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EXISTENCE THEOREMS IN QUASILINEAR OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper we consider the one-step and the two-step quasilinear optimal control problems whith moving ends in the sense of the maximal speed and prove the existence theorems. We assume that admissible controls are functions belonging to $L_2[0,T]$ and to the convex and compact set U, and that the initial set M_0 and the terminal set M_T are compact subsets of R^n in the one-step problem, and $M_0 \subset R^{n_1}$, $M_T \subset R^{n_2}$ in the two-step problem.

Key words and phrases: optimal control, quasilinear proces, bundles

1. INTRODUCTION

There are many optimal control problems which can be reduced to optimal control problems with bundles of trajectories, like a control problem of charget particles and the others; we discuss here problems of the existence of optimal solutions in a quasilinear optimal control problem with bundles of trajectories. We will prove existence theorems using Filippov's result in **[2]** and Arzela's theorem.

2. ONE-STEP PROBLEM

Consider the quasilinear process

$$\dot{x} = A(t,x) + B(t,x)u(t), \qquad u(t) \in U \subset R^m, \quad x(t) \in R^n. \tag{1}$$

DEFINITION 1. A function $X(t, x_0)$ which is a solution of system (1) with the initial condition $X(0, x_0) = x_0$ for every $x_0 \in M_0$, is named the bundle of trajectories of process (1) for the initial set M_0 and the moment $t = 0$.

Suppose that the following conditions are satisfied

$$\begin{aligned} &A(t,x), B(t,x) \in C^1([0,T] \times R^n), \\ &\langle x, (A(t,x) + B(t,x)u(t))\rangle \leq c(|x|^2 + 1), \quad c = \text{const.} \end{aligned} \tag{2}$$

We assume that admissible controls are functions $u(t) \in L_2[0,T]$ whose values belong to the convex and compact set U, and that the initial and terminal sets M_0, M_T are compact subsets of the space R^n.

Under the assumption that there exists such an admissible control $u_T(t)$ that bundle of trajectories $X(t, x_0)$, with initial set M_0 and corresponding to control $U_T(t)$, attains the terminal set M_T during the finite time T, $0 < T < +\infty$, it is possible to prove that

$$X(t, x_0) \in C^1([0, T] \times R^n).$$

Let us prove the next existence theorem.

THEOREM 1. *If there exists an admissible control displacing the bundle of trajectories $X(t, x_0)$ of process (1) from the initial compact set M_0 to the terminal compact set M_T during a finite time interval, then there exists an optimal admissible control displacing the considered bundle of trajectories from M_0 to M_T during the minimal time interval.*

Proof: If we denote

$$Q = \max_{x_0 \in M_0} (|x_0|^2 + 1),$$

then from (2) it follows

$$|X(t, x_0)| \leq Q^{0.5} e^{cT} \tag{3}$$

(proved in [**2**]).

The set of bundles of trajectories $X(t, x_0)$ corresponding to system (1), with the initial set M_0, satisfying the condition

$$X(\tau, x_0) \in M_T, \qquad x_0 \in M_0, \quad 0 < \tau \leq T$$

is nonempty; if we suppose that this set is infinite, then we can selct a subsequence $\{X_i(t, x_0)\}$ in which the corresponding ends of time intervals $t_i \to t_*$ ($t_* = \inf t_i$).

From the relation (3) if follows that every such subsequence is uniformly bounded on $[0, t_*] \times M_0$; it could be proved that this subsequence is equicontinuous.

For arbitrary values $t^1, t^2 \in [0, t_*]$ and phase states $x_0^1, x_0^2 \in M_0$, using continuous differentiability of the function $X(t, x_0)$ with respect to the variable x_0 and the mean value theorem (Ref. [**3**]) we obtain that

$$\begin{aligned} |X_i(t^1, x_0^1) - X_i(t^2, x_0^2)| &\leq |X_i(t^1, x_0^1) - X_i(t^2, x_0^1)| + |X_i(t^2, x_0^1) - X_i(t^2, x_0^2)| \\ &\leq \max_{[0,t_*]\times \mathrm{co}\, M_0} \left| \frac{\partial X_i(t, x_0)}{\partial t} \right| |t^1 - t^2| + \max_{[0,t_*]\times \mathrm{co}\, M_0} \left\| \frac{\partial X_i(t, x_0)}{\partial x_0} \right\| |x_0^1 - x_0^2|, \end{aligned} \tag{4}$$

where

$$\left\| \frac{\partial X_i(t, x_0)}{\partial x_0} \right\| = \sum_{k=1}^{n} \left| \frac{\partial X_i(t, x_0)}{\partial x_0^k} \right|. \tag{$*$}$$

For the first additive of the inequality (4), based on condition (3), for $0 \leq t \leq t_*$, $u \in U$, we have the evaluation

$$\max_{[0,t_*]\times M_0} \left| \frac{\partial X_i(t, x_0)}{\partial t} \right| = |A(t, X(t, x_0)) + B(t, X(t, x_0))u(t)| \leq L_1,$$
$$(L_1 = \text{const.} > 0). \tag{5}$$

If

$$\delta X_i^k(t, x_0) = \partial X_i(t, x_0)/\partial x_0^k, \qquad k = 1, \dots, n, \quad i = 1, 2, \dots, \tag{$**$}$$

then the vector function $\delta X_i^k(t, x_0)$ satisfies the system in variations on $[0, t_*]$ (Ref. [4])

$$\dot{p} = (A_x(t, X_i(t, x_0)) + B_x(t, X_i(t, x_0))u_i(t))p \tag{6}$$

with the initial condition $p(0) = (0, \dots, 1, 0, \dots, 0)^{\mathrm{T}}$, where the k-th coordinate is equal to 1.

We shall prove that for the system of equations in variations (6) the inequality of the type (2) holds.

We have that

$$\begin{aligned}
&\langle p, (A_x(t, X_i(t, x_0)) + B_x(t, X_i(t, x_0)))u_i(t)p\rangle \\
&= \langle (A_x^{\mathrm{T}}(t, X_i(t, x_0)) + u_i^{\mathrm{T}}(t)B_x^{\mathrm{T}}(t, X_i(t, x_0)))p, p\rangle \\
&\le |\langle (A_x^{\mathrm{T}}(t, X_i(t, x_0)) + u_i^{\mathrm{T}}(t)B_x^{\mathrm{T}}(t, X_i(t, x_0)))p, p\rangle| \\
&\le \max_{[0,t_*]\times M_0} \|(A_x^{\mathrm{T}}(t, X_i(t, x_0)) + u_i^{\mathrm{T}}(t)B_x^{\mathrm{T}}(t, X_i(t, x_0)))\|\, |p|^2 \\
&\le C(|p|^2 + 1),
\end{aligned}$$

where

$$C = \max_{t\in[0,t_*]} \|A_x(t, X(t, x_0)) + B_x(t, X(t, x_0))u(t)\|,$$

assuming that the inequality (3) holds.

Hence, for the bundles of trajectories $\delta X_i^k(t, x_0)$ of solutions of system (6), we have the inequality of the type (3)

$$|\delta X_i^k(t, x_0)| \le 2^{0.5} e^{cT},$$

which implies, using ($**$), the next evaulation of the norm ($*$)

$$\|\partial X_i(t, x_0)/\partial x_0\| \le n 2^{0.5} e^{cT}. \tag{7}$$

It follows from (4), (5) and (7) that

$$|X_i(t^1, x_0^1) - X_i(t^2, x_0^2))| \le L_1|t^1 - t^2| + L_2|x_0^1 - x_0^2|,$$

where $L_2 = n2^{0.5}e^{ct_*}$; which means that the sequence $\{X_i(t, x_0)\}$ is equicontinuous.

From the Arzela theorem, it follows that the sequence of bundles of trajectories $\{X_i(t, x_0)\}$ is uniformly approaching the bundle $X(t, x_0)$, for which it can be proved that in the time instant t_* satisfies the terminal condition

$$X(t_*, x_0) \subset M_T.$$

Let the control sequence $\{u_i(t)\}$ corespond to the chosen sequence of bundles of trajectories $\{X_i(t, x_0)\}$; from the sequence $\{u_i(t)\}$ a weakly convergent subsequence with limit $u(t)$ can be selected.

If we substitute the differential equation (1) on interval $[0, t_*]$ with the corresponding integral equation

$$X_i(t, x_0) = x_0 + \int_0^t A(\tau, X_i(\tau, x_0))\, d\tau + \int_0^t B(\tau, X_i(\tau, x_0))u_i(\tau)\, d\tau \tag{8}$$

and consider on the given interval the difference between the expression on the right side of (8) and the expression

$$x_0 + \int_0^t A(\tau, X(\tau, x_0))\, d\tau + \int_0^t B(\tau, X(\tau, x_0))u(\tau)\, d\tau \tag{9}$$

we obtain the expression

$$\begin{aligned} &\int_0^t (A(\tau, X_i(\tau, x_0)) - A(\tau, X(\tau, x_0)))\, d\tau \\ &+ \int_0^t (B(\tau, X_i(\tau, x_0)) - B(\tau, X(\tau, x_0)))u_i(\tau)\, d\tau \\ &+ \int_0^t (B(\tau, X(\tau, x_0))u_i(\tau) - B(\tau, X(\tau, x_0))u(\tau))\, d\tau. \end{aligned} \tag{10}$$

Using the uniform convergence of $\{X_i(t, x_0)\}$ to $X(t, x_0)$ and the continuity of A and B, for any $\varepsilon > 0$ there exixsts such a positive integer N_1 that for each $i \geq N_1$ holds

$$\begin{aligned} &\int_0^t |A(\tau, X_i(\tau, x_0)) - A(\tau, X(\tau, x_0))|\, d\tau < \varepsilon/3 \\ &\int_0^t |(B(\tau, X_i(\tau, x_0)) - B(\tau, X(\tau, x_0)))u_i(\tau)|\, d\tau < \varepsilon/3. \end{aligned} \tag{11}$$

and using the weak convergence of the sequence $\{u_i(t)\}$ to $u(t)$[1], we obtain the weak convergence of the sequence $\{B(t, X(t, x_0))u_i(t)\}$ to $B(t, X(t, x_0))u(t)$. It can be proved (See Ref. [5]) that the sequence

$$\left\{\int_0^t B(\tau, X(\tau, x_0))u_i(\tau)\, d\tau\right\}$$

strongly converges to

$$\int_0^t B(\tau, X(\tau, x_0))u(\tau)\, d\tau$$

in the norm of the space $C[0, t_*]$; so, there exists such a positive integer N_2, that for every $i \geq N_2$

$$\left|\int_0^t B(\tau, X_i(\tau, x_0))u_i(\tau)\, d\tau - \int_0^t B(\tau, X(\tau, x_0))u(\tau)\, d\tau\right| < \frac{\varepsilon}{3}. \tag{12}$$

From the relations (10)–(12) it follows that the expression

$$X_i(t, x_0) = x_0 + \int_0^t A(\tau, X_i(\tau, x_0))\, d\tau + \int_0^t B(\tau, X_i(\tau, x_0))u_i(\tau)\, d\tau$$

uniformly converges to $X(t, x_0)$ when $i \to \infty$, i.e., using the uniqueness of the limit, we get that

$$X(t, x_0) = x_0 + \int_0^t A(\tau, X(\tau, x_0))\, d\tau + \int_0^t B(\tau, X(\tau, x_0))u(\tau)\, d\tau,$$

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[1] $u(t) \in U$ (See Ref. [6, Theorem 1, p. 78, Lemma 1A, p. 169])

for every $t \in [0, t_*]$; by this, the existence of on optimal solution is proved.

3. TWO-STEP PROBLEM

Suppose that the considered quasilinear process is defined on the interval $[0, \tau]$, $0 < \tau < T$, by the system

$$\dot{x} = A_1(t, x) + B_1(t, x)u(t), \qquad x \in R^{n_1}, \tag{13}$$

and on interval $[\tau, T]$, by the system

$$\dot{y} = A_2(t, y) + B_2(t, y)v(t), \qquad y \in R^{n_2}. \tag{14}$$

The initial set M_0 generates the bundles of trajectories $X(t, x_0)$ of the system (13), which at the moment τ transforms to a bundle of trajectories $Y_1(t, x_0)$ of the system (14), corresponding to the joining condition

$$Y_1(\tau, x_0) = k(X(\tau, x_0)), \qquad \forall x_0 \in M_0, \tag{15}$$

where the bundle of trajectories of system (14) at the time instant τ starts from the set $M_\tau = k(X(\tau, M_0))$ and attains the terminal set M_T during the finite time T.

We assume that the functions $A_1(t, x)$, $B_1(t, x)$, $A_2(t, x)$, $B_2(t, x)$, $k(x)$ are continuous and countinuously differentiable with respect to phase variables on $[0, T] \times R^{n_1}$, $[0, T] \times R^{n_2}$, R^{n_1} respectively.

Let the following conditions hold

$$\begin{aligned} \langle x, (A_1(t, x) + B_1(t, x)u(t))\rangle &\le c_1(|x|^2 + 1), \\ \langle y, (A_2(t, y) + B_2(t, y)v(t))\rangle &\le c_2(|y|^2 + 1). \end{aligned} \tag{16}$$

where c_1, c_2 are positive constants.

DEFINITION 2. An admissible control for two-step problem (13)–(15) is a triple $(u(t), v(t), \tau)$ where $u(t)$ and $v(t)$ are functions from $L_2[0, T]$, with values from the convex and compact sets $U \subset R^{m_1}$, $V \subset R^{m_2}$, $\tau \in [0, T]$ and the sets M_0 and M_T are compact, $M_0 \subset R^{n_1}$, $M_T \subset R^{n_2}$.

The next theorem asserts the existence of an optimal solution in the two-step quasilinear problem (13)–(15).

THEOREM 2. *If there exists an admissible control translating quasilinear object (13)–(15) from the compact initial set $M_0 \subset R^{n_1}$ to the compact terminal set $M_T \subset R^{n_2}$ in a finite time, then there exists an optimal admissible control translating the considered object from M_0 to M_T in the minimal time.*

Proof. Suppose that there exists an admissible control $(u_T(t), v_T(t), \tau_T)$, corresponding to bundles of trajctories $X_T(t, x_0)$, $Y_T(t, x_0)$ of systems (13), (14), satisfying condition (15) and the terminal condition $Y_T(t_T, x_0) \subset M_T$, $\forall x_0 \in M_0$, $0 < t_T < +\infty$.

It can be proved that then there exists an optimal control corresponding to the minimal final moment.

Using condition (16), we get evaluations of the type (3) for the bundles of trajectories

$$|X(t,x_0)| \le Q^{0.5}e^{c_1T}, \qquad t \in [0,\tau_T],\ x_0 \in M_0,$$
$$|Y(t,x_0)| \le P^{0.5}e^{c_2T}, \qquad t \in [\tau_T,T],\ x_0 \in M_0.$$

where $P = \max(|k(X(\tau_T,x_0))|^2 + 1)$.

Consider the set of amissible controls and the set of the corresponding bundles of trajectorijes of systems (13), (14), for which the corresponding final moments $t' \le T$. These sets are non-empty, suppose that they are infinite.

From the set of final moments select a subsequence $\{t_i\}$, $t_i \to t_*$, $i \to \infty$, $t_* = \inf\{t'\}$.

This sequence $\{t_i\}$ corresponds to the control sequence $\{(u_i(t), v_i(t), \tau_i)\}$ and to the sequences of bundles of trajectories $\{X_i(t,x_0)\}$, $\{Y_i(t,x_0)\}$, defined on the intervals $[0,\tau_i]$, $[\tau_i,t_i]$ and connected with condition (15).

Select a convergent subsequence from the bounded sequence $\{\tau_i\}$; denote it by $\{\tau_i\}$ and its limit by τ.

We shall consider the following cases: 1. $\tau = 0$; 2. $\tau = t_*$; 3. $0 < \tau < t_*$.

For the first and the second case, we have the one-step quasilinear problem, corresponding to Theorem 1.

Consider the third case: suppose that $\{\tau_i\} \to \tau$ from the left side.

For every $i = 1,2,\dots,$ complete the definition of bundles of trajectories $X_i(t,x_0)$

$$X_i'(t,x_0) = \begin{cases} X_i(t,x_0), & t \in [0,\tau_i], \\ X_i(\tau_i,x_0), & t \in [\tau_i,\tau]. \end{cases}$$

Using the proof of Theorem 1, it can be proved that the sequences $\{X_i'(t,x_0)\}$, $\{Y_i(t,x_0)\}$ are equicontinuous on $[0,\tau] \times M_0$ and $[\tau,t_*] \times M_0$ respectively.

According to Arzela's theorem, from uniformly bounded and equicontinuous sequences $\{X_i'(t,x_0)\}$, $\{Y_i(t,x_0)\}$ uniformly convergent sequences can be selected; assume that $\{X_i'(t,x_0)\}$ converges uniformly to $X(t,x_0)$ and $\{Y_i(t,x_0)\}$ converges uniformly to $Y_T(t,x_0)$; then $Y_T(t_*,x_0) \subset M_T$ and joining condition (15) holds.

Let the sequences of admissible controls $\{(u_i(t), v_i(t), \tau_i)\}$, corresponding to the decreasing sequence of final moments $\{t_i\} \to t_*$ be selected, such that $\tau_i \to \tau$ from the left side and $X_i'(t,x_0) \to X(t,x_0)$, $Y_i(t,x_0) \to Y_T(t,x_0)$, when $i \to \infty$, uniformly on $(t,x_0) \in [0,\tau] \times M_0$, $(t,x_0) \in [\tau,t_*] \times M_0$.

From the sequences $\{u_i(t)\}$, $\{v_i(t)\}$ select weakly convergent subsequences (see Ref. [2]) and denote the corresponding limits with $u(t)$, $v(t)$, respectively.

Let us substitute the systems of differential equations (13), (14) on intervals $[0,\tau]$, $[\tau,t_*]$, respectively, with the corresponding integral equations:

$$X_i'(t,x_0) = x_0 + \int_0^t A_{1i}'(s, X_i'(s,x_0))\,ds + \int_0^t B_{1i}'(s, X_i(s,x_0))u_i(s)\,ds, \qquad (17)$$

$$Y_i(t,x_0) = x_0 + \int_\tau^t A_2(s, Y_i(s,x_0))\, ds + \int_\tau^t B_2(s, Y_i(s,x_0)) v_i(s)\, ds, \tag{18}$$

where

$$A'_{1i} = \begin{cases} A_1(t,x), & t \in [0,\tau_i] \\ 0, & t \in (\tau_i, \tau], \end{cases}$$
$$B'_{1i} = \begin{cases} B_1(t,x), & t \in [0,\tau_i] \\ 0, & t \in (\tau_i, \tau], \end{cases}$$

When $i \to \infty$, the expressions (17), (18) uniformly converge with respect to $(t, x_0) \in [0,\tau] \times M_0$, $(t, x_0) \in [\tau, t_*] \times M_0$ to the next expressions (as in the proof of Theorem 1):

$$X(t,x_0) = x_0 + \int_0^t A_1(s, X'_i(s,x_0))\, ds + \int_0^t B_1(s, X_i(s,x_0)) u_i(s)\, ds,$$

$$Y_T(t,x_0) = Y_T(\tau,x_0) + \int_\tau^t A_2(s, Y_T(s,x_0))\, ds + \int_\tau^t B_2(s, Y_T(s,x_0)) v(s)\, ds.$$

By this, theorem 2 is proved.

4. CONCLUSION

In this paper we formulated and proved the existence theorems for the one-step and the two-step quasilinear optimal control problems with moving ends in the sense of maximal speed, using Arzela's theorem and Filippov's results (Ref. **[2]**). In the two-step problem, coefficient k, appearing in the joining condition, depends on the technical characteristics of the considered object.

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