Spirals on surfaces of revolution

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Abstract

In this paper some spirals on surfaces of revolution and the corresponding helicoids are presented.

Keywords: helix; spiral; surface of revolution; ruled surface; helicoid
MSC 2010: 14H50, 53A04, 53A05

1 Introduction

One of the most known spiral in space is the cylindrical helix (Figure 1). It is a a curve for which the tangent makes a constant angle with a fixed line (Oz-axis in our case) [1]. The parametric equations of the helix are given by:

$$\begin{cases} x = r\cos t \\ y = r\sin t \\ z = ct \end{cases}, \quad t \in \mathbb{R}, \tag{1}$$

where r is the radius of the cylinder and c is a positive parameter such that $2\pi c$ is a constant giving the vertical separation of the helix's loops and controls the number of the rotations.

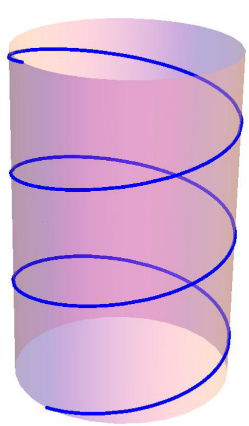

Figure 1: A cylindrical helix

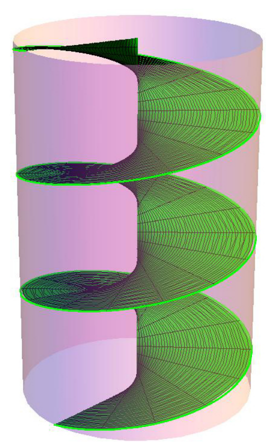

Figure 2: A circular helicoid

The circular helicoid (Figure 2) is a ruled surface having a cylindrical helix as its boundary [1]. In fact, the cylindrical helix is the curve of intersection between the cylinder and the helicoid. The circular helicoid bounded by the planes $z = a$ and $z = b$ is given in parametric form by:

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = cv \end{cases}, \quad (u,v) \in [0,r] \times \left[\frac{a}{c}, \frac{b}{c}\right] \tag{2}$$

The aim of this work is to present similar curves on the other surfaces of revolution, the similar helicoids and their equations. Also, some planar curves which looks similar with known plane curves are obtained.

2 Spirals, helicoids and surfaces of revolution

Let f be a positive continuous function. A surface of revolution generated by rotating the curve $y = f(z)$, $z \in [a, b]$ around the Oz-axis has the equation [1]:

$$x^2 + y^2 = f^2(z) \tag{3}$$

and a parametric representation can be given by

$$\begin{cases} x = f(t)\cos s \\ y = f(t)\sin s \\ z = t \end{cases}, \quad (t, s) \in [a, b] \times [0, 2\pi). \tag{4}$$

As the intersection of the circular helicoid and cylinder is a cylindrical helix as intersection of the helicoid and another surface of revolution gives rise to a three-dimensional spiral.

Equating x, y, respectively z from equations (2) and (4) one gets

$$\begin{cases} u\cos v = f(t)\cos s \\ u\sin v = f(t)\sin s \\ cv = t \end{cases} \tag{5}$$

Imposing the condition $f(t) \in [0, r]$ it follows $u = f(t)$, whence the equations of the three-dimensional spiral are given by:

$$\begin{cases} x = f(t)\cos\dfrac{t}{c} \\ y = f(t)\sin\dfrac{t}{c} \\ z = t \end{cases}, \quad t \in [a, b] \tag{6}$$

or equivalent

$$\begin{cases} x = f(ct)\cos t \\ y = f(ct)\sin t \\ z = ct \end{cases}, \quad t \in \left[\frac{a}{c}, \frac{b}{c}\right]. \tag{7}$$

Spiral (6) and Oz-axis generate a ruled surface. Taking into account the vectorial equation of a ruled surface [1], the equations of a ruled surface having the aforementioned spiral as its boundary are given by :

$$\begin{cases} x = uf(v)\cos\dfrac{v}{c} \\ y = uf(v)\sin\dfrac{v}{c} \\ z = v \end{cases}, \quad (u, v) \in [0, 1] \times [a, b] \tag{8}$$

or in equivalent form

$$\begin{cases} x = uf(cv)\cos v \\ y = uf(cv)\sin v \\ z = cv \end{cases}, \quad (u, v) \in [0, 1] \times \left[\frac{a}{c}, \frac{b}{c}\right]. \tag{9}$$

The projection of spiral (6) on xOy plane is given by

$$\begin{cases} x = f(t)\cos\dfrac{t}{c} \\ y = f(t)\sin\dfrac{t}{c} \end{cases}, \quad t \in [a, b] \tag{10}$$

or in equivalent form

$$\begin{cases} x = f(ct)\cos t \\ y = f(ct)\sin t \end{cases}, \quad t \in \left[\frac{a}{c}, \frac{b}{c}\right]. \tag{11}$$

Equation (11) can be written in polar coordinates as follows:

$$\rho = f(c\theta), \quad \theta \in \left[\frac{a}{c}, \frac{b}{c}\right], \tag{12}$$

which sometimes represents the equation of a planar spiral [1]. This property allows us to consider some particular planar spirals [2] and then to obtain the corresponding three-dimensional spirals.

Pappus' conical spiral

The first considered planar spiral is the Archimedean spiral (Figure 3) described by the equation

$$\rho = \beta + \alpha\theta,\ \theta \in \mathbb{R},$$

where α and β are real numbers. In this case $f(z) = \beta + \frac{\alpha}{c}z$, $z \in \mathbb{R}$ (red line in Figure 4) and the surface of revolution around Oz-axis (black line in Figure 4) is a cone: $x^2 + y^2 = \left(\beta + \frac{\alpha}{c}z\right)^2$. The equations of spiral (7), known as Pappus' conical spiral [3] (Figure 4, just a half of the cone is shown), and of ruled surface (9), called conical helicoid (Figure 5) are respectively:

$$\begin{cases} x = (\beta + \alpha t)\cos t \\ y = (\beta + \alpha t)\sin t \\ z = ct \end{cases},\ t \in \mathbb{R} \quad \text{and} \quad \begin{cases} x = u(\beta + \alpha v)\cos v \\ y = u(\beta + \alpha v)\sin v \\ z = cv \end{cases},\ (u, v) \in [0, 1] \times \mathbb{R}.$$

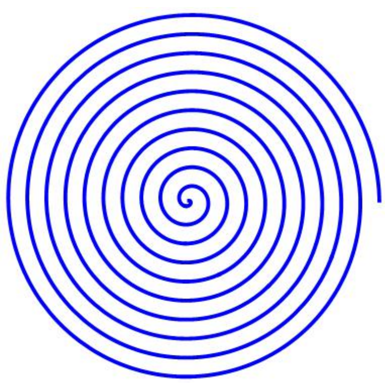

Figure 3: An Archimedean spiral

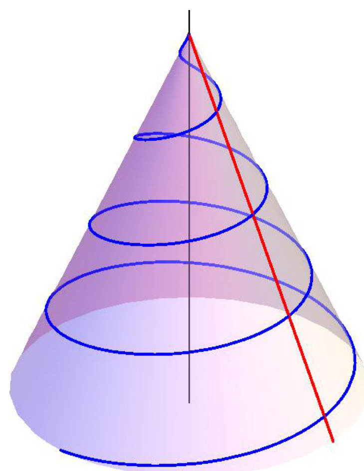

Figure 4: A conical spiral

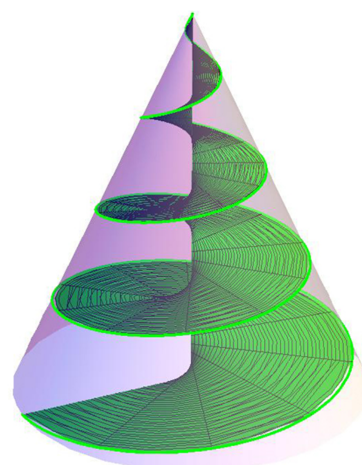

Figure 5: A conical helicoid

Three-dimensional Fermat's spiral

Considering Fermat's spiral or the parabolic spiral (Figure 6) described by the equation

$$\rho = \alpha\sqrt{\theta},\ \theta \in [0, \infty),$$

where α is a real number, the function $f(z) = \alpha\sqrt{\frac{1}{c}z}$, $z \in [0, \infty)$ generates a paraboloid of revolution: $x^2 + y^2 = \frac{\alpha^2}{c}z$. The equations of paraboloidal spiral (three-dimensional parabolic spiral, three-dimensional Fermat's spiral) (7) and paraboloidal helicoid (9) are given by:

$$\begin{cases} x = \alpha\sqrt{t}\cos t \\ y = \alpha\sqrt{t}\sin t \\ z = ct \end{cases},\ t \in [0, \infty) \quad \text{and} \quad \begin{cases} x = \alpha u\sqrt{v}\cos v \\ y = \alpha u\sqrt{v}\sin v \\ z = cv \end{cases},\ (u, v) \in [0, 1] \times [0, \infty)$$

Analogously we can consider the function $f(z) = \alpha\sqrt{b - z}$, $z \in [a, b]$ (red line in Figure 7). One obtain a paraboloid of revolution: $x^2 + y^2 = \alpha^2(b - z)$. The equations of paraboloidal spiral (Figure 7) and paraboloidal helicoid (Figure 8) are given by:

$$\begin{cases} x = \alpha\sqrt{b-t}\cos\dfrac{t}{c} \\ y = \alpha\sqrt{b-t}\sin\dfrac{t}{c} \\ z = t \end{cases}, \ t \in [a,b] \quad \text{and} \quad \begin{cases} x = \alpha u\sqrt{b-v}\cos\dfrac{v}{c} \\ y = \alpha u\sqrt{b-v}\sin\dfrac{v}{c} \\ z = v \end{cases}, \ (u,v) \in [0,1]\times[a,b].$$

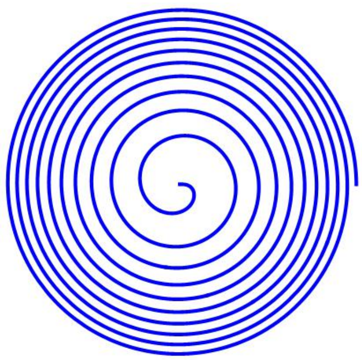

Figure 6: A Fermat's spiral

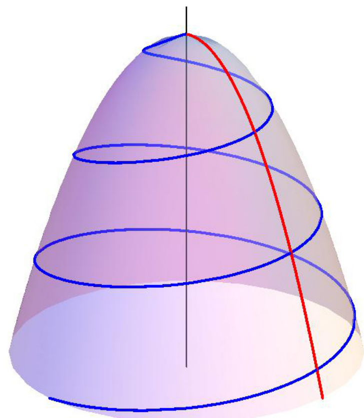

Figure 7: A paraboloidal spiral

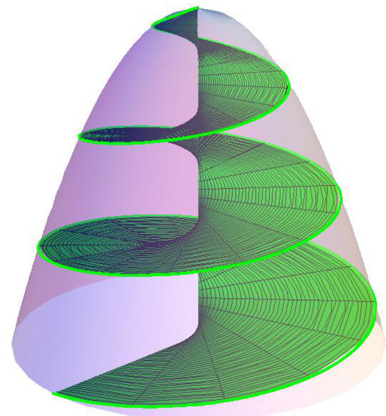

Figure 8: A paraboloidal helicoid

Three-dimensional hyperbolic spiral

Next planar spiral is the hyperbolic spiral (Figure 9) given by the equation

$$\rho = \frac{\alpha}{\theta},\ \theta \in (0,\infty),$$

where α is a real numbers.

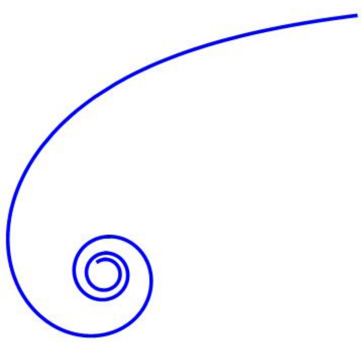

Figure 9: A hyperbolic spiral

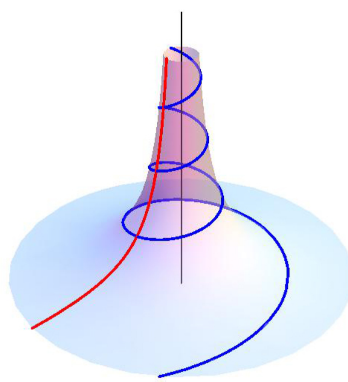

Figure 10: 3D hyperbolic spiral

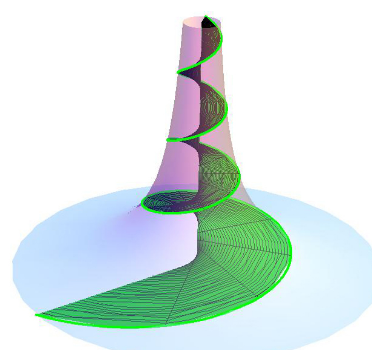

Figure 11: A hyperbolic helicoid

In this case $f(z) = \dfrac{c\alpha}{z}$, $z \in (0,\infty)$ (red line in Figure 10) generates the surface of revolution around Oz-axis given by $x^2+y^2 = \dfrac{c^2\alpha^2}{z^2}$ (Figure 10, a rectangular hyperbola with horizontal/vertical asymptotes rotates around its vertical asymptote). The three-dimensional hyperbolic spiral (Figure 10) and the hyperbolic helicoid (Figure 11) are given by the following equations:

$$\begin{cases} x = \dfrac{\alpha}{t}\cos t \\ y = \dfrac{\alpha}{t}\sin t \\ z = ct \end{cases}, \ t \in (0,\infty) \quad \text{and} \quad \begin{cases} x = \alpha\dfrac{u}{v}\cos v \\ y = \alpha\dfrac{u}{v}\sin v \\ z = cv \end{cases}, \ (u,v) \in [0,1]\times(0,\infty).$$

Three-dimensional lituus spiral

Let us consider the lituus spiral (Figure 12) given by the equation

$$\rho = \frac{\alpha}{\sqrt{\theta}},\ \theta \in (0,\infty),$$

where α is a real numbers. In this case $f(z) = \frac{\alpha\sqrt{c}}{\sqrt{z}}$, $z \in (0,\infty)$ (red line in Figure 13) generates the surface of revolution around Oz-axis given by $x^2 + y^2 = \frac{c\alpha^2}{z}$ (Figure 13). The three-dimensional lituus spiral (Figure 13) and the lituus helicoid (Figure 14) are given by the following equations:

$$\begin{cases} x = \frac{\alpha}{\sqrt{t}}\cos t \\ y = \frac{\alpha}{\sqrt{t}}\sin t \\ z = ct \end{cases},\ t \in (0,\infty) \quad \text{and} \quad \begin{cases} x = \alpha\frac{u}{\sqrt{v}}\cos v \\ y = \alpha\frac{u}{\sqrt{v}}\sin v \\ z = cv \end{cases},\ (u,v) \in [0,1] \times (0,\infty).$$

Figure 12: A lituus spiral

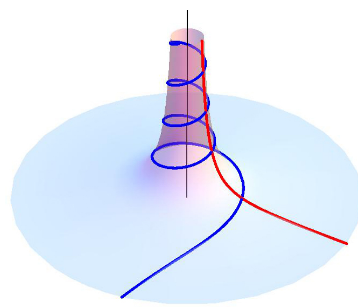

Figure 13: 3D lituus spiral

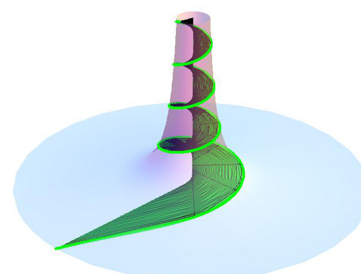

Figure 14: A lituus helicoid

Three-dimensional logarithmic spiral

The last considered planar spiral is the logarithmic spiral (Figure 15) described by the equation

$$\rho = \alpha e^{\beta\theta},\ \theta \in \mathbb{R},$$

where α and β are real numbers.

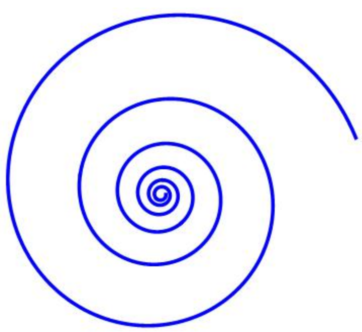

Figure 15: A logarithmic spiral

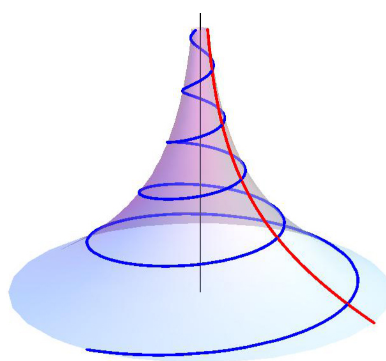

Figure 16: 3D logarithmic spiral

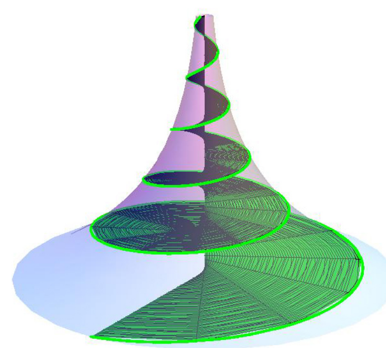

Figure 17: A logarithmic helicoid

In this case $f(z) = \alpha e^{\beta\frac{1}{c}z}$, $z \in \mathbb{R}$ (red line in Figure 16) and the surface of revolution around Oz-axis has the equation $x^2 + y^2 = \alpha^2 e^{\frac{2\beta}{c}z}$. The equations of three-dimensional logarithmic spiral (Figure 16) and logarithmic helicoid (Figure 17) are given by:

$$\begin{cases} x = \alpha e^{\beta t}\cos t \\ y = \alpha e^{\beta t}\sin t \\ z = ct \end{cases},\ t \in \mathbb{R} \quad \text{and} \quad \begin{cases} x = \alpha u e^{\beta v}\cos v \\ y = \alpha u e^{\beta v}\sin v \\ z = cv \end{cases},\ (u,v) \in [0,1] \times \mathbb{R}.$$

In the following we consider other important surfaces of revolution and we present the corresponding three-dimensional spirals (6) and ruled surfaces (8).

Spherical helicoid

The sphere of radius r is generated by the function $f : [-r, r] \to [0, r]$, $f(z) = \sqrt{r^2 - z^2}$ (red line in Figure 18). Then the spherical helical curve (Figure 18) and the spherical helicoid (Figure 20) are given by:

$$\begin{cases} x = \sqrt{r^2 - t^2}\cos\dfrac{t}{c} \\ y = \sqrt{r^2 - t^2}\sin\dfrac{t}{c} \\ z = t \end{cases}, \ t \in [-r, r] \quad \text{and} \quad \begin{cases} x = u\sqrt{r^2 - v^2}\cos\dfrac{v}{c} \\ y = u\sqrt{r^2 - v^2}\sin\dfrac{v}{c} \\ z = v \end{cases}, \ (u, v) \in [0, 1] \times [-r, r].$$

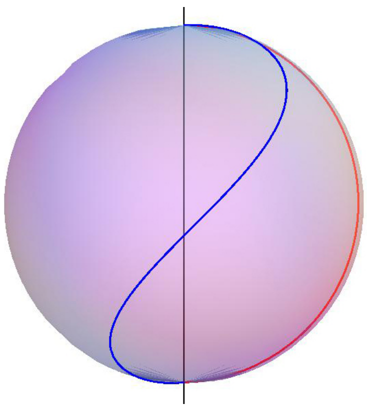

Figure 18: A spherical helical curve ($c = 1$)

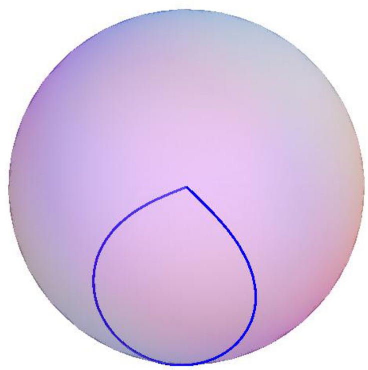

Figure 19: $x - y$ projection of the spiral ($c = 1$)

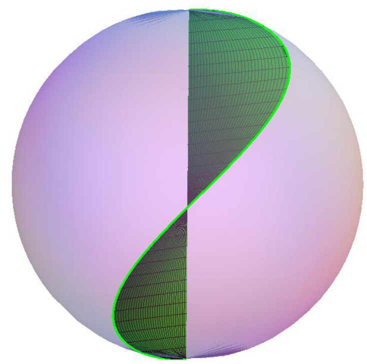

Figure 20: A spherical helicoid ($c = 1$)

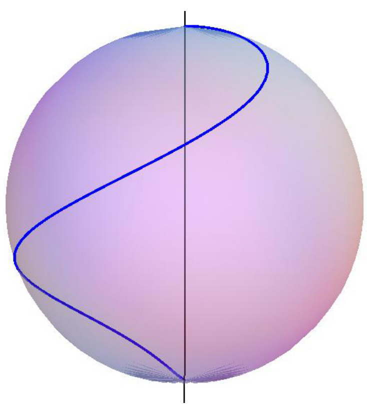

Figure 21: A spherical helical curve ($c = \frac{1}{2}$)

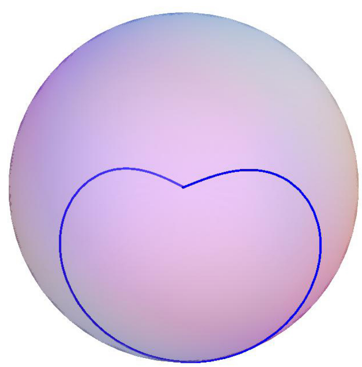

Figure 22: $x - y$ projection of the spiral ($c = \frac{1}{2}$)

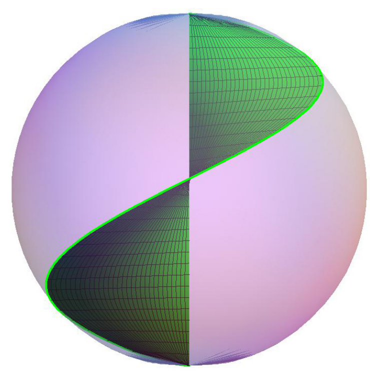

Figure 23: A spherical helicoid($c = \frac{1}{2}$)

Let us observe that if we take different values of c, then we obtain different shapes of the spherical helical curve (Figures 18,21,24). Moreover, the $x - y$ projections of these curves are different (Figures 19,22,25). The family of these curves, which reminds us of the Limacon of Pascal [2], is given by:

$$\begin{cases} x = \sqrt{r^2 - t^2}\cos\dfrac{t}{c} \\ y = \sqrt{r^2 - t^2}\sin\dfrac{t}{c} \end{cases}, \ t \in [-r, r] \tag{13}$$

or in polar coordinates

$$\rho = \sqrt{r^2 - c^2\theta^2}\ , \ \theta \in \left[-\frac{r}{c}, \frac{r}{c}\right], \tag{14}$$

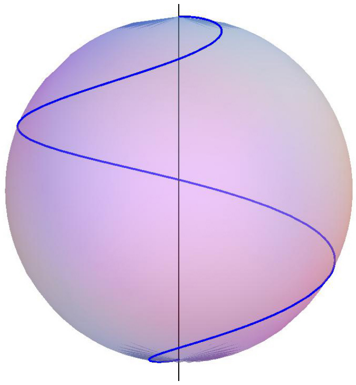

Figure 24: Spherical helical curve ($c = \frac{1}{4}$)

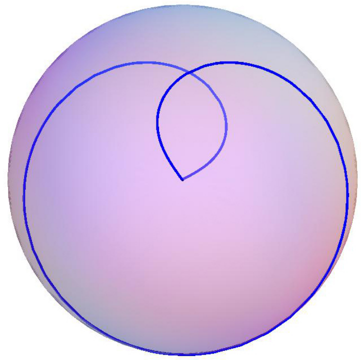

Figure 25: $x - y$ projection of the spiral ($c = \frac{1}{4}$)

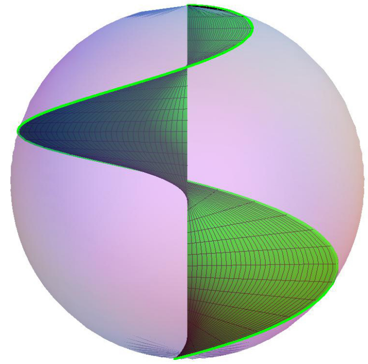

Figure 26: A spherical helicoid($c = \frac{1}{4}$)

where $c \in (0, \infty)$ is its parameter. In Figure 27 we overlap the shapes of the above curve in the cases $c = 1, c = \frac{1}{2}, c = \frac{1}{4}$. In Figures 28 and 29 the above curve in the case $c = \frac{1}{6}$, respectively $c = \frac{1}{8}$, is shown.

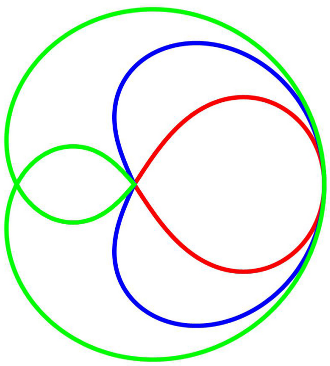

Figure 27: The curve (14): red ($c = 1$), blue ($c = \frac{1}{2}$), green ($c = \frac{1}{4}$)

Figure 28: The curve (14), $c = \frac{1}{6}$

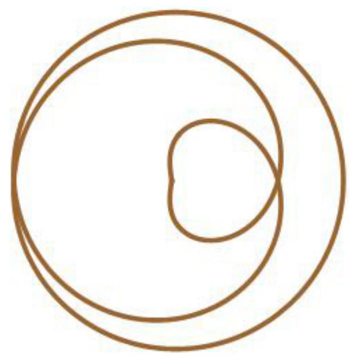

Figure 29: The curve (14), $c = \frac{1}{8}$

Hyperboloidal helicoid I

The one-sheeted hyperboloid $\dfrac{x^2}{p^2} + \dfrac{y^2}{p^2} - \dfrac{z^2}{q^2} = 1$ is a surface of revolution generated by the function $f : \mathbb{R} \to \mathbb{R}$, $f(z) = p\sqrt{\dfrac{z^2}{q^2} + 1}$ (red line in Figure 30), where p, q are positive numbers. Then the hyperboloidal helical curve I (Figure 30) and the hyperboloidal helicoid I (Figure 32) are given by:

$$\begin{cases} x = p\sqrt{\dfrac{t^2}{q^2} + 1}\cos\dfrac{t}{c} \\ y = p\sqrt{\dfrac{t^2}{q^2} + 1}\sin\dfrac{t}{c} \\ z = t \end{cases}, \ t \in \mathbb{R} \quad \text{and} \quad \begin{cases} x = pu\sqrt{\dfrac{v^2}{q^2} + 1}\cos\dfrac{v}{c} \\ y = pu\sqrt{\dfrac{v^2}{q^2} + 1}\sin\dfrac{v}{c} \\ z = v \end{cases}, \ (u, v) \in [0, 1] \times \mathbb{R}.$$

The $x - y$ projection of the hyperboloidal helical curve I in the case $z \in [0, \infty)$ (Figure 31) is a planar

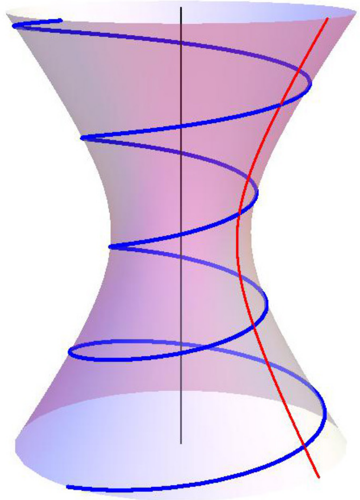

Figure 30: A hyperboloidal helical curve I

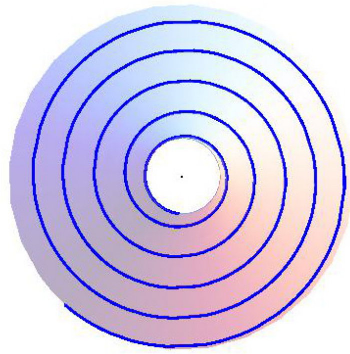

Figure 31: $x-y$ projection of the spiral for $z \in [0, 10\pi]$, $c=1$

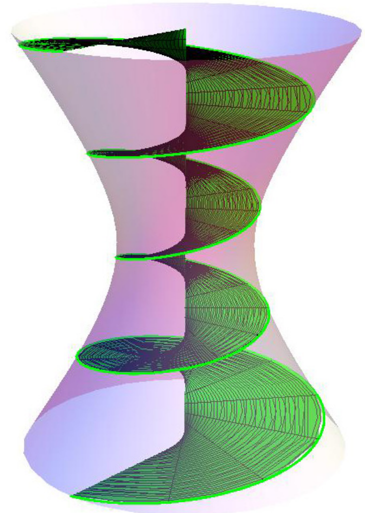

Figure 32: A hyperboloidal helicoid I

spiral which looks similar with an Archimedean spiral. Its equation is given by:

$$\begin{cases} x = p\sqrt{\dfrac{t^2}{q^2}+1}\cos\dfrac{t}{c} \\ y = p\sqrt{\dfrac{t^2}{q^2}+1}\sin\dfrac{t}{c} \end{cases}, \ t \in [0,\infty) \tag{15}$$

or in polar coordinates

$$\rho = p\sqrt{\frac{c^2\theta^2}{q^2}+1}\ ,\ \theta \in [0,\infty). \tag{16}$$

Hyperboloidal helicoid II

The two-sheeted hyperboloid $\dfrac{x^2}{p^2}+\dfrac{y^2}{p^2}-\dfrac{z^2}{q^2}=-1$ is a surface of revolution generated by the function $f:(-\infty,-q]\times[q,\infty)\to\mathbb{R}$, $f(z)=p\sqrt{\dfrac{z^2}{q^2}-1}$ (red line in Figure 33, in the case $z\in(-\infty,-q]$), where p,q are positive numbers. Considering the case when $z\in(-\infty,-q]$, the hyperboloidal helical curve II (Figure 33) and the hyperboloidal helicoid II (Figure 35) are given by:

$$\begin{cases} x = p\sqrt{\dfrac{t^2}{q^2}-1}\cos\dfrac{t}{c} \\ y = p\sqrt{\dfrac{t^2}{q^2}-1}\sin\dfrac{t}{c} \\ z = t \end{cases}, \ t\in(-\infty,-q] \quad \text{and} \quad \begin{cases} x = pu\sqrt{\dfrac{v^2}{q^2}-1}\cos\dfrac{v}{c} \\ y = pu\sqrt{\dfrac{v^2}{q^2}-1}\sin\dfrac{v}{c} \\ z = v \end{cases}, \ (u,v)\in[0,1]\times(-\infty,-q].$$

As we see in Figure 34, the $x-y$ projection of the hyperboloidal helical curve II in the case $z\in(-\infty,-q]$ is a planar spiral which looks similar with an Archimedean spiral. Its equation is given by:

$$\begin{cases} x = p\sqrt{\dfrac{t^2}{q^2}-1}\cos\dfrac{t}{c} \\ y = p\sqrt{\dfrac{t^2}{q^2}-1}\sin\dfrac{t}{c} \end{cases}, \ t\in(-\infty,-q] \tag{17}$$

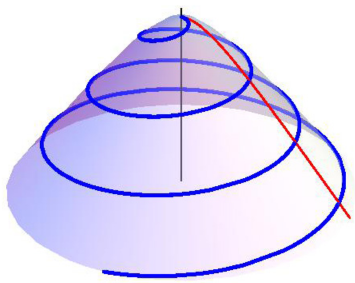

Figure 33: A hyperboloidal helical curve II

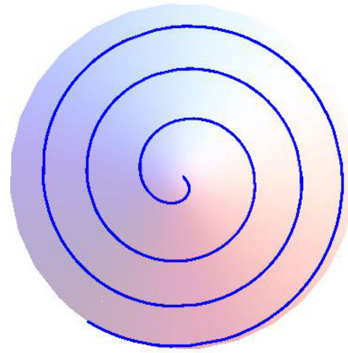

Figure 34: $x - y$ projection of the spiral for $z \in [0, 10\pi]$, $c = 1$

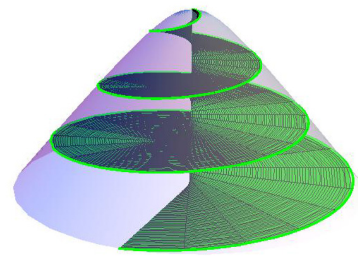

Figure 35: A hyperboloidal helicoid II

or in polar coordinates

$$\rho = p\sqrt{\frac{c^2\theta^2}{q^2} - 1}\ ,\ \theta \in (-\infty, -q]. \tag{18}$$

Catenoidal helicoid I

The last example analysed in present paper is given by a hyperbolic function, namely $f : \mathbb{R} \to \mathbb{R}$, $f(z) = p\cosh(qz)$ (red line in Figure 36), where p, q are positive numbers. It generates a surface of revolution called (generalized) catenoid: $x^2 + y^2 = p^2\cosh^2(qz)$. The catenoidal helical curve (Figure 30) and the catenoidal helicoid (Figure 32) are given by:

$$\begin{cases} x = p\cosh(qt)\cos\dfrac{t}{c} \\ y = p\cosh(qt)\sin\dfrac{t}{c} \\ z = t \end{cases},\ t \in \mathbb{R} \quad \text{and} \quad \begin{cases} x = pu\cosh(qv)\cos\dfrac{v}{c} \\ y = pu\cosh(qv)\sin\dfrac{v}{c} \\ z = v \end{cases},\ (u, v) \in [0, 1] \times \mathbb{R}.$$

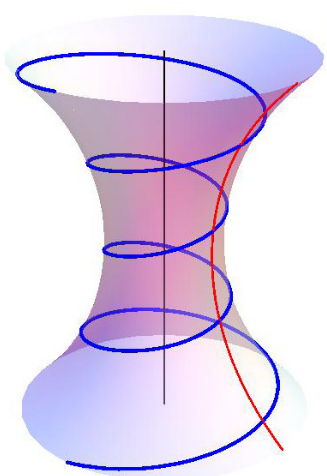

Figure 36: A catenoidal helical curve

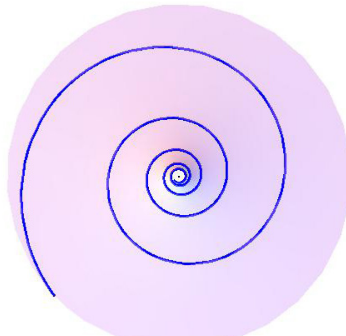

Figure 37: $x - y$ projection of the spiral for $z \in [0, 10\pi]$, $c = 1$

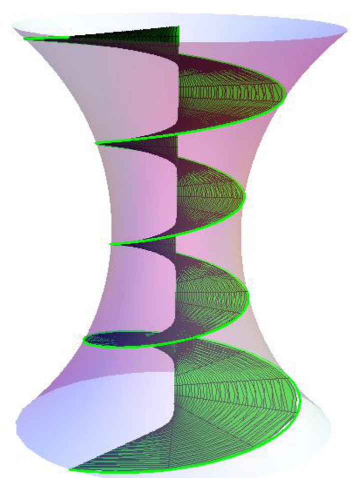

Figure 38: A catenoidal helicoid

The $x - y$ projection of the catenoidal helical curve in the case $z \in [0, \infty)$ (Figure 37) is a planar spiral which looks similar with a logarithmic spiral. Its equation is given by:

$$\begin{cases} x = p\cosh(qt)\cos\dfrac{t}{c} \\ y = p\cosh(qt)\sin\dfrac{t}{c} \end{cases} \quad , \ t \in [0, \infty) \tag{19}$$

or in polar coordinates

$$\rho = p\cosh(qc\theta) \quad , \ \theta \in [0, \infty). \tag{20}$$

References

[1] Gray, A., *Modern Differential Geometry of Curves and Surfaces with Mathematica, 2nd ed.* Boca Raton, FL: CRC Press, 1997.

[2] Yates, R.C., *A Handbook on Curves and Their Properties*, Ann Arbor, MI: J.W.Edwards, 1952.

[3] Ferréol, R., http://www.mathcurve.com/courbes3d/spiraleconic/pappus.shtml