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DIDACTICAL ANALYSIS OF PRIMARY GEOMETRIC CONCEPTS

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Abstract. Inspired by Freudenthal's didactical phenomenology, in this and a series of subsequent papers, didactical analysis of geometric concepts in primary school programmes is undertaken.

In this paper, spontaneous concept of shape is reflected historically, covering shortly handworks and decorations of human beings from the far archaeological past. Some samples of inherent geometrical ideas from ethnomathematics are also included. At the end, the birth of the first Greacian schools is touched and the beginnings of science are reflected.

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1. Introduction

The spontaneous concept of shape is used in a quite diffuse way and it is impossible to define mathematically all varieties of its meaning. For example, two triangles may have different shapes when the meaning is taken as totality of all Euclidean properties and, in the same time, they may be considered as having the same triangular shape. Not to speak of the meaning associated with the phrases as "shape of a boot", "shape of a pointnig finger", etc. Nevertheless, in geometry at the bottom level, recognition by shape as a whole is the main activity which leads to the formation of the intuitive meaning of some fundamental geometric concepts. By modelling and drawing, the formed geometric ideas are materialized and, as regards cognition, a child in primary school undergoes a process similar to that human underwent in the remote prehistoric past.

Planning to undertake a didactical analysis of primary school geometry concepts in this and subsequent papers, first we cover shortly handworks from the far archaeological past, made by beings belonging to the human genus. (And for a student at an educational institution, an excellent complement to the reading of this paper should be a visit to an archaeological museum.)

At the early stage, the geometric ideas of school children are inherent (in the real world objects) what is also the case with the peopel living in still untouched primitive civilaztions. That is why we also include some samples from ethnomathematics.

The rest of the paper touches the birth of the first Greacian schools, abstract comprehension of mathematical concepts and the beginnings of science.

2. The remote past when geometry begins

Where and when the first handworks were made, why and how they were created are the questions which anthropology and archaeology try to answer. Beings from the remote past who knew to select smooth, round or flat, pieces of stone and to use them for some purposes certainly belong to the genus *Homo*. These, not hand-shaped objects of stone are called manuports and they can be considered as the earliest human beings' conveyors of geometric meanings.

Excavations in Harar, in present day Ethiopia bring to light that these human beings who knew how to dress and shape stone lived some 3 million years ago. The oldest human dwellings discovered along Olduvai Gorge, today in South-East Tanzania are dated back to 1.9–1.5 million years. A circular construction of stone was dug up there as well as tools made of petrified lava and quartz. The science of human origins classifies the human beings from this remote period as *Homo habilis* whose activity demonstrates that their thinking processes were leading to the ideas of circular and spherical shapes.

Some 1.6 million years ago, also in the southern regions of Asia and Europe, a type of human species emerged whose physical characteristics were close to those of modern human beings (e.g., height 140–160 cm and volume of brain 800–1200 cm^3).

Science classifies this type of being as *Homo erectus*. This being thrived in different climatic zones and natural ambiences and searched for new materials that were harder and could be cut more regularly, in particular for flint. Besides different tools having convex, concave or straight blades the most characteristic shape was the hand wedge that often had a considerable aesthetic value. With a length over 40 cm and a weight of several kilograms, these finely manufactured tools could serve as the signs of power and authority, mighty fetishes or cult instruments.

Fig. 1. The Regions of the Old World which were populated by Homo erectus

When cutting stone, *Homo erectus* succeeded in transforming showers of sparks into burning flames, that is, into fire. Pebbles found arranged in the centres of their dwellings used to be the fireplaces that unified the groups, enriched intellectual and emotional life and stimulated verbal communication. In a period from 300 000 to 120 000 years ago, this human being evolved into a new anthropological type called *Homo sapiens*, who also created a new culture.

The development of archaic *sapiens* diverged in two directions, in Asia, towards the type more close to the modern human being and, in Europe, towards a specific human species called the Neanderthal man. This type of being established cults of the dead as a foundation of religious systems and was the constructor of the first sacral objects. Besides more and more symmetric spherical tools of stone, tools in conical form have been found in their dwellings. Clods of mineral colours found in archaeological digs suggest that the Neanderthal beings used them for colouring tools and decoration of their bodies.

The modern human species (*Homo sapiens sapiens*) emerges in the period from 100 000 to 50 000 years ago. Colonizing Europe some 30 000 years ago, modern human beings drove away the Neanderthal man who slowly became extinct. This long period which is characterized by finely made tools and weapons of stone is divided into Palaeolith (early part of the Stone Age) lasting until 10 000 years B.C., Mesolith (middle part) until 6500 and Neolith (later part of the Stone age) until 3000 years B.C. The period to the present is taken as the time of historical civilizations.

Besides fashioning a large variety of shapes in stone and using colours, the Palaeolithic human beings produced a rich world of visual forms found as paintings on the cave walls. The thought pattern of this species accompanied by mental imagery give evidence of high intelligence.

In the following picture

Fig. 2. (taken from `www.uncg.edu/rom/courses/dafein/civ/timeline.htm`)

the ceiling of the cave Lascaux (Dordogne), France, is seen with the figures of horses and cows painted in red and black.

The Mesolithic human species produced different tools of stone of small size, with handles of wood or bone, shaped in the form of rectangles, trapezoids, rhomboids or circular segments. The human spices from Neolith discovered a new technique of whetting and smoothing stone, using quartz sand and in doing so produced hemispherical dishes, grindstones and wheels. In building sacred objects, pieces of stone were cut into blocks shaped as quaders (parallelepipeds).

In many parts of the world, excavations from the Mesolithic period contain a lot of remains of ceramics which is evidence for an even richer world of forms. Especially rich geometric contents are seen in the form of decorations on ceramic objects: parallel and perpendicular lines, equilateral triangles, circles, series of lines repeating rhythmically and many other plane geometrical shapes.

The power of this human species to imagine different shapes, to produce them as solid objects or to represent them pictorially manifests itself as the fundamental geometric ability present to a bigger or less extent from prehistory to the present. And those remote human beings, were they producers of goods, sculptors, painters or geometers? It is best if we say, they were all of those.

3. Shape and the real world objects

There are so many objects in our surrounding environment whose visual manifestations are conceived as pure geometric shapes. We can think of several kinds of spherical fruits, cylinder-like trunks of trees, circular crowns of flowers and the structure of some kinds of minerals, full of regularity and symmetry, called crystals, to name but a few. There is no need to add to this list all such objects made by contemporary man, surrounding us everyday. These objects, the things we use, have their own names and a child, (or grown up person), in contact with them, develops abstract geometric ideas which correspond to notions of geometry.

It is often heard that a thing has the shape of boot, a cook, a flask, a pointing finger and so on. Such shapes stay as inseparable properties of the perceived objects and, as such, they are not abstracted as geometric notions. These shapes are said to be inherent, so these are distinguished from pure geometric shapes. For human beings from prehistory, each shape was inherent, what also is the case with a child at some stage of development, as well as it is so for people who live within still untouched, primitive civilizations.

Paulus P. J. Gerdes, in his paper [1], reports on attractive ground designs executed by the Tchokwe people of Northeast Angola or Tamils in South India. Their pictograms, besides being beautiful creations, are also complex constructions for which, it is often hard to find the underlying geometric algorithm. These are the subject of research in the area of so-called ethnomathematics.

In Figure 3, the Tchokwe sand drawing represents the unity of a married couple. It is composed of two symmetrically positioned, overlapping curves. The symmetry of the design expresses a “social symmetry”.

Fig. 3

During the harvest month, Tamil women draw designs in front of the thresholds of their houses. Such a design is seen in the following figure

Fig. 4

as being evidently composed of three overlapping continuous curves. (Both figures are taken from [1]). There exist many other examples exhibiting magical art which, we could say, is a primitive science.

Through planned activities or play, a child identifies or distinguishes objects according to their shape, so gathering pregeometric ideas. The environment in which he or she lives and the variety of forms of objects existing there, influences essentially the development of his or her visual representations and spatial relations which are formed in the mind. Speaking of ontogenetic development, early childhood is the period when some fundamental geometric ideas begin to be created. At this point in development, such ideas are associated with the objects of material world, which are conveying them. The way from these rudimentary ideas to fully developed concepts of geometry is not spontaneous and direct and such a progress is only possible as a result of learning.

The interesting book of R. Pinxten, ([2]), treats the ambient conditions in which a preschool child grows and develops indigenous spatial perceptions. The context, studied by Pinxten, is the one in which Navajo Indians in Southwest of North America live (or some of them still live). According to their system of beliefs, a moving spirit exists in all things and thus, their representations of reality consist of events, changes, processes and much less of objects and states. This essentially dynamical view of the world has resulted in their language consisting predominantly of verbs and constructions derived from verbs. Our Indo-European languages are noun-forming with nouns denoting things and their classes and therefore, our pregeometry is different.

4. The ancient times when abstract geometry begins

All things were in chaos when mind arose and made order.
Anaxagoras

In the historical civilizations of Babylon and Egypt, geometry was the knowledge needed for practical purposes of measuring time, areas of plots of land or volumes of some objects. The rules describing these procedures were given rhetorically (in words) with specific numbers illustrating the method in general. Without being exact, these rules determined only approximate values and the relation with the kinds of measured objects (say, shapes of lots of land) was not clearly fixed. Let us illustrate it by examples taken from Egyptian geometry.

On the walls of the Temple in Edu, the lots of land of quadrilateral shape are drawn as donations to the sacred place. If we use the contemporary, algebraic notation and take pairs of numbers a, b and c, d to denote lengths of pairs of opposite sides of a quadrilateral lot, again using contemporary ways of symbolic expression by means of formulas, its area is given by

$$A = \frac{a+b}{2} \cdot \frac{c+d}{2}$$

which becomes less and less accurate as the shape of the lot deviates more and more from the rectangular one. Contrary to this example, we could really be impressed with the accuracy with which the ancient Egyptians new how to determine the volume of a truncated pyramid with square bases. Their rule of calculation, translated into symbolic language, gives the formula

$$V = \frac{h}{3}(a^2 + ab + b^2),$$

where a and b are the lengths of sides of the two bases and h is the height of the pyramid. This is a correct formula as we know it. These examples illustrate connection between geometric ideas and reality, existing in the ancient Egyptian Civilization. And in the spiritual world in which that man lived, a pyramid, as a sacred place, could mean more than the exact area of a lot of land.

In the epic period of the ancient Greek Civilization, as we know it from Homer's poems, the world was imagined as an enormous plate surrounded by a water channel, called *okeanos*, upon which the heavens lean. Some four centuries after Homer, the Greeks began to create a system of thoughts based on reason, which led to the discovering of things as they are, not as they seem to be. This change in their thought was essentially related to their interest to extend further mathematical knowledge gained from other surrounding cultures of that time. The Grecian dialectical mind brought the cause-effect relation in the centre of their reasoning and then, mathematics became a closed system containing causes of its own facts. And so, the logical thinking began.

In the manuscripts belonging to ancient civilizations other than Greek, mathematical facts were only stated. In the Hindu manuscripts the statements are also accompanied by corresponding drawings with a "Behold!", just to activate the readers mind.

Thales of Miletus (625–545, B.C.) was the first man in history to produce a mathematical proof. Namely, he proved that an angle inscribed in a semicircle is a right angle. Having an exclusive place in the history of science, this proof will be exposed here in a form supposed to correspond to the original version.

Thales uses the fact that the angles of a triangle make one straight or, what is the same, two right angles. (Probably this fact had been already known before him).

Fig. 5

(The angles 1 and $1'$, 2 and $2'$ are equal.)

He also postulates (takes as evident) the fact that the base angles of an isosceles triangle are equal.

Fig. 6

To point out why this fact is acceptable by itself, Thales observes that such a triangle, when turned around its centre line, coincides with itself, as well as two angles, 1 and $1'$, do each to the other.

Here is how the proof goes (and we suggest you to follow it, looking at Fig. 7).

Fig. 7

The segments OA, OB and OC are equal, being the radii of the circle. The base angles 1 and $1'$ of the isosceles triangle AOC are equal, as are the angles 2 and $2'$ of the triangle BOC. The angles $1'$ and $2'$ make the angle with the vertex C of the triangle ABC, with 1 and 2 being two of its other angles. Together, they make two right angles. Since the pairs of angles 1, 2 and $1'$, $2'$ make equal angles, each of which will be equal to a right angle. Therefore, the angle with the vertex C is a right angle.

A conscious recognition of the fact that mathematical objects are abstractions, ideas formed by the mind and sharply distinguished from the physical objects, is attributed to Pythagoras (582–497, B.C.) and his followers. Having learned from Thales, he founded his own school in ancient Tarentum (today Taranto, in Sounth-East Italy). Since no written works by the Pythagoreans have been left, we know about them through writings of the other Greek thinkers. Existing through his thoughts and teachings, Pythagoras was made into a legendary semi-god of science and philosophy.

Certainly, the most famous of all mathematical theorems is the Pythagorean theorem which states that in a right triangle, the area of the square constructed on the hypotenuse is equal to the sum of areas of the squares constructed on the two legs.

Fig. 8

(Area of C is equal to the sum of areas of A and B.)

There exist hundreds of different proofs of this theorem. Which was the original one is not known and we include here a proof which is easy to understand.

Take two contiguous squares with the side length equal to the sum of the leg lengths of the triangle T in Figure 8. In each of them, construct four triangles congruent to T, as it is exhibited in Figure 9.

Fig. 9

Removing the first four of these triangles (1, 2, 3 and 4) from the first and the other four (5, 6, 7 and 8) from the second square, equal areas remain, the former being the area of the square on the hypotenuse and the letter the sum of the areas of squares on the legs of the triangle T.

The function of geometric drawings is twofold: they serve as simple graphical representations of some real worlds objects or as iconic signs whereby we represent our own mental images. In prehistory and generally the period preceding the ancient Greek Civilization, the former function was the only conscious act.

With Pythagoras and his disciples, geometric concepts started to be conceived abstractly and the drawn figures represented ideas rather than objects. That was a big step towards a new system of thought which we call science. Something can be absolutely true or accurate only when dealing with ideal objects. These are attributes which are often assigned to mathematics.

In his excellent book of short essays [3], Ernesto Sabato writes about Pythagoras and his discovery of "the eternal world of numbers and forms". There he poetically says: "Under the sunny sky of Calabria, it was Pythagoras' powerful mind first to discern that *topus uranium*" (place in the heavens).

When a material thing was measured at the time of Pythagoras, it was usually compared with another one of smaller size (called the unit of measurement) and the result of such comparison was a whole number. When two things of the same kind were measured by a unit, two whole numbers resulted. Then, the comparison of their sizes was expressed as the ratio of these numbers. The smaller the unit, the more accurate is the measuring and, up to a reasonable threshold, each measurement is sufficiently accurate.

At that ancient time, induced from practical activities of measuring, it was also supposed as a matter of course that for two geometric magnitudes, say, for two line segments, there exists a third one being their common measure, which is a whole number of times contained in each of these magnitudes. Thus, the ratios of whole numbers appeared to be adequate to express comparison of any two geometric magnitudes. In addition, in the philosophical system of Pythagoras numbers were considered as being the building elements of everything.

When the existence of a pair of line segments that did not have a common measure was discovered by some Pythagoras' disciples, it caused a deep crises in the

Grecian system of thought. In particular, the quite abstract idea of number that Pythagoreans had developed, was ejected from the foundations of mathematics, and, probably, the beginnings of algebraic thinking were postponed for quite some time.

The phenomenon of existence of a pair of segments without a common measure, technically called incommensurable, deserves our attention and we will provide here a quite intelligible proof.

Take a square $ABCD$ and consider the pair of segments AB, AC (a side and a diagonal of the square). Suppose AB and AC have a common measure KL. Then, $AB = m \cdot KL$, $AC = n \cdot KL$, where m and n are two whole numbers. Evidently, $m < n$, and, from $AB + BC > AC$, it also follows that $2m > n$. Take the point E on AC such that $AB = AE$ (see Figure 10). Let F be the point on BC such that EF is perpendicular to AC. The triangle FEC is isosceles (has two angles equal to $45°$), and $EC = EF$. Draw the segment AF.

Fig. 10

The triangles ABF and AEF are congruent (both right angled, $AE = AB$ and AF is their common side). Hence, $EF = BF$. Construct the square $EFGC$ on the side EC and call it the square associated with the square $ABCD$. Then

$$EC = AC - AE = (n - m) \cdot KL,$$
$$FC = BC - BF = m \cdot KL - (n - m) \cdot KL = (2m - n) \cdot KL.$$

Put $n - m = m_1$, $2m - n = n_1$. Then the side and the diagonal of the associated square have also KL as their common measure and

$$EC = m_1 \cdot KL, \quad FC = n_1 \cdot KL, \quad \text{with} \quad m > m_1 \text{ and } n > n_1.$$

The process of associating smaller and smaller squares does not terminate but produces two infinite sequence of smaller and smaller numbers

$$m > m_1 > m_2 > \cdots,$$
$$n > n_1 > n_2 > \cdots.$$

But, this is impossible since there exist at most $m - 1$ natural numbers smaller than m (and $n - 1$ numbers smaller than n).

This contradiction proves that the side and the diagonal of a square cannot have a common measure.

Since in an endless process, only ideal objects can be imagined as being infinitely divided into smaller and smaller parts, the above proof is indeed a reliable indicator that Pythagoreans conceived geometry in an abstract way.

From the point of view of the modern conception of numbers, we react to the phenomenon of incommensurable segments as, historically, the first proof of the existence of an irrational number. Namely, if d is the length of the diagonal and a is that of the side of a square, then the relation $d = a\sqrt{2}$ holds, and the ratio of these two magnitudes is the irrational number $\sqrt{2}$, which is not equal to the ratio of any two whole numbers. So we could say that this story brought about the discovery of $\sqrt{2}$.

For the Greeks, mathematical laws were the essence of the design of the universe and knowledge how the human mind works. To support this claim, let us quote the words of a famous, fifth century Pythagorean, Philolaus who says: "Were it not for number and its nature, nothing that exists would be clear to anybody, either in itself or in its relation to other things."

To compare the apparent, vague and fantastic Homerian epic image of the world with the real, determinate and intelligible scientific one from later period of the ancient Greek Civilization, consider the way how Erathosthenes of Cyrene (284–192, B.C.) calculated the length of the circumference of the earth.

The later Pythagoreans were the first to believe that the earth was spherical and with such a supposition Erathosthenes begins his calculation. Alexandria and Syene (a city today called Aswan) are almost on the same meridian. At noon on the summer solstice (June 21 in the northern hemisphere), the sun was observed to mirror in the water of a very deep well in Syene. At the same time, in Alexandria, the angle between a vertical stick and the sun rays was 1/50 of full angle. (Two legs of the right triangle make the stick and its shadow.) In Figure 11, the angle AOS is also 1/50 of full angle and hence, the arc AS is 1/50 of the circumference of the earth. (Being the sun so far from the earth, the falling rays are practically parallel.)

Fig. 11

The camel trains were estimated to travel 100 stadia a day and they took 50 days to reach Syene from Alexandria. Hence. the length of the arc AS was 5000 stadia, and, therefore, the circumference of the earth was 250 000 stadia. Originally a stadium was a paved, semicircular path build in front of antique amphitheatres and then, the unit of length estimated to be approximately 157 meters. Though based on somewhat imprecise estimations, the accuracy of Eratosthenes' calculation is quite impressive.

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