, \quad z_2 \geq 0. \end{aligned} \tag{1.2}$$

This is a classical example introduced in [6] to demonstrate that the so-called constraint qualification may not be satisfied.

For the sake of convenience we denote the programs, that are "convexifiable-by-a-splitting" in the above sense, by

$$(P, \theta) \qquad \begin{aligned} &\operatorname*{Min}_{s.t.} \ f^0(x, \theta) \\ &f^i(x, \theta) \leq 0, \qquad i \in \mathcal{P} \end{aligned}$$

Note that, for a fixed θ, (P, θ) is a convex program. The paper provides a solution to the following problem: Given a feasible point $z^* = (x^*, \theta^*)$ of (P), where $x^* = \tilde{x}(\theta^*)$ is an optimal solution of the convex program (P, θ^*), find a condition that is both necessary *and* sufficient for z^* to be globally optimal for (P). The answer follows by deduction from some recent studies in parametric optimization and point-to-set mappings, e.g., [**10**, **11**, **12**].

2. A CHARACTERIZATION OF OPTIMALITY

We consider an arbitrary convexifiable-by-a-splitting program (P), written in the form (P, θ). For every θ we denote by

$$F(\theta) = \{x : f^i(x, \theta) \leq 0, \ i \in \mathcal{P}\}$$

the corresponding feasible set in the variable x and by

$$\mathcal{F} = \{\theta : F(\theta) \neq \emptyset\}$$

the set of all θ's for which the feasible set of (P) is nonempty. Clearly $F : R^p \to R^n$ is a point-to-set mapping.

Around a fixed candidate for optimality $z^* = (\tilde{x}(\theta^*), \theta^*)$, we also consider, at the θ level, another point-to-set mapping $F_*^= : R^p \to R^n$, defined by

$$F_*^=(\theta) = \{x : f^i(x, \theta) \leq 0, \ i \in \mathcal{P}^=(\theta^*)\}.$$

Here we recall, at any given θ, the notation

$$\mathcal{P}^=(\theta) = \{i \in \mathcal{P} : x \in F(\theta) \Rightarrow f^i(x, \theta) = 0\}.$$

In our characterization of optimality we require that the mapping F be lower semi-continuous. The latter notion is recalled for an arbitrary mapping (see, e.g., [**1**, **4**, **12**]):

2.1. DEFINITION. A point-to-set mapping $\Gamma : R^p \to R^n$ is lower semicontinuous at $\theta^* \in \mathcal{F}$, relative to $\mathcal{F} \subset R^p$, if for each open set $\mathcal{A}$, satisfying $\mathcal{A} \cap \Gamma(\theta^*) \neq \varnothing$, there exists a neighbourhood $N(\theta^*)$ of θ^* such that $\mathcal{A} \cap \Gamma(\theta) \neq \varnothing$ for each $\theta \in N(\theta^*) \cap \mathcal{F}$.

REMARK. Since the mapping F for (P, θ) is closed, lower semicontinuity of F implies its continuity (e.g., [**1**, **4**, **12**]).

We need more notation. The complement of the index set $P^=(\theta)$ is denoted by

$$\mathcal{P}^<(\theta) = \mathcal{P} \setminus \mathcal{P}^=(\theta).$$

The Lagrangian function to be used in describing optimality of $z^* = (\tilde{x}(\theta^*), \theta^*)$ is

$$\mathcal{L}_*^<(z, u) = f^0(z) + \sum_{i \in \mathcal{P}^<(\theta^*)} u_i f^i(z).$$

We will study its behaviour on the set

$$\{F_*^=(\theta), \mathcal{F}\}$$

in $R^n \times R^p$. Denote by $c = \operatorname{card} \mathcal{P}^<(\theta^*)$, the cardinality of the set $\mathcal{P}^<(\theta^*)$ and by $R_+^c = \{x \in R^c : x \geq 0\}$ the non-negative orthant in R^c. The main result of the paper follows.

2.2. THEOREM. *Consider a convexifiable-by-a-splitting program (P) and its feasible point $z^* = (\tilde{x}(\theta^*), \theta^*)$, where $\tilde{x}(\theta^*)$ is an optimal solution of the convex program (P, θ^*). Assume that the point-to-set mapping F is lower semicontinuous at θ^* relative to $\mathcal{F}$. Then z^* is a globally optimal solution of (P) if, and only if, there exists a non-negative vector function $\tilde{u} = \tilde{u}(\theta) \in R_+^c$ such that*

$$\mathcal{L}_*^<(z^*, u) \leq \mathcal{L}_*^<(z^*, \tilde{u}(\theta^*)) \leq \mathcal{L}_*^<(z, \tilde{u}(\theta)) \tag{2.1}$$

for every $z \in \{F_*^=(\theta), \mathcal{F}\}$ and every non-negative $u \in R_+^c$.

Proof. (Necessity:) Let z^* be a globally optimal solution of (P). Then

$$f^0(z^*) = f^0(\tilde{x}(\theta^*), \theta^*) \leq f^0(x, \theta)$$

for every $x \in F(\theta)$ and $\theta \in \mathcal{F}$. We follow the general ideas from the proof of [**12**, Theorem 4.2] but with several modifications. First, for the sake of simplicity, assume that $\mathcal{P}^<(\theta^*) =: \{1, \ldots, c\}$.

For every $\theta \in \mathcal{F}$, define the two sets in R^{c+1}:

$$K_1(\theta) = \{y : y \geq [f^0(x, \theta) \quad f^1(x, \theta) \quad \cdots \quad f^c(x, \theta)]^{\mathrm{T}} \text{ for at least one } x \in F_*^=(\theta)\}$$

and

$$K_2 = \{y : y < [f^0(z^*) \quad 0 \quad \ldots \quad 0]^{\mathrm{T}}\}.$$

Since $F_*^=(\theta)$ is convex, so is $K_1(\theta)$. Convexity of K_2 is obvious. We claim that the two sets do not have a common point, i.e.,

$$K_1(\theta) \cap K_2 = \varnothing$$

for every $\theta \in \mathcal{F}$. Otherwise we would have

$$f^0(x,\theta) < f^0(z^*)$$
$$f^i(x,\theta) < 0, \qquad i \in \mathcal{P}^<(\theta^*)$$

for some $\theta \in \mathcal{F}$ and $x \in F_*^=(\theta)$. Hence $x \in F(\theta)$ and the optimality of z^* is violated.

The two convex sets are now separated by a hyperplane: There exists a nonzero vector $\tilde{u} = \tilde{u}(\theta)$ and a scalar $\alpha = \alpha(\theta)$ such that

$$\tilde{u}^{\mathrm{T}} y^1 \geq \alpha \geq \tilde{u}^{\mathrm{T}} y^2 \tag{2.2}$$

for all $y^1 \in K_1$ and $y^2 \in \mathrm{cl}\, K_2$ (closure of K_2). Clearly, all components of $\tilde{u}$ are non-negative. (Otherwise $\tilde{u}^{\mathrm{T}} y^2 \to +\infty$ by an appropriate choice of $y^2 \in K_2$). After specification

$$y^1 = \begin{bmatrix} f^0(x,\theta) \\ f^1(x,\theta) \\ \cdots \\ f^c(x,\theta) \end{bmatrix}, \qquad y^2 = \begin{bmatrix} f^0(z^*) \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

for some $x \in F_*^=(\theta)$, the inequalities (2.2) yield

$$\tilde{u}_0 f^0(z^*) \leq \tilde{u}_0 f^0(z) + \sum_{i \in \mathcal{P}^<(\theta^*)} \tilde{u}_i f^i(z). \tag{2.3}$$

The next crucial claim is that $\tilde{u}_0$ is positive for every above fixed θ. If not, then $\tilde{u}_0 = 0$ and (2.3) yields

$$\sum_{i \in \mathcal{P}^<(\theta^*)} \tilde{u}_i f^i(z) \geq 0. \tag{2.4}$$

Recall that $z = (x,\theta)$ with $x \in F_*^=(\theta)$ and take $\theta \in \mathcal{F}$ close to θ^* and

$$\hat{x} \in F(\theta) \subset F_*^=(\theta)$$

such that

$$f^i(\hat{x},\theta) < 0, \qquad i \in \mathcal{P}^<(\theta).$$

Then, in particular,

$$f^i(\hat{x},\theta) < 0, \qquad i \in \mathcal{P}^<(\theta^*). \tag{2.5}$$

The last step follows from the fact that F is assumed lower semicontinuous, implying $\mathcal{P}^<(\theta^*) \subset \mathcal{P}^<(\theta)$ for every $\theta \in \mathcal{F}$ sufficiently close to θ^*, e.g., [**7**, **12**]. But (2.5) contradicts (2.4). So we can set $\tilde{u}_0 = 1$, and (2.3) becomes

$$f^0(z^*) \leq f^0(z) + \sum_{i \in \mathcal{P}^<(\theta^*)} \tilde{u}_i(\theta) f^i(z) \tag{2.6}$$

for every z in the set $\{F_*^=(\theta), \mathcal{F}\}$.

Let us specify $z = z^*$ in (2.6). We obtain

$$\sum_{i \in \mathcal{P}^<(\theta^*)} \tilde{u}_i(\theta^*) f^i(z^*) \geq 0.$$

But also the reverse sign holds, by the feasibility of z^*. Hence

$$\sum_{i\in\mathcal{P}^<(\theta^*)} \tilde{u}_i(\theta^*)f^i(z^*) = 0 \tag{2.7}$$

and, after adding this term to the left-hand side of (2.6), we find that

$$\mathcal{L}_*^<(z^*, \tilde{u}(\theta^*)) \leq \mathcal{L}_*^<(z, \tilde{u}(\theta)) \tag{2.8}$$

i.e., z^* minimizes $\mathcal{L}_*^<(z,u)$ for some $u = \tilde{u}(\theta) \geq 0$ on the set $\{F_*^=(\theta), \mathcal{F}\}$. The necessary condition for optimality is proved.

(Sufficiency:) Going in reverse, we assume that (2.8) holds on the set $\{F_*^=, \mathcal{F}\}$. Because of (2.7), which follows after setting $u = 0$ in (2.1), we have (2.6) holding for every feasible z. (Note that $F(\theta) \subset F_*^=(\theta)$ for every $\theta \in \mathcal{F}$.) For such z's we have $f^i(z) \leq 0$, $i \in \mathcal{P}$, so, definitely, $f^0(z^*) \leq f^0(z)$, establishing global optimality of z^*.

Remark. From the proof of Theorem 2.2 we see that if z^* is optimal, then z^* optimizes $\mathcal{L}_*^<(z, \tilde{u}(\theta))$ on $z \in \{F_*^=(\theta), \mathcal{F}\}$ for some $\tilde{u}(\theta) \geq 0$. The left-hand side inequality in (2.1) is used only in the proof of sufficiency to guarantee the complementarity (2.7).

Conditions under which the set

$$\{F_*^=(\theta), \mathcal{F}\}$$

in the *necessary part* of Theorem 2.2 can be replaced by the simpler condition

$$\{F^=(\theta^*), \mathcal{F}\} \tag{2.9}$$

are called "input constraint qualifications" (abbreviated: ICQ). Conditions under which that set can be replaced by

$$\{F^=(\theta), \mathcal{F}\}$$

are called "modified input constraint qualifications" (abbreviated: MICQ). For a study of these conditions in input optimization see [9]. The results given in that paper are easily adopted to convexifiable-by-a splitting programs. A sample result follows.

2.3. Theorem. *Consider a convexifiable-by-a-splitting program (P) and its feasible point $z^* = (\tilde{x}(\theta^*), \theta^*)$, where $\tilde{x}(\theta^*)$ is an optimal solution of the convex program (P, θ^*). Assume that the point-to-set mapping F is lower semicontinuous at θ^* relative to $\mathcal{F}$. Also asume that an input constraint qualification holds at θ^*. If z^* is a globally optimal solution of (P), then z^* minimizes the Lagrangian $\mathcal{L}_*^<$ on the set (2.9) for some non-negative functions $u_i = \tilde{u}_i(\theta) \geq 0$, $i \in \mathcal{P}^<(\theta^*)$.*

Some of ICQ's and MICQ's are essentially different from the ones known in usual (nonparametric) mathematical programming, as the following adaptation (from [9]) demonstrates.

2.4. ADOPTATION: Consider a convexifiable-by-a-splitting program (P). If

$$\{\mathcal{F} \cap N(\theta^*)\} \subset \{\theta : F(\theta^*) \subset F_*^=(\theta)\}$$

for some neighbourhood $N(\theta^*)$ of θ^*, then the following condition is an ICQ:

For every $\theta \in \mathcal{F} \cap N(\theta^*)$ and for every $x \in F^=(\theta)$ such that

$$f^i(x,\theta) < 0, \qquad i \in \mathcal{P}^<(\theta^*)$$

it also follows that

$$f^i(x,\theta^*) \le 0, \qquad i \in \mathcal{P}^<(\theta^*).$$

The above is a condition on the constraints at the x-level. On the other hand, some ICQ conditions are familiar, such as Slater's condition:

$$f^i(\hat{x},\theta^*) < 0, \quad i \in \mathcal{P} \quad \text{for some} \quad \hat{x} \in R^n.$$

This condition is both an ICQ and a MICQ. Moreover, under this condition

$$F_*^=(\theta) = F^=(\theta) = F^=(\theta^*) = R^n$$

for every $\theta \in N(\theta^*) \cap \mathcal{F}$, where $N(\theta^*)$ is some neighbourhood of θ^*.

Let us illustrate the above results on the examples from Section 1.

2.5. EXAMPLE. We want to know whether $z^* = (1,0,1)^{\mathrm{T}}$ is a globally optimal solution of the program (1.1) from Example 1.1.

After specifying $z_1 = x_1$, $z_2 = x_2$, $z_3 = \theta$, the program is convexified by a splitting:

$$\begin{aligned} \operatorname*{Min}_{s.t.}\ & f^0 = -x_1/\theta \\ & f^1 = \ \ x_1 + x_2 - 1 \le 0 \\ & f^2 = -x_1 - x_2\theta + 1 \le 0 \\ & f^3 = -x_1 \le 0 \\ & f^4 = -x_2 \le 0 \\ & f^5 = -\theta + 1 \le 0. \end{aligned} \tag{2.10}$$

The point to be checked for optimality is

$$x_1^* = 1, \quad x_2^* = 0, \quad \theta^* = 1.$$

Since $\mathcal{P}^<(\theta^*) = \{3,4\}$, the Lagrangian is

$$\mathcal{L}_*^<(x,u;\theta) = -\frac{x_1}{\theta} + u_3(-x_1) + u_4(-x_2). \tag{2.11}$$

The mapping F is lower semicontinuous at θ^* relative to perturbations in $\mathcal{F} = [1,\infty)$. The mapping $F_*^=$ is determined by

$$F_*^=(\theta) = \{x : x_1 + x_2 \le 1,\ x_1 + \theta x_2 \ge 1\}.$$

For the choice $u_3 = \tilde{u}_3(\theta) = 0$, $u_4 = \tilde{u}_4(\theta) = 1/\theta$, the complementarity condition (2.7) is satisfied and z^* is optimal if it solves

$$\begin{aligned} \operatorname*{Max}_{s.t.} \ & (1/\theta)(x_1 + x_2) \\ & x_1 + x_2 \le 1 \\ & x_1 + \theta x_2 \ge 1 \\ & \theta \ge 1. \end{aligned}$$

Clearly, this is the case, so z^* is a global minimum for (2.10).

In the next example we demonstrate how ICQ's can significantly simplify identification of minima.

2.6. EXAMPLE. Again we consider the program (1.1) and the same z^* as in Example 2.5. Since the constraints f^i, $i \in \mathcal{P}^<(\theta^*) = \{3,4\}$ do not depend on θ, the ICQ from Adoptation 2.3 is satisfied. Theorem 2.3 now says that if z^* is indeed a locally optimal solution of (1.1), then z^* must optimize (2.11) on the set (2.9) for some $u_3 = \tilde{u}_3(\theta) \ge 0$ and $u_4 = \tilde{u}_4(\theta) \ge 0$. Again, choosing $\tilde{u}_3(\theta) = 0$ and $\tilde{u}_4(\theta) = 1/\theta$, the equivalent program becomes

$$\begin{aligned} \operatorname*{Max}_{s.t.} \ & (1/\theta)(x_1 + x_2) \\ & x_1 + x_2 = 1 \\ & \theta \ge 1. \end{aligned} \tag{2.12}$$

Clearly, $x_1^* = z_1^* = 1$, $x_2^* = z_1^* = 0$, $\theta^* = z_3^* = 1$ is an optimal solution of (2.12), so z^* passes our necessary condition for optimality.

2.7. EXAMPLE. Finally, consider the program (1.2). We want to check how $z_1^* = 1$, $z_2^* = 0$ passes our necessary condition for optimality. After the splitting $z_1 = \theta$, $z_2 = x$, the program is convexified:

$$\begin{aligned} \operatorname*{Min}_{s.t.} \ & f^0 = (\theta - 2)^2 + x^2 \\ & f^1 = (\theta - 1)^3 + x \le 0 \\ & f^2 = -x \le 0 \\ & f^3 = -\theta \le 0. \end{aligned}$$

Now we have to check $z_1^* = \theta^* = 1$ and $z_2^* = x^* = 0$ for optimality. Since $\mathcal{P}^<(\theta^*) = \{3\}$ and there are no x's in the third constraint, the ICQ from Adoptation 2.3 is satisfied. If z^* is indeed an optimal solution of (1.2), then z^* must minimize

$$\mathcal{L}_*^<(x, u; \theta) = (\theta - 2)^2 + x^2 + u_3(-\theta)$$

on the set $\{F^=(\theta^*), \mathcal{F}\}$ for some $u_3 = \tilde{u}_3(\theta) \ge 0$. Since

$$F^=(\theta^*) = \{x : (\theta^* - 1)^3 + x = 0, \ -x = 0\} = \{0\}$$

the latter means that z^* must solve

$$\begin{aligned} \operatorname*{Min}_{s.t.} \ & (\theta - 2)^2 + x^2 + \tilde{u}_3(\theta)(-\theta) \\ & x = 0 \\ & \theta \le 1 \end{aligned} \tag{2.13}$$

for some $\tilde{u}_3(\theta)$. Specifying, e.g., $\tilde{u}_3(\theta) = 0$, it is obvious that $\theta^* = 1$, $x^* = 0$ solve (2.13). Hence $z_1^* = 1$, $z_2^* = 0$ passes the optimality test. In order to actually establish optimality of z^* one can use Theorem 2.2.

3. IMPORTANCE OF LOWER SEMICONTINUITY

The characterization of optimality in Theorem 2.2 requires lower semicontinuity of the point-to-set mapping F. The result is not valid if this assumption is dropped as demonstrated by the following example.

3.1 EXAMPLE. Consider the program borrowed from [**8**, **10**]:

$$\begin{aligned} \min_{s.t.} \ & z_1 z_3 \\ & z_2 \le 0 \\ & (z_1^2 - z_2) z_3^2 \le 0 \\ & \max\{z_1^2 + z_2^2 - 1, 0\} \le 0. \end{aligned} \tag{3.1}$$

This program is convexifiable by a splitting. Namely, choosing $z_1 = x_1$, $z_2 = x_2$, $z_3 = \theta$, the program becomes

$$\begin{aligned} \min_{s.t.} \ & f^0 = x_1 \theta \\ & f^1 = x_2 \le 0 \\ & f^2 = (x_1^2 - x_2)\theta^2 \le 0 \\ & f^3 = \max\{x_1^2 + x_2^2 - 1, 0\} \le 0. \end{aligned} \tag{3.2}$$

The feasible set at the (x_1, x_2)-level, as θ varies, is

$$F(\theta) = \begin{cases} [0, \alpha]^T, & 0 \le \alpha \le -1, \quad \text{for } \theta = 0 \\ [0, 0]^T, & \text{for } \theta \ne 0. \end{cases}$$

Clearly, $z^* = 0$ is an optimal solution of (3.1), i.e., $x_1^* = 0$, $x_2^* = 0$, $\theta^* = 0$ is an optimal solution of (3.2). Note that F is not lower semicontinuous at $\theta^* = 0$. Does $x_1^* = 0$, $x_2^* = 0$, $\theta^* = 0$ minimize

$$\mathcal{L}_*^<(x, u; \theta) = x_1\theta + u_2 x_2$$

on the set $\{F_*^=(\theta), \mathcal{F}\}$, for some $u_2 = \tilde{u}_2(\theta) \ge 0$? It the answer was affirmative, we would have

$$\mathcal{L}_*^<(z, \tilde{u}_2(\theta)) \ge \mathcal{L}_*^<(z^*, \tilde{u}_2(\theta^*)) = 0$$

i.e.,

$$\tilde{u}_2(\theta) \ge -\frac{x_1}{x_2}\theta = \frac{\theta}{|x_1|}$$

for a fixed $\theta \ne 0$ along the path $x_2 = x_1^2$, $x_1 < 0$ in $F_*^=(\theta)$. Now $x_1 \to 0$ would imply $\tilde{u}_2(\theta) \to +\infty$. We conclude that such Lagrange multiplier does not exist.

4. CONNECTIONS WITH CONVEX OPTIMIZATION

If (P) is a convex program then no splitting of z is needed to characterize optimality. In that case we can identify $z = x$ independently of θ. The mapping $F_*^=$ reduces to

$$\begin{aligned} F_*^= &= \{z : f^i(z) \leq 0, i \in \mathcal{P}^=\} \\ &= \{z : f^i(z) = 0, i \in \mathcal{P}^=\} \\ &= F^= \end{aligned}$$

where

$$\mathcal{P}^= = \{i \in \mathcal{P} : z \in F \Rightarrow f^i(z) = 0\}$$

and

$$F = \{z : f^i(z) \leq 0, i \in \mathcal{P}\}$$

is the feasible set. Since, in this situation, F does not depend on θ, the mapping is trivially lower semicontinuous. Moreover, one can remove the nonactive constraints from the Lagrangian. We recall that the *active constraints* at z^* correspond to the index set

$$\mathcal{P}(z^*) = \{i \in \mathcal{P} : f^i(z^*) = 0\}$$

while the constraints f^i, $i \in \mathcal{P} \setminus \mathcal{P}(z^*)$ are *nonactive*. From (2.7) it follows that

$$\tilde{u}_i(\theta^*) = 0, \qquad i \in \mathcal{P} \setminus \mathcal{P}(z^*).$$

Moreover, in the fixed θ situation, the corresponding Lagrange multipliers in the right-hand side of (2.6) are also equal to zero:

$$\tilde{u}_i(\theta) = \tilde{u}_i(\theta^*) = 0, \qquad i \in \mathcal{P} \setminus \mathcal{P}(z^*).$$

Thus we have the following special case:

4.1. COROLLARY. *Consider a convex program (P). A feasible point z^* is optimal if, and only if, z^* minimizes the Lagrangian*

$$\mathcal{L}_*^<(z, u) = f^0(z) + \sum_{i \in \mathcal{P}(z^*) \setminus \mathcal{P}^=} u_i f^i(z)$$

on the set $F^=$ for some non-negative numbers $u_i = \tilde{u}_i \geq 0$, $i \in \mathcal{P}(z^*) \setminus \mathcal{P}^=$.

Since $\mathcal{L}_*^<(\cdot, u) : R^n \to R$ is a convex function for $u = \tilde{u} = (\tilde{u}_i) \geq 0$ and $F^=$ is a convex set, the statement of Corollary 4.1 is equivalent to saying that there exists a subgradient

$$h \in \partial \mathcal{L}_*^<(z^*, \tilde{u})$$

such that

$$h(z - z^*) \geq 0 \quad \text{for every} \quad z \in F^=$$

or, more formally, h belongs to the following polar set

$$h \in \{F^= - z^*\}^+ = \{z - z^* : z \in F^=\}^+. \tag{4.1}$$

(We recall that the polar set of an arbitrary set M is defined by $M^+ = \{v : v^T x \geq 0$ for every $x \in M\}$, e.g., [12]).

In order to see the light in this situation, we recall the definition of the cone of directions of constancy of $f^i : R^n \to R$ at an arbitrary $z^* \in R^n$:

$$D_i^=(z^*) = \{d : f^i(z^* + \alpha d) = f^i(z^*), 0 \leq \alpha \leq \overline{\alpha}, \text{ for some } \overline{\alpha} > 0\}$$

see, e.g., [3]. The intersection of all cones of constancy of the functions belonging to the index set $\mathcal{P}^=$, at z^*, is denoted by

$$D^=(z^*) = \bigcap_{i \in \mathcal{P}^=} D_i^=(z^*).$$

Although

$$F^= - z^* \subset D^=(z^*) \tag{4.2}$$

generally with a strict inclusion, it is curious that the polars of these two sets concide:

4.2. LEMMA. *Consider the convex program (P). Then, at any feasible z^*,*

$$\{F^= - z^*\}^+ = \{D^=(z^*)\}^+.$$

Proof: From (4.2) we conclude, using the properties of polar sets, that

$$\{D^=(z^*)\}^+ \subset \{F^= - z^*\}^+$$

so only the reverse inclusion has to be proved. To this end, take any $d \in D^=(z^*)$. Since

$$f^i(z^* + \alpha d) = f^i(z^*) = 0, \qquad i \in \mathcal{P}^=$$

it follows that

$$z^* + \alpha d \in F^=$$

for all sufficiently small $\alpha > 0$. Hence

$$\alpha d = (z^* + \alpha d) - z^* \in F^= - z^*. \tag{4.3}$$

At this point, using (4.3), we oberve that for an arbitrary $g \in \{F^= - z^*\}^+$ we have $g^T(\alpha d) \geq 0$ and $g^T d \geq 0$ since $\alpha > 0$. Hence

$$g \in \{D^=(z^*)\}^+.$$

We have, indeed, shown that

$$\{F^= - z^*\} \subset \{D^=(z^*)\}^+$$

which completes the proof.

Using Lemma 4.2, the relation (4.1) can be rewitten as

$$h \in \{D^=(z^*)\}^+$$

and Corollary 4.1 assumes the following form:

4.3. COROLLARY. *Consider the convex program (P). A feasible point z^* is optimal if, and only if, the system*

$$h^0 + \sum_{i \in \mathcal{P}(z^*) \setminus \mathcal{P}^=} u_i h^i \in \{D^=(z^*)\}^+$$
$$u_i \geq 0, \quad i \in \mathcal{P}(z^*) \setminus \mathcal{P}^=$$

is consistent for some $h^i \in \partial f^i(z^*)$, $i \in \{0\} \cap \{\mathcal{P}(z^*) \setminus \mathcal{P}^=\}$.

If the convex functions in (P) are all differentiable, then the above yields another result from [3], but here proved using different arguments:

4.4. COROLLARY. *Consider the convex program (P), where all functions are assumed differentiable. Then a feasible point z^* is optimal if, and only if, the system*

$$\nabla f^0(z^*) + \sum_{i \in \mathcal{P}(z^*) \setminus \mathcal{P}^=} u_i \nabla f^i(z^*) \in \{D^=(z^*)\}^+$$
$$u_i \geq 0, \quad i \in \mathcal{P}(z^*) \setminus \mathcal{P}^=$$

is consistent.

This deduction would be incomplete without mentioning Slater's condition:

$$f^i(\hat{z}) < 0, \qquad i \in \mathcal{P} \quad \text{for some } \hat{z} \in R^n.$$

The condition, for convex programs, is equivalent to $\mathcal{P}^= = \varnothing$, in which case

$$D^=(z^*) = R^n \quad \text{and} \quad \{D^=(z^*)\}^+ = 0.$$

The statement of Corollory 4.4 is further simplified:

4.5. COROLLARY. *Consider the convex program (P), where all functions are assumed differentiable. Also, assume that Slater's condition is satisfied. Then a feasible point z^* is optimal if, and only if, the system*

$$\nabla f^0(z^*) + \sum_{i \in \mathcal{P}(z^*)} u_i \nabla f^i(z^*) = 0$$
$$u_i \geq 0, \qquad i \in \mathcal{P}(z^*)$$

is consistent.

The latter is the classical result of Karush, Kuhn and Tucker (e.g., [**3**, **5**, **6**]).

Of course, one can state corresponding results for convexifiable-by-a-splitting programs, assuming that the set $\{F_*^=(\theta), \mathcal{F}\}$ is convex.

REFERENCES

[1] Bank, B., J. Guddat, D. Klatte, B. Kummer and K. Tammer: *Nonlinear Parametric Optimization*, Akademie-Verlag, Berlin, 1982.

[2] Ben-Israel, A. and B. Mond: *First-order optimality conditions for generalized convex functions: A feasible directions approch*, Utilitas Mathematica **25** (1984), 249–262.

[3] Ben-Israel, A., A. Ben-Tal and S. Zlobec: *Optimality in Nonlinear Programming: A Feasible Directions Approach*, Wiley-Interscience, New York, 1981.

[4] Hogan, W. W.: *Point-to-set maps in mathematical programming*, SIAM Review **15** (1973), 591–603.

[5] Kuhn, H. W. and A. W. Tucker: *Nonlinear programming*, in: *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, (J. Neyman, editor), University of California Press, Berkeley, California.

[6] Mangasarian, O. L: *Nonlinear Programming*, McGraw-Hill, New York, 1969.

[7] Semple, J. and S. Zlobec: *On a necessary condition for stability in perturbed linear and convex programming*, Zeitshrift für Operations Research, (Series A: Theorie) **31** (1987), 161–172.

[8] van Rooyen, M.: *Characterizing an Optimal Input in Perturbed Convex Programming*, M. Sc. Thesis, University of Witwatersrand, Department of Computational and Applied Mathematics, Johannesburg, 1987.

[9] van Rooyen, M. and S. Zlobec: *Constraint qualifications in input optimization*, Journal of the Australian Mathematical Society, Series B, (1989), 326–342.

[10] van Rooyen, M. and S. Zlobec: *A complete characterization of optimal economic systems with respect to stable perturbations*, Glasnik Matematički **25** (**45**) (1990), 235–253.

[11] Zlobec, S.: *Input Optimization: I. Optimal realization of mathematical models*, Mathematical Programming **31** (1985), 245–268.

[12] Zlobec, S: *Characterizing optimality in mathematical programming models*, Acta Applicandae Mathematicae **12** (1988), 113–180.

[13] Zlobec, S. and J. Petrić: *Nelinearno Programiranje*, Naučna Knjiga, Beograd, 1989.