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THE DISTANCE FROM A POINT TO A LINE OR A PLANE IN COORDINATE GEOMETRY: A REVISIT WITH THREE ELEMENTARY SOLUTIONS

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Abstract. The popular problem of the distance from a point to a line in two-dimensional coordinate geometry has been presented in many textbooks, books, and articles with multiple solutions. In this paper, we revisit this problem by presenting three interesting elementary solutions. In addition, we discuss some applications of these solutions in solving some related three-dimensional coordinate geometry problems. These elementary solutions to the two-dimensional and three-dimensional geometry problems can be used suitably in teaching mathematics at high school or junior college level.

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1. Introduction

In this article, we are first interested in the following two-dimensional coordinate geometry problem.

PROBLEM 1. *Find the distance from a given point $M(x_0, y_0)$ to a given line l: $Ax + By + C = 0$ in the coordinate plane, where $A^2 + B^2 > 0$.*

Multiple forms of the equation of a straight line were presented in books, including the slope intercept form, point-slope form, two-point form, intercept form, normal or perpendicular form, general form, and parametric form; see, for example, [1], [2] or [6].

Many solutions to this problem have appeared in the literature. González, Hinthorn and Martínez [4] discussed a proof using the similarity of two right triangles after drawing some extra segments. Gore [5] derived four elementary methods (algebraic method, using vectors, a method based on coordinate geometry and using areas of polygons) and two (advanced) optimization-based methods (a simple calculus-based derivation and a constrained optimization method). The slope intercept form of the straight-line equation was used in these two articles.

In this article, we present three interesting elementary solutions to the problem, including finding the coordinates of the perpendicular projection point by using a parametric equation of a straight line (Solution 1), using the smallest value of a quadratic function (Solution 2) and using an inequality (Solution 3). We use the general form and the parametric form of a straight-line equation.

Moreover, we discuss the applications of these solutions in solving two related analogous problems in three-dimensional coordinate geometry.

PROBLEM 2. *Find the distance from a given point $M(x_0, y_0, z_0)$ to a given line l:* $\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$ *in the three-dimensional space, where t is a parameter and $a^2 + b^2 + c^2 > 0$.*

PROBLEM 3. *Find the distance from a given point $M(x_0, y_0, z_0)$ to a given plane $Ax+By+Cz+D = 0$ in the three-dimensional space, where $A^2+B^2+C^2 > 0$.*

Solution 1 can be used similarly to solve both Problem 2 and Problem 3. In the same manner used in Solution 2, we can address Problem 2. Using a similar lemma applied in Solution 3, we can find a solution to Problem 3.

Solution 1: Finding the coordinates of the perpendicular projection point by using a parametric equation of a straight line

Denote the line passing through M and perpendicular to line l by d, and let H be the intersection point of lines l and d (Fig. 1). Then, the distance from M to line l is the length of segment MH.

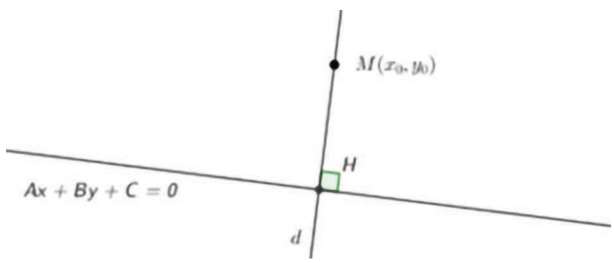

Figure 1

The parametric equation of line d is $\begin{cases} x = x_0 + At \\ y = y_0 + Bt \end{cases}$, where t is a parameter. Since H lies on line d, its coordinates have the form $H(x_0 + At_1, y_0 + Bt_1)$. We have

$$MH^2 = (x_0 + At_1 - x_0)^2 + (y_0 + Bt_1 - y_0)^2 = (A^2 + B^2)t_1^2. \tag{1}$$

Moreover, as line l passes through H, we have the following equation with the unknown t_1:

$$A(x_0 + At_1) + B(y_0 + Bt_1) + C = 0.$$

By solving the equation above, we obtain the following:

$$t_1 = -\frac{Ax_0 + By_0 + C}{A^2 + B^2}. \tag{2}$$

From (1) and (2), we have $MH^2 = \dfrac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}$. Hence, the distance from M to line $Ax + By + C = 0$ is

$$MH = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

REMARK 1. In a similar manner, we can also solve Problems 2 and 3.

To solve Problem 2, denote by P the plane passing through $M(x_0, y_0, z_0)$ and perpendicular to line l; then, write the the equation of plane P having the normal vector $\vec{n} = (a, b, c)$ in the form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

The coordinates of the intersection point H of plane P and line l are $H(x_1 + at_1, y_1 + bt_1, z_1 + ct_1)$, where

$$t_1 = -\frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{a^2 + b^2 + c^2}.$$

Continuing similarly as in the solution of Problem 1, after some calculation, we obtain the distance of point M to line l as

$$MH = \frac{\sqrt{\begin{array}{c}[c(y_1 - y_0) - b(z_1 - z_0)]^2 + [a(z_1 - z_0) - c(x_1 - x_0)]^2 \\ +[b(x_1 - x_0) - a(y_1 - y_0)]^2\end{array}}}{\sqrt{a^2 + b^2 + c^2}}. \tag{3}$$

To solve Problem 3, denote by d the line passing through $M(x_0, y_0, z_0)$ and perpendicular to plane $Ax + By + Cz + D = 0$ (i.e., having a direction vector $\vec{u} = (A, B, C)$) and write its parametric equations in the form $\begin{cases} x = x_0 + At \\ y = y_0 + Bt \\ z = z_0 + Ct. \end{cases}$ Denoting by H the intersection point of d and the given plane and proceeding analogously to the solution of Problem 1, we obtain the desired distance as

$$MH = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}. \tag{4}$$

Solution 2. Using the least value of a quadratic function

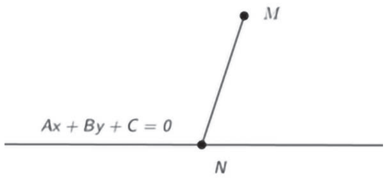

Figure 2

Let N be any point on line $Ax + By + C = 0$ (Fig. 2). Since $A^2 + B^2 > 0$, without loss of generality, suppose that B is different from 0. A parametric equation of line $Ax + By + C = 0$ can be written in the form $\begin{cases} x = t \\ y = -\frac{C}{B} - \frac{A}{B}t \end{cases}$, where t is a parameter. Thus, point N has coordinates in the form $N\Big(t, -\dfrac{C}{B} - \dfrac{A}{B}t\Big)$.

Applying the distance formula, we obtain

$$MN^2 = (x_0 - t)^2 + \Big(y_0 + \frac{A}{B}t + \frac{C}{B}\Big)^2 = f(t).$$

We find the lowest value of this quadratic function. It can be written as

$$f(t) = \Big(\frac{A^2}{B^2} + 1\Big)t^2 + 2\Big[\Big(y_0 + \frac{C}{B}\Big)\frac{A}{B} - x_0\Big]t + \Big[\Big(y_0 + \frac{C}{B}\Big)^2 + x_0^2\Big].$$

We have

$$\Delta' = \Big[\Big(y_0 + \frac{C}{B}\Big)\frac{A}{B} - x_0\Big]^2 - \Big(\frac{A^2}{B^2} + 1\Big)\Big[\Big(y_0 + \frac{C}{B}\Big)^2 + x_0^2\Big] = -\frac{[Ax_0 + By_0 + C]^2}{B^2}.$$

Therefore, the least value of the function is

$$-\frac{\Delta'}{\frac{A^2}{B^2} + 1} = \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}.$$

As a consequence, the distance between point M and line $Ax + By + C = 0$ is $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$.

Remark 2. In a similar manner, we can solve Problem 2 in the following way.

Let N be an arbitrary point on line $\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$. Using the distance formula, we obtain

$$MN^2 = (x_1 + at - x_0)^2 + (y_1 + bt - y_0)^2 + (z_1 + ct - z_0)^2.$$

By rearranging the term, we can find the distance between point M and line l by finding the smallest value of the following quadratic function with variable t:

$$f(t) = [at + (x_1 - x_0)]^2 + [bt + (y_1 - y_0)]^2 + [ct + (z_1 - z_0)]^2.$$

As the result we obtain formula (3).

Solution 3: Using an inequality

LEMMA 1. The following is valid for all real numbers m, n, p, q:

$$(m^2 + p^2)(n^2 + q^2) \geq (mn + pq)^2. \tag{5}$$

Proof. We have

$$\begin{aligned} &(m^2 + p^2)(n^2 + q^2) \geq (mn + pq)^2 \\ \iff\ & m^2n^2 + p^2q^2 + m^2q^2 + n^2p^2 \geq m^2n^2 + p^2q^2 + 2mnpq \\ \iff\ & m^2q^2 + n^2p^2 - 2mnpq \geq 0 \\ \iff\ & (mq - np)^2 \geq 0. \end{aligned}$$

Since the last inequality is obvious for real numbers m, n, p, q, this completes the proof of inequality (5). The equality occurs if and only if $mq = np$. ■

Now, we use inequality (5) to solve Problem 1.

SOLUTION 3 TO PROBLEM 1.

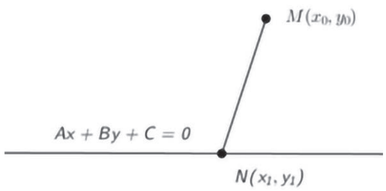

Figure 3

Denote by $N(x_1, y_1)$ an arbitrary point on line $Ax + By + C = 0$ (Fig. 3); then we have

$$Ax_1 + By_1 + C = 0. \tag{6}$$

Applying inequality (5), we obtain

$$(A^2 + B^2)[(x_1 - x_0)^2 + (y_1 - y_0)^2] \geq [A(x_1 - x_0) + B(y_1 - y_0)]^2. \tag{7}$$

We can rewrite $[A(x_1 - x_0) + B(y_1 - y_0)]^2$ in the form

$$[(Ax_1 + By_1 + C) - (Ax_0 + By_0 + C)]^2.$$

Using identity (6), we have

$$[A(x_1 - x_0) + B(y_1 - y_0)]^2 = (Ax_0 + By_0 + C)^2. \tag{8}$$

From (7) and (8) and noting that $MN^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$, we obtain

$$MN^2 \geq \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}.$$

The equality occurs when

$$A(y_1 - y_0) = B(x_1 - x_0). \tag{9}$$

As a consequence, from (6) and (9), the pair of coordinates (x_1, y_1) of the perpendicular projection point is the root of the system of two equations

$$\begin{cases} Ax_1 + By_1 = -C \\ Bx_1 - Ay_1 = Bx_0 - Ay_0. \end{cases}$$

Solving the system, one gets

$$x_1 = \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2}, \quad y_1 = \frac{A^2y_0 - ABx_0 - BC}{A^2 + B^2}.$$

Substituting into $MN^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$ one obtains

$$MN^2 = \Big(\frac{Ax_0 + By_0 + C}{A^2 + B^2}\Big)^2.$$

Therefore, the least value of MN is $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$.

Remark 3. Analogously, we can solve Problem 3 and obtain formula (4) by using the following lemma.

Lemma 2. The following is valid for all real numbers m, n, p, q, r, s:

$$(m^2 + p^2 + r^2)(n^2 + q^2 + s^2) \geq (mn + pq + rs)^2,$$

with equality if and only if $mq = pn$, $ms = rn$ and $ps = rq$.

Conclusion

Since Solution 1 can be similarly used in solving Problem 1 and Problem 2, it is highly applicable. Although both Solution 2 and Solution 3 use "the least value", these solutions do not need derivative knowledge.

With the idea of didactical phenomenology, see [3], some elementary solutions presented in the articles to these problems can be suitably selected for teaching mathematics at the high school or junior college level.

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REFERENCES

[1] L. P. Eisenhart, *Coordinate Geometry*, Dover Publications, INC, 2005.

[2] H. B. Fine, H. D. Thompson, *Coordinate Geometry*, The Macmillan Company, 1911.

[3] H. Freudenthal, *Didactical Phenomenology of Mathematical Structures*, Dordrecht: Reidel, 1983.

[4] R. González, R. Hinthorn, G. Martínez, *The distance from a point to a line*, The Mathematics Teacher, **88**, 9, (1995).

[5] B. W. Gore, *On finding the shortest distance of a point from a line: Which method do you prefer?*, Resonance, July 2017.

[6] G. Tewani, *Mathematics Coordinate Geometry*, Cengage Learning, 2012.

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