THE TEACHING OF MATHEMATICS
2024, Vol. XXVII, No. 2, pp. 79–103
DOI: 10.57016/TM-NXML6989

MIXING PROBLEMS REPRESENTED BY QUASI-DIGRAPHS

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Abstract. A notable class of problems often employed in undergraduate courses on differential equations is that of mixing problems: those involving a number of brine-filled tanks equipped with a number of brine-transporting pipes. Closed mixing problems, which feature neither filler nor drainer pipes, have been studied on a general level, where the flow networks are represented by digraphs [A. Slavík, *Mixing problems with many tanks*, American Mathematical Monthly, 120 (2013), 806–821]. In this paper, we extend the study to open mixing problems, which may feature filler and/or drainer pipes, representing the flow networks by a generalization of digraphs: quasi-digraphs. We formulate sufficient conditions under which such a mixing problem can be modeled as a system of linear ordinary differential equations whose coefficient matrix is the negative of the transpose of the Laplacian of the associated quasi-digraph. Subsequently, we formulate the analogues for mixing problems represented by weighted quasi-digraphs, and by cascade-type multilayer weighted quasi-digraphs. At the end of this paper, we propose suggestions for instructors on how our materials could be distilled to form a set of taught materials or a mini-project enriching an undergraduate differential equations course.

MathEduc Subject Classification: I45

AMS Subject Classification: 97I40

Key words and phrases: Differential equation; mixing problem; digraph; quasi-digraph; multilayer.

1. Introduction

Undergraduate courses on ordinary differential equations often involve the study of modest continuous-time mathematical models of real-world problems. A notable class of such problems is that which involves a number of brine-filled tanks, equipped with a number of pipes which allow not only tank-to-tank flows but also the filling and draining of some or all of the tanks. Typically, the task is to determine the mass of salt in each tank as a function of time, and subsequently the long-time limits of these masses.

Such problems, referred to as *mixing problems*, are featured in numerous textbooks on ordinary differential equations. For instance, the well-known book by Boyce, et al. [5] introduces single-tank mixing problems within a section on linear equations [5, Sec. 2.3] while employing multi-tank problems as several exercises in sections on linear systems [5, Sec. 7.1, 7.5]. On the other hand, the book by Edwards and Penney [7], which also introduces single-tank mixing problems within a section on linear equations [7, Sec. 1.5], employ multi-tank problems more substantially: not only in exercises but also along the extensive discussions on linear systems [7, Sec. 5.1, 5.2, 5.4, 5.8]. Finally, the book by Polking, et al. [9] introduces single-tank mixing problems in a separate section [9, Sec. 2.5] and, as [7], employs a

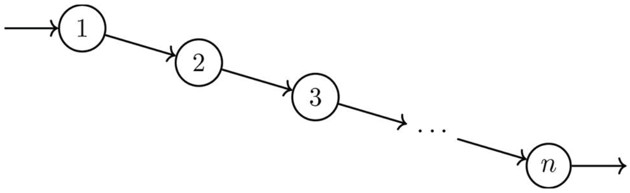

Fig. 1. A cascade of n tanks 1, ... , n, as studied in [11, Sec. 6]

substantial variety of multi-tank problems in the discussions and exercises on linear systems [9, Sec. 8.5, 9.2, 9.4–9.6, 9.9].

In 2013, Slavík [11], after compiling and solving a collection of multi-tank mixing problems [11, Sec. 2–4], formulated a general multi-tank problem, in which the existing tank-to-tank flows are specified in the form of a digraph [11, Sec. 5]. The author proved that, under the three-fold assumptions of constant and equal brine volumes in all tanks, constant and equal tank-to-tank flow rates, as well as a weakly connected digraph of flows, the salt masses in all tanks converge to the same long-term limit: the average of their initial values [11, Cor. 4]. Unfortunately, mixing problems represented by digraphs are closed, meaning that they involve neither the filling nor the draining of any of the tanks. Addressing this deficiency, Slavík subsequently considered a prototypical open mixing problem [11, Sec. 6]: *a cascade of tanks* (Fig. 1), and proved that, if the flow into the uppermost tank is that of salt-free water, then the salt masses in all tanks converge to zero for all initial values. Notice that the diagram in Fig. 1 is not precisely a digraph, since not every edge in the diagram joins exactly two vertices.

The purpose of the present paper is to advance the study of open mixing problems. As noted above, such problems are represented by digraph-like diagrams which allow an edge to be attached only to a single vertex. We shall refer to such diagrams as *quasi-digraphs*.

Our discussion is organized as follows. In the upcoming Section 2, we begin by recalling the standard definitions of a digraph and its Laplacian. We restate Slavík's sufficient conditions [11, p. 814] under which a mixing problem represented by a digraph with Laplacian $\mathbf{L}$ is modelable by the linear system $\mathbf{x}' = -\mathbf{L}^\top\mathbf{x}$. In the subsequent Section 3, we define a quasi-digraph, and define its Laplacian in such a way that, under similar sufficient conditions, a mixing problem represented by a quasi-digraph with Laplacian $\mathbf{L}$ is also modelable by the linear system $\mathbf{x}' = -\mathbf{L}^\top\mathbf{x}$. In Section 4, we consider mixing problems represented by weighted quasi-digraphs, again establishing sufficient conditions for the analogous modelability.

In Section 5, we consider mixing problems represented by a specific class of weighted quasi-digraphs: cascade-type multilayer weighted quasi-digraphs. We demonstrate that, under the stated sufficient conditions, the Laplacians of such quasi-digraphs are near-triangular, the diagonal blocks being precisely the Laplacians of the quasi-digraphs occupying the respective layers. In the final Section 6, we suggest two different ways for incorporating materials on mixing problems — including those discussed in [11] and in the present paper— into an undergraduate differential equations course, i.e., as a part of the course's taught materials and as a mini-project.

2. Mixing problems represented by digraphs

As announced in the previous section, we shall first revisit mixing problems represented by digraphs—whose vertices represent tanks and edges represent tank-to-tank flows—which have been studied in [11]. Let us begin by recalling the standard definition of a digraph. Throughout this paper, we shall only deal with finite digraphs.

Definition 1. [10, p. 103] A *digraph* on $\mathcal{V} = \{1, \dots, n\}$ is an ordered pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is a finite multiset of ordered pairs of elements of $\mathcal{V}$. The elements of $\mathcal{V}$ and of $\mathcal{E}$, respectively, are referred to as the *vertices* (or *nodes*) and the *edges* (or *links*) of the digraph.

For example, the ordered pair $(\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = \{1, 2, 3, 4\} \quad \text{and} \quad \mathcal{E} = \{(1, 2), (2, 3), (3, 4), (4, 1)\}, \tag{1}$$

is a digraph, which can be drawn as the diagram in Fig. 2: each vertex is drawn as a circle, and each edge (i, j) as an arrow from i to j. Every vertex in a digraph is characterized by two important quantities, referred to as the *in-degree* and *out-degree* of the vertex, defined as follows.

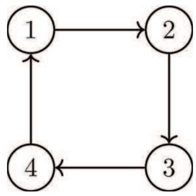

Fig. 2. The digraph $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are given in (1)

Definition 2. [10, p. 103] Consider a digraph on $\mathcal{V} = \{1, \dots, n\}$. Let $i \in \mathcal{V}$. The *in-degree* k_i^{in} of i is the number of edges going into i, while the *out-degree* k_i^{out} of i is the number of edges going out from i.

Therefore, for the digraph $(\mathcal{V}, \mathcal{E})$ in Fig. 2, we have that $k_i^{\text{in}} = k_i^{\text{out}} = 1$ for every $i \in \mathcal{V}$.

Let us next recall three key matrices associated to a digraph, i.e., its *out-degree matrix*, *adjacency matrix*, and *Laplacian*.

Definition 3. [6, p. xxix] Consider a digraph on $\mathcal{V} = \{1, \dots, n\}$. The *out-degree matrix* $\mathbf{D}$ of the digraph is the $n \times n$ diagonal matrix whose i-th diagonal entry is k_i^{out}. The *adjacency matrix* $\mathbf{A}$ of the digraph is the $n \times n$ matrix whose (i, j)-entry is the number of edges from i to j. The *Laplacian* of the digraph is the $n \times n$ matrix $\mathbf{L} := \mathbf{D} - \mathbf{A}$.

Remark 1. The analogous definition of the *in-degree matrix* of a digraph is omitted as it is not used in this paper.

The out-degree and adjacency matrices of the digraph in Fig. 2 are thus

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

so that the digraph's Laplacian is

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Let us now consider the mixing problem represented by the digraph in Fig. 2. Once again, vertices represent tanks, and edges represent tank-to-tank flows. As in [11, p. 814], we assume that all tanks maintain the same constant brine volume V, and that all tank-to-tank brine flows occur at the same constant rate f. Denoting by $x_i(t)$ the mass of salt in tank i at time $t \geqslant 0$, for every $i \in \{1, 2, 3, 4\}$, one finds that the problem is modeled by the linear system

$$\begin{cases} x_1'(t) = \dfrac{f}{V} x_4(t) - \dfrac{f}{V} x_1(t), \\ x_2'(t) = \dfrac{f}{V} x_1(t) - \dfrac{f}{V} x_2(t), \\ x_3'(t) = \dfrac{f}{V} x_2(t) - \dfrac{f}{V} x_3(t), \\ x_4'(t) = \dfrac{f}{V} x_3(t) - \dfrac{f}{V} x_4(t). \end{cases}$$

Letting $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^\top$, one rewrites the system as

$$\begin{aligned} \mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{pmatrix} &= \begin{pmatrix} -f/V & 0 & 0 & f/V \\ f/V & -f/V & 0 & 0 \\ 0 & f/V & -f/V & 0 \\ 0 & 0 & f/V & -f/V \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} \\ &= -\frac{f}{V} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}^\top \mathbf{x}(t) = -\frac{f}{V} \mathbf{L}^\top \mathbf{x}(t), \end{aligned}$$

where $\mathbf{L}$ denote the previously computed Laplacian (2) of the digraph in Fig. 2. The time-rescaling $\tau = (f/V)t$ transforms the model into

$$\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau), \tag{3}$$

which implies that the asymptotic behavior of the solutions of the above mixing problem can be determined by examining the eigenvalues of the Laplacian $\mathbf{L}$ of the associated digraph [9, Thm. 7.4]. Having this observation in mind, let us now turn our attention to quasi-digraphs.

3. Mixing problems represented by quasi-digraphs

As informally introduced, by a quasi-digraph we mean a structure resembling a digraph, except that some of its edges may be attached only to a single vertex. We shall refer to such anomalous edges as *quasi-edges*. By regarding the dangling endpoints of all quasi-edges as coinciding on a single fictitious vertex, let us formally define quasi-digraphs as follows.

Definition 4. A *quasi-digraph* on $\mathcal{V} = \{1, \dots, n\}$ is a digraph $(\mathcal{V} \cup \{0\}, \mathcal{E})$. Every edge containing the *fictitious vertex* 0 is called a *quasi-edge* of the quasi-digraph.

Thus, Fig. 1 depicts a quasi-digraph on $\{1, \dots, n\}$, with multiset of edges

$$\{(0,1), (1,2), (2,3), \dots, (n-1,n), (n,0)\},$$

where $(0,1)$ and $(n,0)$ are quasi-edges.

Our next goal is to define the in-degree and out-degree of a vertex in a quasi-digraph, and subsequently the out-degree matrix, adjacency matrix, and Laplacian of a quasi-digraph. We shall achieve this in such a way that mixing problems represented by a quasi-digraph, in which all tanks maintain the same constant volume and all flows occur at the same constant rate, is also modelable by the system (3). First, we define the in-degree and out-degree of a vertex in a quasi-digraph as follows.

Definition 5. Consider a quasi-digraph on $\mathcal{V} = \{1, \dots, n\}$. Let $i \in \mathcal{V}$. The *in-degree* k_i^{in} of i is the number of edges (including quasi-edges) going into i, while the *out-degree* k_i^{out} of i is the number of edges (including quasi-edges) going out from i.

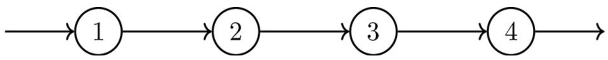

Fig. 3. The quasi-digraph in Fig. 1 in the special case $n = 4$

We could further define the out-degree matrix of the same quasi-digraph as the $n \times n$ diagonal matrix whose i-th diagonal entry is k_i^{out}. However, we shall state this formally later, for the sake of organization. Meanwhile, let us seek for suitable definitions of the adjacency matrix and the Laplacian of a quasi-digraph. For this purpose, consider the mixing problem represented by the quasi-digraph in Fig. 1 in the special case $n = 4$ (Fig. 3). In this case, the out-degree matrix is

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As before, let us assume a constant brine volume V in each tank and a constant brine flow rate f along each edge. Additionally, let us assume that the ingoing quasi-edge $(0,1)$ represents the flow of salt-free water. Then, the problem is modeled by the linear system

$$\begin{cases} x_1'(t) = -\dfrac{f}{V}x_1(t), \\ x_2'(t) = \dfrac{f}{V}x_1(t) - \dfrac{f}{V}x_2(t), \\ x_3'(t) = \dfrac{f}{V}x_2(t) - \dfrac{f}{V}x_3(t), \\ x_4'(t) = \dfrac{f}{V}x_3(t) - \dfrac{f}{V}x_4(t), \end{cases}$$

where, as before, $x_i(t)$ denotes the mass of salt in tank i at time $t \geqslant 0$, for every $i \in \{1,2,3,4\}$. Letting $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^\top$, we can rewrite the system as

$$\begin{aligned} \mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{pmatrix} &= \begin{pmatrix} -f/V & 0 & 0 & 0 \\ f/V & -f/V & 0 & 0 \\ 0 & f/V & -f/V & 0 \\ 0 & 0 & f/V & -f/V \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} \\ &= -\frac{f}{V} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\top \mathbf{x}(t) = -\frac{f}{V}\mathbf{L}^\top \mathbf{x}(t), \end{aligned}$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{D} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here we observe that

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is the adjacency matrix of the digraph in Fig. 4, which can be obtained from the quasi-digraph in Fig. 3 by merely *deleting all quasi-edges*. This motivates the following definition.

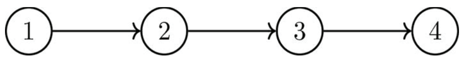

Fig. 4. The digraph obtained from the quasi-digraph in Fig. 3 by deleting all quasi-edges

DEFINITION 6. Consider a quasi-digraph on $\mathcal{V} = \{1, \dots, n\}$. The *out-degree matrix* $\mathbf{D}$ of the quasi-digraph is the $n \times n$ diagonal matrix whose i-th diagonal entry is k_i^{out}. The *adjacency matrix* $\mathbf{A}$ of the quasi-digraph is the $n \times n$ adjacency matrix of the digraph obtained by deleting all quasi-edges. The *Laplacian* of the quasi-digraph is the $n \times n$ matrix $\mathbf{L} := \mathbf{D} - \mathbf{A}$.

Noticing that linear systems of the form $\mathbf{x}'(t) = -(f/V)\mathbf{L}^\top \mathbf{x}(t)$ can be transformed into $\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau)$ via the time-rescaling $\tau = (f/V)t$, that all solutions of the latter system converge to its trivial fixed point if all eigenvalues of $-\mathbf{L}^\top$ have negative real parts [9, Thm. 7.4], and that the eigenvalues of $-\mathbf{L}^\top$ are the negatives of the eigenvalues of $\mathbf{L}$, we obtain the following theorem.

THEOREM 1. *Suppose that a mixing problem is represented by a quasi-digraph with Laplacian* $\mathbf{L}$, *and satisfies the following assumptions.*

- *All tanks maintain the same constant brine volume.*
- *Brine flows along all edges occur at the same constant rate.*
- *All ingoing quasi-edges represent the flow of salt-free water.*

Then the problem is modeled by the system $\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau)$. *Consequently, if all eigenvalues of* $\mathbf{L}$ *have positive real parts, then all solutions of the mixing problem converge to the state where all tanks are salt-free.*

REMARK 2. Notice that the first two assumptions in Theorem 1 necessarily imply that every vertex in the quasi-digraph has equal in-degree and out-degree. In addition, notice that Theorem 1 covers all mixing problems represented by digraphs, since every digraph is a quasi-digraph having no quasi-edges.

REMARK 3. For a mixing problem represented by a quasi-digraph on $\{1, \dots, n\}$ with quasi-edges $(0, 1), \dots, (0, n)$ representing the flow of brine with non-negative constant salt concentrations $c_1, \dots, c_n$, under the assumptions of a constant brine volume V in each tank and a constant brine flow rate along each edge, we have the system

$$\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau) + V\mathbf{c},$$

where $\mathbf{L}$ denotes the Laplacian of the quasi-digraph, and $\mathbf{c} = (c_1, \dots, c_n)^\top$.

Let us next study as an example a specific mixing problem represented by a quasi-digraph. To determine the asymptotic behavior of its solutions, we shall use the following theorem by Gershgorin.

THEOREM 2. [8, p. 420] *Let* $\mathbf{L}$ *be an* $n \times n$ *matrix with* (i, j)*-entry* $l_{i,j} \in \mathbb{C}$. *For every* $i \in \{1, \dots, n\}$, *let*

$$D(l_{i,i}, r_i) := \{z \in \mathbb{C} : |z - l_{i,i}| \leqslant r_i\}$$

be the closed disk centered at $l_{i,i}$ *with radius*

$$r_i := \sum_{\substack{j=1 \\ j \neq i}}^{n} |l_{i,j}|,$$

i.e., the sum of the absolute values of the i*-th row off-diagonal entries of* $\mathbf{L}$. *Then all eigenvalues of* $\mathbf{L}$ *belong to the set* $D(l_{1,1}, r_1) \cup \dots \cup D(l_{n,n}, r_n)$.

EXAMPLE 1. Consider the n-vertex version of the digraph in Fig. 2, and attach to each vertex in it an ingoing quasi-edge and an outgoing quasi-edge. The result is the quasi-digraph in Fig. 5. Consider the mixing problem represented by this quasi-digraph which satisfies the three assumptions in Theorem 1.

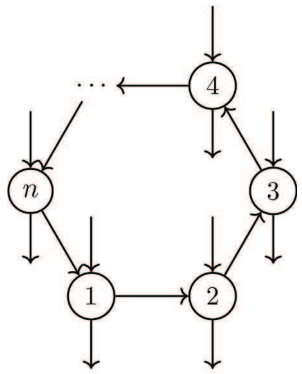

Fig. 5. An n-vertex quasi-digraph, obtained from the n-vertex version of the digraph in Fig. 2 by attaching to each vertex an ingoing quasi-edge and an outgoing quasi-edge.

The out-degree and adjacency matrices of the quasi-digraph are the $n \times n$ matrices

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

so that its Laplacian is the $n \times n$ matrix

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 2 \end{pmatrix}.$$

The sum of the absolute values of the i-th row off-diagonal entries of $\mathbf{L}$ is $r_i = |-1| = 1$. By Gershgorin's theorem, it follows that all eigenvalues of $\mathbf{L}$ belong to the set

$$\underbrace{\mathrm{D}(2,1) \cup \cdots \cup \mathrm{D}(2,1)}_{n} = \mathrm{D}(2,1),$$

which implies that all of these eigenvalues have positive real parts. Therefore, all solutions of the problem converge to the state where all tanks are salt-free. △

Our examples thus far employ *simple* digraphs, which feature neither multi-edges (two or more edges joining the same ordered pair of vertices) nor loops (edges from a vertex to itself). Clearly, the addition of loops into a digraph representing a mixing problem leads to no change in the associated system. Let us conclude this section with an example involving multi-edges.

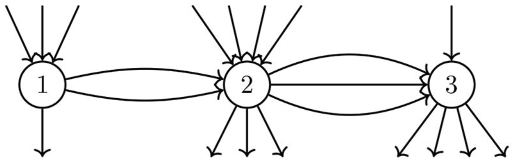

Fig. 6. A quasi-digraph featuring multi-edges

Example 2. Consider the mixing problem represented by the quasi-digraph in Fig. 6. Suppose that the problem satisfies the three assumptions in Theorem 1. The out-degree and adjacency matrices of the quasi-digraph are

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

so that its Laplacian is

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 6 & -3 \\ 0 & 0 & 4 \end{pmatrix}.$$

The eigenvalues of $\mathbf{L}$ are 3, 6, and 4, all of which have positive real parts. Thus, all solutions converge to the state where all tanks are salt-free. △

4. Mixing problems represented by weighted quasi-digraphs

Notice that the quasi-digraph in Fig. 6 can be represented as a weighted quasi-digraph in Fig. 7, in which the weight of an edge from vertex i to vertex j is specified to be the number of multi-edges from vertex i to vertex j in the original quasi-digraph. On the general level, let us define a weighted quasi-digraph as follows.

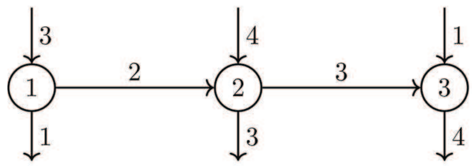

Fig. 7. A weighted quasi-digraph representing the quasi-digraph in Fig. 6

Definition 7. A *weighted quasi-digraph* is a quasi-digraph equipped with a *weight function*, i.e., a positive-valued function over its multiset of edges.

Thus, the weighted quasi-digraph in Fig. 7 is the quasi-digraph on $\mathcal{V} = \{1, 2, 3\}$ with multiset of edges

$$\mathcal{E} = \{(1,2), (2,3), (0,1), (0,2), (0,3), (1,0), (2,0), (3,0)\},$$

equipped with the weight function $w : \mathcal{E} \to (0, \infty)$ given by $w(1,0) = w(0,3) = 1$, $w(1,2) = 2$, $w(2,3) = w(0,1) = w(2,0) = 3$, and $w(0,2) = w(3,0) = 4$. Next, the in-degree and out-degree of a vertex in a weighted quasi-digraph can be defined as follows.

Definition 8. Consider a weighted quasi-digraph on $\mathcal{V} = \{1, \dots, n\}$. Let $i \in \mathcal{V}$. The *in-degree* k_i^{in} of i is the total weight of all edges (including quasi-edges) going into i, while the *out-degree* k_i^{out} of i is the total weight of all edges (including quasi-edges) going out from i.

For the weighted quasi-digraph in Fig. 7, we have that

$$k_1^{\text{in}} = 3, \qquad k_2^{\text{in}} = 2 + 4 = 6, \qquad k_3^{\text{in}} = 3 + 1 = 4,$$

and that

$$k_1^{\text{out}} = 1 + 2 = 3, \qquad k_2^{\text{out}} = 3 + 3 = 6, \qquad k_3^{\text{out}} = 4.$$

Notice that we have $k_i^{\text{in}} = k_i^{\text{out}}$ for every $i \in \{1, 2, 3\}$, since the weighted quasi-digraph in Fig. 7 represents the quasi-digraph in Fig. 6, which also satisfies $k_i^{\text{in}} = k_i^{\text{out}}$ for every $i \in \{1, 2, 3\}$, as stated in Remark 2. The definitions of the out-degree matrix, adjacency matrix, and Laplacian of weighted quasi-digraphs are derived immediately from those of unweighted quasi-digraphs (cf. [6, p. xxix]).

Definition 9. Consider a weighted quasi-digraph on $\mathcal{V} = \{1, \dots, n\}$. The *out-degree matrix* $\mathbf{D}$ of the weighted quasi-digraph is the $n \times n$ diagonal matrix whose i-th diagonal entry is k_i^{out}. The *adjacency matrix* $\mathbf{A}$ of the weighted quasi-digraph is the $n \times n$ matrix whose (i, j)-entry is the total weight of all edges from i to j in the weighted digraph obtained by deleting all existing quasi-edges. The *Laplacian* of the weighted quasi-digraph is the $n \times n$ matrix $\mathbf{L} := \mathbf{D} - \mathbf{A}$.

From the above definition of the Laplacian of a weighted quasi-digraph, it follows that we have the following analogue of Theorem 1 for mixing problems represented by weighted quasi-digraphs, in which the brine flow along each edge occurs at a rate which is proportional to the edge's weight.

Theorem 3. *Suppose that a mixing problem is represented by a weighted quasi-digraph with Laplacian* $\mathbf{L}$*, and satisfies the following assumptions.*

- *All tanks maintain the same constant brine volume.*
- *Brine flows along all edges occur at rates proportional to the weights of the respective edges.*
- *All ingoing quasi-edges represent the flow of salt-free water.*

Then the problem is modeled by the system $\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau)$*. Consequently, if all eigenvalues of* $\mathbf{L}$ *have positive real parts, then all solutions of the mixing problem converge to the state where all tanks are salt-free.*

Remark 4. As in the case of unweighted quasi-digraphs (Theorem 1), the first two assumptions in Theorem 3 necessarily imply that every vertex in the weighted quasi-digraph has equal in-degree and out-degree. In addition, notice that the above theorem also covers mixing problems represented by unweighted quasi-digraphs, since every unweighted quasi-digraph is a weighted quasi-digraph with unit-weight edges.

5. Mixing problems represented by cascade-type multilayer weighted quasi-digraphs

By specifying a partition $\{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ of its vertex set $\mathcal{V}$—a set of pairwise-disjoint non-empty subsets of $\mathcal{V}$ whose union is $\mathcal{V}$—every digraph can be viewed as a *multilayer digraph* [3, Sec. 5.2], where, for every $i \in \{1, \ldots, m\}$, the i-th layer is occupied by the digraph having vertex set $\mathcal{L}_i$ and edge set containing all edges joining vertices which are both in $\mathcal{L}_i$. Such edges are *intralayer edges*, meaning that they join vertices belonging to the same layer. By contrast, the other edges are *interlayer edges*, meaning that they join vertices belonging to two different layers.

Similarly, by specifying a partition of its vertex set, every quasi-digraph can be viewed as a *multilayer quasi-digraph*, and its non-quasi-edges can be classified as intralayer and interlayer edges. For instance, the partition $\{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$ of $\{1, \ldots, n\}$, where $\mathcal{L}_i = \{i\}$ for every $i \in \{1, \ldots, n\}$, makes Slavík's quasi-digraph in Fig. 1 an n-layer quasi-digraph, where every layer is occupied by a single-vertex quasi-digraph and every non-quasi-edge is an interlayer edge since it emanates from a vertex in $\mathcal{L}_i$ and terminates at a vertex in $\mathcal{L}_j$ where $i \neq j$ (Fig. 8). In fact, this n-layer quasi-digraph possesses a special property that every interlayer edge emanates from a vertex in $\mathcal{L}_i$ and terminates at a vertex in $\mathcal{L}_j$ where $i < j$, and it is precisely this property that makes it fitting for the tank configuration to be referred to as a *cascade* [11, Sec. 6]. Accordingly, let us define the following.

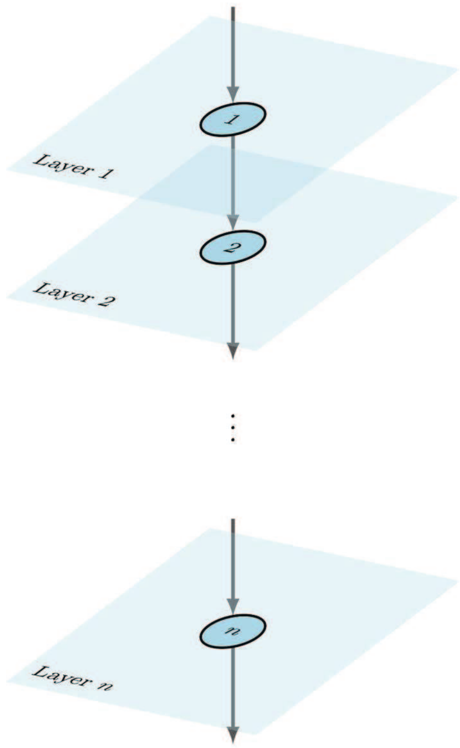

Fig. 8. The quasi-digraph in Fig. 1, viewed as a multilayer quasi-digraph

Definition 10. Let $\mathcal{D}$ be a (weighted) quasi-digraph on a vertex set $\mathcal{V}$.

- By specifying a partition $\{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ of $\mathcal{V}$, whose elements are to be called the *layers* of $\mathcal{D}$, one makes $\mathcal{D}$ an *m-layer* (*weighted*) *quasi-digraph*, and classifies each non-quasi-edge as either an *intralayer edge*, which joins two vertices from the same layer, or an *interlayer edge*, which joins two vertices from different layers.

- By the (*weighted*) *quasi-digraph occupying layer* $\mathcal{L}_i$ we mean the (weighted) quasi-digraph with vertex set $\mathcal{L}_i$ obtained by including all (weighted) edges in $\mathcal{D}$ whose components both belong to $\mathcal{L}_i$ as its non-quasi-edges and all (weighted) edges in $\mathcal{D}$ with exactly one component belonging to $\mathcal{L}_i$ as its quasi-edges.
- If every interlayer edge in $\mathcal{D}$ emanates from a vertex in $\mathcal{L}_i$ and terminates at a vertex in $\mathcal{L}_j$ where $i < j$, then $\mathcal{D}$ is called a *cascade-type* m*-layer* (*weighted*) *quasi-digraph*.

For example, the weighted quasi-digraph in Fig. 9 is a cascade-type three-layer weighted quasi-digraph with vertex-set partition $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$, where

$$\mathcal{L}_1 = \{1\}, \qquad \mathcal{L}_2 = \{2,3\}, \qquad \text{and} \qquad \mathcal{L}_3 = \{4,5,6\}.$$

The out-degree and adjacency matrices, by Definition 9, are

$$\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} = \left(\begin{array}{c|cc|ccc} \mathbf{D}_1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{D}_2$}} & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{D}_3 & \\ 0 & 0 & 0 & & & \end{array}\right)$$

where

$$\mathbf{D}_1 = (\,4\,), \qquad \mathbf{D}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{D}_3 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

are the out-degree matrices of the quasi-digraphs occupying layers 1, 2, and 3, and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} = \left(\begin{array}{c|cc|ccc} \mathbf{A}_1 & 1 & 1 & 1 & 0 & 1 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{A}_2$}} & 0 & 1 & 0 \\ 0 & & & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{A}_3 & \\ 0 & 0 & 0 & & & \end{array}\right)$$

where

$$\mathbf{A}_1 = (\,0\,), \qquad \mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{A}_3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

are the adjacency matrices of the quasi-digraphs occupying layers 1, 2, and 3. The Laplacian of the weighted quasi-digraph is thus

(4)

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} 4 & -1 & -1 & -1 & 0 & -1 \\ 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 6 & -2 \\ 0 & 0 & 0 & 0 & -2 & 3 \end{pmatrix} = \left(\begin{array}{c|cc|ccc} \mathbf{L}_1 & -1 & -1 & -1 & 0 & -1 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{L}_2$}} & 0 & -1 & 0 \\ 0 & & & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{L}_3 & \\ 0 & 0 & 0 & & & \end{array}\right),$$

where

(5) $$\mathbf{L}_1 = (\,4\,)\,, \qquad \mathbf{L}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{L}_3 = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$

are the Laplacians of the quasi-digraphs occupying layers 1, 2, and 3. Let us refer to matrices of the form (4) as being *near-upper-triangular.*

Definition 11. A matrix $\mathbf{L} \in \mathbb{R}^{n\times n}$ is *near-upper-triangular* if there are matrices $\mathbf{L}'_{i,j}$ and square matrices $\mathbf{L}_i$ such that

$$\mathbf{L} = \left(\begin{array}{c|c|c|c} \mathbf{L}_1 & \mathbf{L}'_{1,2} & \cdots & \mathbf{L}'_{1,m} \\ \hline \mathbf{O} & \mathbf{L}_2 & \cdots & \mathbf{L}'_{2,m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{O} & \mathbf{O} & \cdots & \mathbf{L}_m \end{array}\right)$$

(6)

Next, since for square matrices $\mathbf{A}$ and $\mathbf{C}$ we have

$$\det\left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{O} & \mathbf{C} \end{array}\right) = \det(\mathbf{A})\det(\mathbf{C})$$

(see, e.g., [1, p. 111]), the near-upper-triangular matrix (6) possesses the convenient property that

$$\det(\lambda\mathbf{I} - \mathbf{L}) = \det(\lambda\mathbf{I} - \mathbf{L}_1)\det(\lambda\mathbf{I} - \mathbf{L}_2)\cdots\det(\lambda\mathbf{I} - \mathbf{L}_m)\,,$$

which implies that the multiset $\sigma(\mathbf{L})$ of all eigenvalues of $\mathbf{L}$ is precisely the union of the multisets $\sigma(\mathbf{L}_1)$, ... , $\sigma(\mathbf{L}_m)$ of all eigenvalues of $\mathbf{L}_1$, ... , $\mathbf{L}_m$ [12, p. 105]:

$$\sigma(\mathbf{L}) = \sigma(\mathbf{L}_1) \uplus \cdots \uplus \sigma(\mathbf{L}_m)\,,$$

where the additive union operator $\uplus$ takes the sum—as opposed to the maximum—of the multiplicities of common elements [4, p. 50]. In summary, we have the following theorem.

Theorem 4. *Suppose that a mixing problem is represented by a cascade-type weighted m-layer quasi-digraph with Laplacian $\mathbf{L}$, and satisfies the following assumptions.*

- *All tanks maintain the same constant brine volume.*
- *Brine flows along all edges occur at rates proportional to the weights of the respective edges.*
- *All ingoing quasi-edges represent the flow of salt-free water.*

Then the problem is modeled by the system $\mathbf{x}'(\tau) = -\mathbf{L}^\top\mathbf{x}(\tau)$, where $\mathbf{L}$ is the $n\times n$ near-upper-triangular matrix whose i-th diagonal block is the Laplacian $\mathbf{L}_i$ of the quasi-digraph occupying layer i. Consequently, if all eigenvalues of $\mathbf{L}_1$, ... , $\mathbf{L}_m$ have positive real parts, then all solutions of the mixing problem converge to the state where all tanks are salt-free.

The eigenvalues of the Laplacians in (5) are given by

$$\sigma(\mathbf{L}_1) = \{4\}, \qquad \sigma(\mathbf{L}_2) = \{1,3\}, \qquad \text{and} \qquad \sigma(\mathbf{L}_3) = \{1,2,4\}.$$

Since these eigenvalues all have positive real parts, all solutions of the mixing problem represented by the quasi-digraph in Fig. 9 satisfying the three assumptions of Theorem 4 converge to the state where all tanks are salt-free.

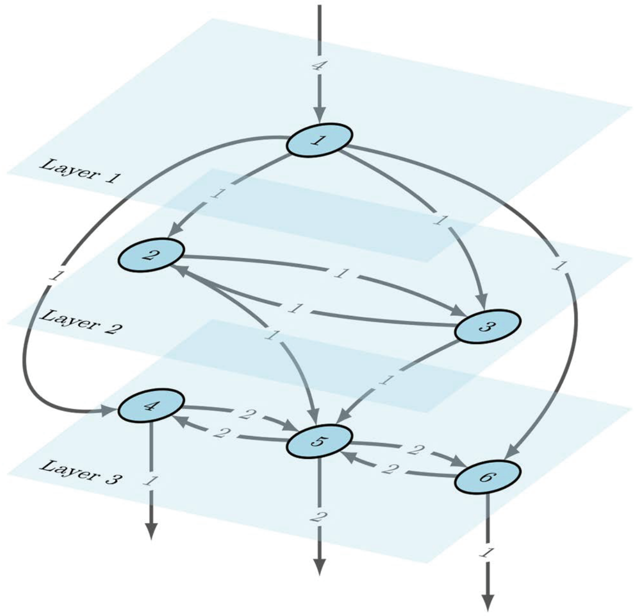

Fig. 9. A cascade-type three-layer weighted quasi-digraph

EXAMPLE 3. Let $m \in \mathbb{N}$. Consider a cascade-type m-layer quasi-digraph $\mathcal{D}_m$ corresponding to the partition $\{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ of its vertex set $\mathcal{V} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{i} \{(i,j)\}$, where

$$\mathcal{L}_i = \{(i,j) : j \in \{1, \ldots, i\}\}$$

for every $i \in \{1, \ldots, m\}$. Suppose that in $\mathcal{D}_m$,

- there is only one ingoing quasi-edge attached to the one vertex in $\mathcal{L}_1$, and m outgoing quasi-edges attached to the m vertices in $\mathcal{L}_m$;
- for every $i \in \{1, \ldots, m-1\}$, every vertex in $\mathcal{L}_i$ is joined with an interlayer edge to every vertex in $\mathcal{L}_{i+1}$;
- for every $i \in \{1, \ldots, m\}$, every ordered pair of distinct vertices in $\mathcal{L}_i$ is joined with an intralayer edge.

Thus, $\mathcal{D}_m$ has

$$(1+m) + \sum_{i=1}^{m-1} i(i+1) + \sum_{i=1}^{m} i(i-1) = \frac{1}{3}(m+1)\left(2m^2 - 2m + 3\right)$$

edges. The quasi-digraph $\mathcal{D}_3$ is depicted in Fig. 10.

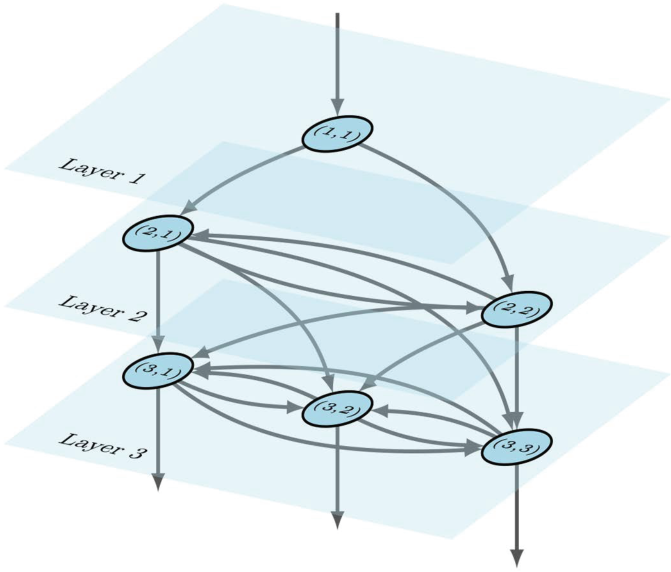

Fig. 10. The quasi-digraph $\mathcal{D}_3$ discussed in Example 3

Let us next assign weights to the edges of $\mathcal{D}_m$ as follows. First, for every $j \in \{1, \dots, m\}$, let the weight of the outgoing quasi-edge attached to the vertex (m, j) be p_j. Next, for every $i \in \{1, \dots, m-1\}$, $j \in \{1, \dots, i\}$, and $k \in \{1, \dots, i+1\}$, let the weight of the interlayer edge $((i, j), (i+1, k))$ be $q_{i,k}$. Finally, for every $i \in \{1, \dots, m\}$ and $j, k \in \{1, \dots, i\}$ with $j \neq k$, let the weight of the intralayer edge $((i, j), (i, k))$ be r.

Consider the mixing problem represented by $\mathcal{D}_m$ satisfying the three assumptions in Theorem 3. Let us show the positivity of the real parts of the eigenvalues of the Laplacian $\mathbf{L}_i$ of the quasi-graph occupying $\mathcal{L}_i$, for every $i \in \{1, \dots, m\}$.

- The degree and adjacency matrices of the quasi-graph occupying $\mathcal{L}_1$ are the 1×1 matrices
$$\mathbf{D}_1 = \left(\sum_{j=1}^{2} q_{1,j} \right) \qquad \text{and} \qquad \mathbf{A}_1 = (\, 0 \,),$$
so that its Laplacian is the 1×1 matrix
$$\mathbf{L}_1 = \mathbf{D}_1 - \mathbf{A}_1 = \left(\sum_{j=1}^{2} q_{1,j} \right),$$
whose only eigenvalue is its only entry, whose real part is positive.
- For every $i \in \{2, \dots, m-1\}$, the out-degree and adjacency matrices of the quasi-graph occupying $\mathcal{L}_i$ are the $i \times i$ matrices
$$\mathbf{D}_i = \begin{pmatrix} \sum_{j=1}^{i+1} q_{1,j} + (i-1)r & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{i+1} q_{2,j} + (i-1)r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{i+1} q_{i,j} + (i-1)r \end{pmatrix}$$

and

$$\mathbf{A}_i = \begin{pmatrix} 0 & r & \cdots & r \\ r & 0 & \cdots & r \\ \vdots & \vdots & \ddots & \vdots \\ r & r & \cdots & 0 \end{pmatrix},$$

so that its Laplacian is the $i \times i$ matrix

$$\begin{aligned} \mathbf{L}_i &= \mathbf{D}_i - \mathbf{A}_i \\ &= \begin{pmatrix} \sum_{j=1}^{i+1} q_{1,j} + (i-1)r & -r & \cdots & -r \\ -r & \sum_{j=1}^{i+1} q_{2,j} + (i-1)r & \cdots & -r \\ \vdots & \vdots & \ddots & \vdots \\ -r & -r & \cdots & \sum_{j=1}^{i+1} q_{i,j} + (i-1)r \end{pmatrix}. \end{aligned}$$

For this matrix $\mathbf{L}_i$, the sum of the absolute values of the k-th row off-diagonal entries is

$$r_k = \underbrace{|-r| + \cdots + |-r|}_{i-1} = (i-1)r.$$

By Gershgorin's theorem, it follows that all eigenvalues of $\mathbf{L}_i$ belong to the set

$$\bigcup_{k=1}^{i} \mathrm{D}\left(\sum_{j=1}^{i+1} q_{k,j} + (i-1)r,\ (i-1)r\right),$$

which readily implies that all of these eigenvalues have positive real parts.

- The degree and adjacency matrices of the quasi-graph occupying $\mathcal{L}_m$ are the $m \times m$ matrices

$$\mathbf{D}_m = \begin{pmatrix} p_1 + (m-1)r & 0 & \cdots & 0 \\ 0 & p_2 + (m-1)r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_m + (m-1)r \end{pmatrix}$$

and

$$\mathbf{A}_m = \begin{pmatrix} 0 & r & \cdots & r \\ r & 0 & \cdots & r \\ \vdots & \vdots & \ddots & \vdots \\ r & r & \cdots & 0 \end{pmatrix},$$

so that its Laplacian is the $m \times m$ matrix

$$\mathbf{L}_m = \mathbf{D}_m - \mathbf{A}_m = \begin{pmatrix} p_1 + (m-1)r & -r & \cdots & -r \\ -r & p_2 + (m-1)r & \cdots & -r \\ \vdots & \vdots & \ddots & \vdots \\ -r & -r & \cdots & p_m + (m-1)r \end{pmatrix}.$$

For this matrix $\mathbf{L}_m$, the sum of the absolute values of the k-th row off-diagonal entries is

$$r_k = \underbrace{|-r| + \cdots + |-r|}_{m-1} = (m-1)r.$$

By Gershgorin's theorem, it follows that all eigenvalues of $\mathbf{L}_m$ belong to the set

$$\bigcup_{k=1}^{m} \mathrm{D}\left(p_k + (m-1)r,\ (m-1)r\right),$$

which readily implies that all of these eigenvalues have positive real parts.

By Theorem 4, it follows that all solutions of the mixing problem converge to the state where all tanks are salt-free. △

6. Suggestions for instructors

Let us conclude this paper by providing instructors with suggestions on how mixing problems could be conveniently inserted as classroom materials, with the aim of not only introducing such problems as already customary but also to allow some extensive analysis inspired by those conducted both in [11] and in the present paper. To this end, let us suggest two approaches for delivering mixing problems in an undergraduate course on ordinary differential equations: as a part of taught materials and as a mini-project. In both approaches, for convenience, we assume that the students have no prior knowledge on mixing problems.

6.1. Mixing problems as a part of taught materials

First let us discuss how mixing problems could be delivered as a part of taught materials. For this purpose, we assume the availability of a 150-minute teaching duration, ideally organized into two sessions, one of 50 and the other of 100 minutes.

To ensure the sufficiency of students' theoretical background, the first, shorter session should be scheduled only after the materials on linear differential equations have been delivered. Since we treat students as having no previous familiarity with mixing problems, this session should begin with a gentle exposition of a standard starter: a single-tank mixing problem with a filler pipe and a drainer pipe, where the brine volume in the tank is kept constant. We thus recommend the following problem.

Problem 1. Consider a tank, call it Tank 1, which initially holds 10 liters of brine containing 20 grams of salt. Suppose that pure water flows into Tank 1 at the rate of 5 liters per minute and the well-mixed brine in Tank 1 flows out at the same rate. See Fig. 11.

(a) Construct a mathematical model governing the evolution of the mass $x_1(t)$ of salt remaining in Tank 1 at time $t \geqslant 0$.

(b) Solve the model to obtain an expression for $x_1(t)$ in terms of t, and justify mathematically that $x_1(t) \overset{t\to\infty}{\longrightarrow} 0$.

Certainly, the desired mathematical model is the initial value problem

$$x_1'(t) = -\frac{1}{2}x_1(t), \qquad x_1(0) = 20, \tag{7}$$

whose solution is given by $x_1(t) = 20\,\mathrm{e}^{-t/2} \overset{t\to\infty}{\longrightarrow} 0$.

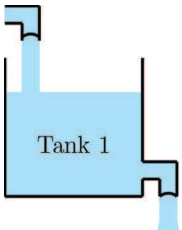

Fig. 11. The tank configuration in Problem 1

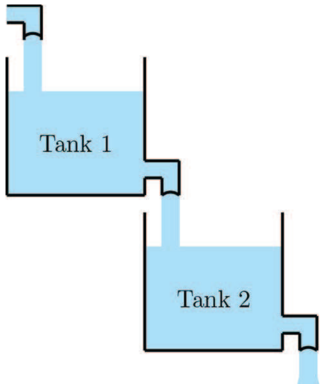

Fig. 12. The tank configuration in Problem 2

The use of the name Tank 1 for the sole tank involved in the above problem, and of the notation $x_1(t)$ as opposed to merely $x(t)$, are intended to set the scene for the subsequent part of the discussion, namely, the following two-tank problem, posed as a natural extension of the above problem. △

Problem 2. Consider two tanks: Tank 1 and Tank 2. As before, Tank 1 initially holds 10 liters of brine containing 20 grams of salt, and pure water flows into Tank 1 at the rate of 5 liters per minute. Tank 2 initially holds 10 liters of pure water, the well-mixed brine in Tank 1 flows into Tank 2 at the rate of 5 liters per minute, and the well-mixed brine in Tank 2 flows out at the same rate. See Fig. 12.

(a) Construct a mathematical model governing the evolution of the masses $x_1(t)$ and $x_2(t)$ of salt remaining in Tank 1 and Tank 2 at time $t \geqslant 0$.

(b) Solve the model to obtain expressions for $x_1(t)$ and $x_2(t)$ in terms of t, and justify mathematically that $x_1(t), x_2(t) \overset{t\to\infty}{\longrightarrow} 0$.

The students should be led to see that the required mathematical model comprises a set of two initial value problems, one of them being precisely (7), the other being

$$x_2'(t) = \frac{1}{2}x_1(t) - \frac{1}{2}x_2(t), \qquad x_2(0) = 0. \tag{8}$$

Here, in the absence of a prior introduction to systems of differential equations, there is an opportunity to mention that the two equations in (7) and (8) constitute such a system, and hence to announce that some forthcoming chapters will be dedicated to the discussion of such systems. Meanwhile, substituting the solution $x_1(t) = 20\,\mathrm{e}^{-t/2}$ of (7) into (8) yields the initial value problem

$$x_2'(t) + \frac{1}{2}x_2(t) = 10\mathrm{e}^{-t/2}, \qquad x_2(0) = 0,$$

whose solution can be obtained using the standard method of integrating factor:

$$x_2(t) = 10t\,\mathrm{e}^{-t/2} \overset{t\to\infty}{\longrightarrow} 0,$$

where we have applied L'Hôpital's rule.

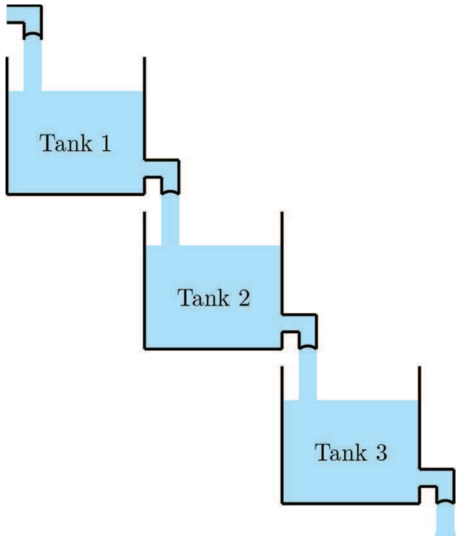

Fig. 13. The tank configuration to be addressed following Problem 2

The final part of the discussion should address, possibly only briefly, the three-tank case (Fig. 13), which adds to (7) and (8) the initial value problem

$$x_3'(t) = \frac{1}{2}x_2(t) - \frac{1}{2}x_3(t), \qquad x_3(0) = 0,$$

whose solution, obtainable as above, is given by $x_3(t) = \frac{5}{2}t^2 e^{-t/2} \overset{t\to\infty}{\longrightarrow} 0$, and subsequently the general n-tank case (cf. [7, p. 55]), governed by the n equations

$$x_1'(t) = -\frac{1}{2}x_1(t) \quad \text{and} \quad x_{i+1}'(t) + \frac{1}{2}x_{i+1}(t) = \frac{1}{2}x_i(t) \;\; \text{for every } i \in \{1, \dots, n-1\},$$

and the initial conditions

$$x_1(0) = 20, \qquad x_2(0) = 0, \qquad \dots, \qquad x_n(0) = 0.$$

It can be shown that the solution in the latter case is given by

$$x_i(t) = \frac{20t^{i-1}e^{-t/2}}{(i-1)!\,2^{i-1}} \overset{t\to\infty}{\longrightarrow} 0 \qquad \text{for every } i \in \{1, \dots, n\},$$

the proof of which could either be omitted or announced as a challenging mathematical induction homework for interested students. Nevertheless, at this point students should be informed that this n-tank problem will be revisited in the second session, after they are equipped with a theorem which could be applied to guarantee that $x_i(t) \overset{t\to\infty}{\longrightarrow} 0$ without deriving an expression of $x_i(t)$ in terms of t. △

Let us now discuss materials to be delivered in the second session. As these materials rely on systems of differential equations, it is advisable to schedule the session only after those which discuss systems of differential equations.

We suggest beginning the session by illustrating the constructions of systems of differential equations which model several modest multi-tank problems. The following closed two-tank and three-tank problems, inspired by [9, pp. 358–359] and [7, pp. 335], could serve as starters.

Problem 3. Consider two tanks: Tank 1 and Tank 2, each containing 10 liters of brine, equipped with two brine-transporting pipes: a pipe from Tank 1 to Tank 2 and a pipe from Tank 2 to Tank 1, through each of which well-mixed brine flows at the rate of 5 liters per minute (Fig. 14). Construct a system of differential equations governing the evolution of the masses $x_1(t)$ and $x_2(t)$ of salt remaining in Tank 1 and Tank 2 at time $t \geqslant 0$.

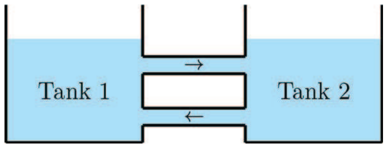

Fig. 14. The tank configuration in Problem 3

Problem 4. Consider three tanks: Tank 1, Tank 2, and Tank 3, each containing 10 liters of brine, equipped with three brine-transporting pipes: a pipe from Tank 1 to Tank 2, a pipe from Tank 2 to Tank 3, and a pipe from Tank 3 to Tank 1, through each of which well-mixed brine flows at the rate of 5 liters per minute (Fig. 15). Construct a system of differential equations governing the evolution of the masses $x_1(t)$, $x_2(t)$, $x_3(t)$ of salt remaining in Tank 1, Tank 2, and Tank 3 at time $t \geqslant 0$.

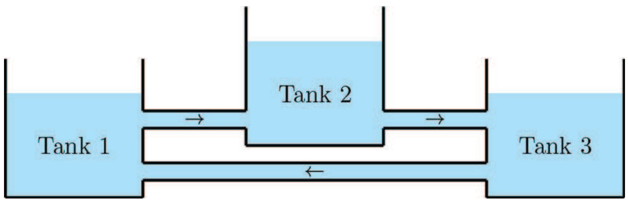

Fig. 15. The tank configuration in Problem 4

The discussion should aim to demonstrate that the above two problems are modeled by the systems of differential equations

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{9}$$

and

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1/2 & 0 \\ 0 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \tag{10}$$

respectively. A brief introduction to digraphs can then follow, covering at least Definitions 1, 2, and 3 in the present paper, before leading students to see that the systems (9) and (10), being represented by the digraphs in Fig. 16 and Fig. 17,

Fig. 16. The digraph representing the system (9) of Problem 3

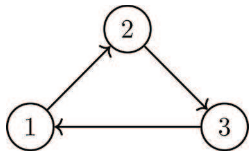

Fig. 17. The digraph representing the system (10) of Problem 4

can be rewritten in the form $\mathbf{x}'(t) = -(1/2)\mathbf{L}^\top \mathbf{x}(t)$, and subsequently in the time-rescaled form $\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau)$, where $\mathbf{L}$ denotes the corresponding digraph's Laplacian, as demonstrated in the final paragraph of Section 2. △

Next, we recommend continuing the model-constructing discussions, to cover cases involving filler and drainer pipes. For this purpose, one could merely modify Problems 3 and 4 by adding a filler pipe and a drainer pipe to each of the existing tanks.

Problem 5. Modify Problem 3 by adding to each tank a filler pipe through which pure water flows in at the rate of 5 liters per minute, and a drainer pipe through which well-mixed brine flows out at the same rate (Fig. 18). Construct a system of differential equations governing the evolution of the masses $x_1(t)$ and $x_2(t)$ of salt remaining in Tank 1 and Tank 2 at time $t \geqslant 0$.

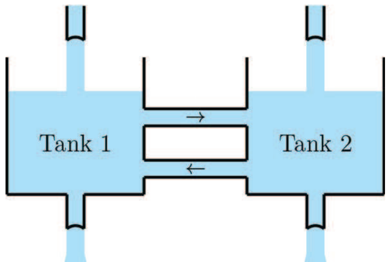

Fig. 18. The tank configuration in Problem 5

Problem 6. Modify Problem 4 by adding to each tank a filler pipe through which pure water flows in at the rate of 5 liters per minute, and a drainer pipe through which well-mixed brine flows out at the same rate (Fig. 19). Construct a system of differential equations governing the evolution of the masses $x_1(t)$, $x_2(t)$, $x_3(t)$ of salt remaining in Tank 1, Tank 2, and Tank 3 at time $t \geqslant 0$.

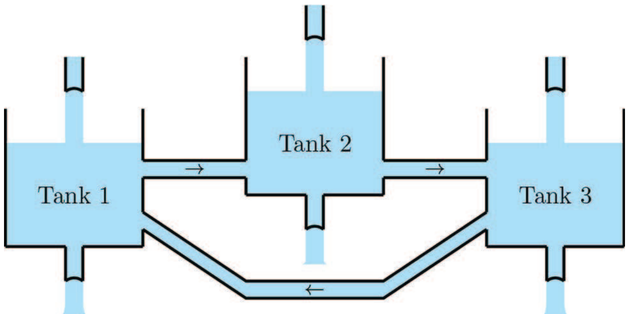

Fig. 19. The tank configuration in Problem 6

Once the students understand that the modification transforms the systems (9) and (10) to

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1/2 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{11}$$

and

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1/2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \tag{12}$$

they are ready for a brief introduction to quasi-digraphs. Paralleling the case of digraphs, the discussion should cover at least Definitions 4, 5, and 6, and should lead students to see that the systems (11) and (12), being represented by the quasi-digraphs in Fig. 20 and Fig. 21, can be rewritten in the analogous form involving the corresponding quasi-digraph's Laplacian. Finally, Gershgorin's theorem could be stated without a proof, and applied to the systems (11) and (12), demonstrating its powerfulness in ensuring the solutions' convergence to the trivial fixed point without the need of solving the systems analytically or even calculating the eigenvalues of their coefficient matrices. △

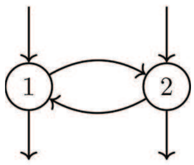

Fig. 20. The quasi-digraph representing the system (11) of Problem 5

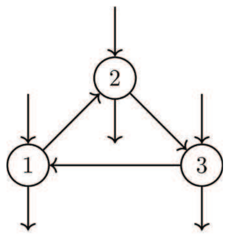

Fig. 21. The quasi-digraph representing the system (12) of Problem 6

To conclude the second session, we recommend revisiting, as previously promised, the n-tank mixing problem considered at the end of the first session, computing the associated quasi-digraph's Laplacian and applying Gershgorin's theorem to establish the solution's convergence to the trivial fixed point. If time constraints arise, this task could instead be assigned as a homework, along with, e.g., our Example 1.

6.2. Mixing problems as a mini-project

In the case of a limited teaching duration, instructors could consider incorporating mixing problems only as a group or individual mini-project. Let us present our design of a mini-project on mixing problems, to be assigned only after the completion of discussions on systems of differential equations.

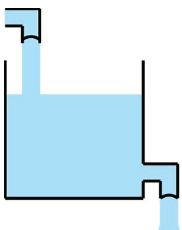

Fig. 22. The tank configuration in our mini-project's first problem

Miniproject

1. Consider a tank which initially holds V liters of brine containing X grams of salt.

Suppose that pure water flows into the tank at the rate of f liters per minute, and that the well-mixed brine in the tank flows out at the same rate. See Fig. 22. Let $x(t)$ be the mass of salt remaining in the tank at time $t \geqslant 0$.

(a) Derive the initial value problem

$$x'(t) = -\frac{f}{V}x(t), \qquad x(0) = X.$$

(b) Show that $x(t) = X\mathrm{e}^{-(f/V)t}$, and compute $\lim_{t\to\infty} x_1(t)$.

2. Consider three tanks: Tank 1, Tank 2, and Tank 3. Tank 1 initially holds V liters of brine containing X grams of salt, and pure water flows into Tank 1 at the rate of f liters per minute. Tank 2 initially holds V liters of pure water, and the well-mixed brine in Tank 1 flows into Tank 2 at the rate of f liters per minute. Tank 3 initially holds V liters of pure water, the well-mixed brine in Tank 2 flows into Tank 3 at the rate of f liters per minute, and the well-mixed brine in Tank 3 flows out at the same rate. See Fig. 13. For every $i \in \{1, 2, 3\}$, let $x_i(t)$ be the mass of salt remaining in Tank i at time $t \geqslant 0$.

(a) Derive the system of differential equations

$$\begin{cases} x_1'(t) = -\dfrac{f}{V}x_1(t), \\ x_2'(t) = \dfrac{f}{V}x_1(t) - \dfrac{f}{V}x_2(t), \\ x_3'(t) = \dfrac{f}{V}x_2(t) - \dfrac{f}{V}x_3(t). \end{cases}$$

(b) By the first question of this mini-project, the solution of the system's first equation satisfying the initial condition $x_1(0) = X$ is given by

$$x_1(t) = X\mathrm{e}^{-(f/V)t}.$$

Substitute this into the system's second equation, and show that the solution of the resulting equation satisfying the initial condition $x_2(0) = 0$ is given by

$$x_2(t) = \frac{f}{V}Xt\,\mathrm{e}^{-(f/V)t}.$$

Then, substitute this into the system's third equation, and show that the solution of the resulting equation satisfying the initial condition $x_3(0) = 0$ is given by

$$x_3(t) = \frac{f^2}{2V^2}Xt^2\mathrm{e}^{-(f/V)t}.$$

Finally, compute $\lim_{t\to\infty} x_i(t)$ for every $i \in \{1, 2, 3\}$.

(c) Letting $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^\top$, determine the 3×3 matrix $\mathbf{M}$ for which the above system can be written as

$$\mathbf{x}'(t) = \mathbf{M}\,\mathbf{x}(t).$$

(d) Define the 3×3 matrix $\mathbf{L}$ whose (i, j)-entry is the negative of the number of pipes transporting brine from tank i to tank j if $i \neq j$, and the number of pipes transporting brine out of tank $i = j$ otherwise. Write down the matrix $\mathbf{L}$, and show that

$$\mathbf{M} = -\frac{f}{V}\mathbf{L}^\top.$$

(e) Show that the time-rescaling $\tau = (f/V)t$ transforms the system in (c) into

$$\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau).$$

According to a theorem by Gershgorin, all eigenvalues of an $n \times n$ complex matrix $\mathbf{L} = (l_{i,j})$ belong to the set

$$\mathrm{D}\left(l_{1,1}, r_1\right) \cup \cdots \cup \mathrm{D}\left(l_{n,n}, r_n\right),$$

where, for every $i \in \{1, \ldots, n\}$,

$$\mathrm{D}\left(l_{i,i}, r_i\right) := \{z \in \mathbb{C} : |z - l_{i,i}| \leqslant r_i\}$$

denotes the closed disk centred at $l_{i,i}$ with radius $r_i := \sum_{\substack{j=1\\ j\neq i}}^{n} |l_{i,j}|$.

(f) Use Gershgorin's theorem to specify a subset of $\mathbb{C}$ containing all eigenvalues of the matrix $\mathbf{L}$ written down in part (d).

(g) What does your answer of part (f) imply with regards to the value of $\lim_{\tau\to\infty} x_i(\tau)$ for every $i \in \{1, 2, 3\}$? Does this agree with your answer of part (b)?

3. Consider three tanks: Tank 1, Tank 2, and Tank 3, each containing V liters of brine, equipped with three brine-transporting pipes: a pipe from Tank 1 to Tank 2, a pipe from Tank 2 to Tank 3, and a pipe from Tank 3 to Tank 1, through each of which well-mixed brine flows at the rate of f liters per minute. In addition, attached to each tank is a filler pipe through which pure water flows in at the rate of f liters per minute, and a drainer pipe through which well-mixed brine flows out at the same rate. See Fig. 19. For every $i \in \{1, 2, 3\}$, let $x_i(t)$ be the mass of salt remaining in Tank i at time $t \geqslant 0$.

(a) Letting $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^\top$, derive a system of differential equations

$$\mathbf{x}'(t) = \mathbf{M}\,\mathbf{x}(t).$$

for some 3×3 matrix $\mathbf{M}$ to be specified.

(b) Define the 3×3 matrix $\mathbf{L}$ whose (i, j)-entry is the negative of the number of pipes transporting brine from tank i to tank j if $i \neq j$, and the number of pipes transporting brine out of tank $i = j$ otherwise. Write down the matrix $\mathbf{L}$, and show that

$$\mathbf{M} = -\frac{f}{V}\mathbf{L}^\top.$$

(c) Show that the time-rescaling $\tau = (f/V)t$ transforms the system in (a) into

$$\mathbf{x}'(\tau) = -\mathbf{L}^\top \mathbf{x}(\tau).$$

(d) Use Gershgorin's theorem to specify a subset of $\mathbb{C}$ containing all eigenvalues of the matrix $\mathbf{L}$ written down in part (b).

(e) What does your answer of part (d) imply with regards to the value of $\lim_{\tau\to\infty} x_i(\tau)$ for every $i \in \{1, 2, 3\}$?

Acknowledgment. The authors are thankful to the anonymous referee for helpful suggestions that helped us to improve the paper.

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Received: 28.08.2024, in revised form 23.09.2024.

Accepted: 25.09.2024