THE TEACHING OF MATHEMATICS
2024, Vol. XXVII, No. 1, pp. 33–51
DOI: 10.57016/TM-NYKZ1813

MEASURING CONCEPTUAL KNOWLEDGE OF BASIC ALGEBRAIC CONCEPTS

Aleksandar Lipkovski, Manuela Muzika Dizdarević, Almasa Odžak

Abstract. Basic algebra is often the first step that enables school students to enter the world of mathematics. Concepts such as relations, equations, and polynomials are considered basic algebraic concepts. Understanding these basic concepts determines the further progress and development of mathematical competencies. After all, some educational systems insist on developing procedural knowledge in mathematics, which is why these and many other fundamental concepts remain underdeveloped. In this paper, we present the research results at two mathematics faculties in the Western Balkans on students' conceptual knowledge of basic algebraic ideas at the beginning of their studies. We also discuss possible explanations of the results.

MathEduc Subject Classification: G45, D65

AMS Subject Classification: 97G40, 97D60

Key words and phrases: Conceptual knowledge; procedural knowledge; mathematics education; algebraic concepts.

1. Introduction

Mathematical skills are undoubtedly among the essential skills that students acquire in school. The importance of mathematical knowledge as a basis for science, technology and engineering is also well known. In a constantly changing world, mainly under the influence of technology, the type of mathematical knowledge that students need is also changing.

Compared to geometry and arithmetic, algebra is a relatively young mathematical discipline that has its beginnings in the works of Al-Khwarizmi around 830 in the Middle East, although some elements of algebraic notation and symbolism were present already in Diophantus' Arithmetics. Today, algebra is considered a central subject to be studied in junior high and high schools in almost all educational systems around the world [1]. The word algebra, originally derived from Al-Khwarizmi's al-Jabr, denotes not only a specific structure in mathematics; the word algebra nowadays denotes three different areas 'school (elementary algebra)', 'linear algebra', and 'abstract algebra'. It is not surprising because, although at first glance, elementary algebra and abstract algebra do not have much in common, they consider the same topics but at different levels of abstraction. Knowledge of elementary algebra is a cornerstone of modern scientific and technological civilization and the foundation on which mathematical and algebraic knowledge is built in the graduate study of mathematics.

The content of algebra has not changed dramatically as the content of biology, chemistry and physics [15]. Although students today learn almost the same algebra content as their parents did, they do not achieve the same level of knowledge in that area. It is surprising because now, much more is known about the methods of teaching mathematics.

We are aware that students in schools have severe cognitive and affective difficulties with algebra. They have difficulties becoming competent at it, and even if they succeed, many fail to see the point of studying it [1]. However, those who decide to study mathematics are expected to possess or are ready to acquire the appropriate basic knowledge quickly. Namely, adopting new and abstract knowledge from algebra is impossible if there is no good foundation or if the gap between school and abstract algebra is too wide. This cognitive discontinuity is a major obstacle in the learning of algebra. Unfortunately, the existence of this gap is not always evident, for students learn to repeat definitions and manipulate symbols even if they are meaningless to them [7].

Mathematical competencies rest on carefully developing two dominant types of mathematical knowledge: conceptual and procedural. Conceptual knowledge is knowledge about concepts that are general and abstract. It can be implicit and explicit. Procedural knowledge is knowledge about procedures, where a procedure means a series of steps or actions that will lead us to solve the task.

The relationship between these two types of knowledge has been researched for years. There is a consensus that these two types of knowledge complement and support each other. There is evidence that the improvement of conceptual knowledge supports the improvement of procedural knowledge, and it is assumed that conceptual-procedural, as the order of teaching, is optimal. However, due to the lack of empirical research, there is a suspicion that this order is only sometimes the best possible.

The education systems in Bosnia and Herzegovina, and Serbia share a legacy from the former Yugoslavia. Since the breakup of Yugoslavia until today, each system has been reformed. However, the reforms in the area of mathematics were more formal and did not result in a significant change in the approach to teaching. Thus, the mathematics curricula in Serbia and Bosnia and Herzegovina at the primary and secondary education levels remained very similar. The similarity is also manifested in using the same textbooks and collections of tasks. Therefore, in the following text, we will not distinguish between the mathematics curriculum from Serbia and Bosnia and Herzegovina. Procedural knowledge is the predominant type of knowledge acquired through primary and secondary school. Textbooks and classroom teaching are focused on solving mathematical problems. As student knowledge levels decrease over the years, the variety and quality of tasks in the materials decrease. Only rare professors explicitly teach concepts in mathematics. The majority believes that the knowledge of the underlying concepts is passively adopted after learning the procedures for solving different tasks.

The processes that characterize the present, and among which the development of technology is accelerated, require a different kind of knowledge. Conceptual knowledge of mathematics is essential not only for mathematicians and future mathematics teachers but also for engineers and experts in the field of computer technologies. Research conducted on prospective mathematics teachers of algebra knowledge showed that the participants have a significantly lower level of conceptual than procedural knowledge [16].

We are convinced, based on the results of the entrance exams and grades in mathematics, that students who enroll in the Faculty of Science, Department of Mathematics and Computer Science in Sarajevo and the Faculty of Mathematics in Belgrade know how to solve different types of equations and perform operations with polynomials. With this research, we intend to examine and classify precisely the conceptual knowledge of these basic algebraic terms.

Based on the literature review, it is easy to conclude that most of the research on the conceptual and procedural understanding of mathematics was conducted on students of lower grades of elementary school. That is why our research, in which significantly older students participated, expanded the existing insight into the conceptual understanding of the basics of mathematics. Encouraged by [2], we offer a slightly different definition and introduce levels of conceptual understanding in mathematics.

Because of its abstract character, algebra is not an easy subject to teach as such, particularly when algebra is often presented without any context but as 'naked' equations and formulas [1]. However, the importance and need for knowledge from algebra is unquestionable nowadays. In Section 2 we elaborate on its importance in the mathematics curricula.

In Section 3 we explain terms 'procedural' and 'conceptual knowledge' in mathematics, we elaborate on the fact that there are no single commonly accepted definitions of these terms. We point out the importance of conceptual knowledge and introduce a way to define it in such way that enables us to measure conceptual understanding of mathematical terms.

Sections 4 and 5 are focused on our practical study designed to measure conceptual knowledge of basic algebraic ideas among first-year students at mathematics faculties in the Western Balkan. In Section 4 we describe the design of the test and give some details about its implementation, while in Section 5 evaluation of the test is presented. A complete overview of the evaluation of the test is given, some selected tasks are elaborated in details and discussion of overall results and some conclusions are presented.

2. The importance of algebra in the mathematics curriculum

As previously said, algebra is a cornerstone of civilization and the foundation on which mathematical knowledge is built in the study of mathematics. In the school curriculum, it actually begins with the introduction of the concept of indeterminate and algebraic operations with indeterminates, which brings us to the concept of polynomial. The solution of quadratic equation is the next major step, although linear equations, done previously, are also important. However, quadratic equation is the first algebraic notion which introduces transformations not so easy to follow, crucially important for further algebraic knowledge – the completion of the square. We dare to state that proper algebra begins from quadratic equation.

Mathematicians agree that the most important discoveries in the medieval and renaissance algebra, which represent the peak of the classical algebraic approach, are the solutions of cubic and quartic equation by Tartaglia, Cardano and Ferrari. Ferrari's solution of the quartic is completely dependent on the completion of the square, which leads to decomposition of the quartic into product of two quadratic equations.

Algebraic procedures actually repeat and generalise arithmetic procedures which should be already mastered at that point. The importance of algebra in the mathematics curriculum arises from the possibility to use formal algebraic expressions and to transform them in accordance with (already mastered) arithmetic rules. So, the knowledge of algebra is of great importance in the process of abstraction which is the most important ingredient of mathematical education.

The importance of algebra within the school curriculum cannot be overestimated. What is the main task of teaching algebra? As to procedural and conceptual contents of algebraic part of mathematical education, one has to admit that both are highly important. However, first things first, and procedures in algebra come before concepts. Children first learn the rules of commutativity and associativity on concrete examples by doing calculations. Only when they master procedures of calculation (or learn how to calculate up to 1000), they can be taught concepts: the corresponding general laws. So, procedural knowledge has to preceed conceptual. But then, it goes back and returns to procedural on the new, higher level, with new procedures and new objects. Children first grasp the ideas of factorisation, least common multiple and greatest common divisors by factoring numbers, and if they don't understand it properly and in a proper time, the respective algebraic notions for polynomials, coming later in the curriculum, will be lost for them. So, these two types of knowledge are interwoven, always beginning with the procedural knowledge.

The main, starting ingredient in algebra is the procedural task how to operate with symbols, which includes parameters, unknowns and variables, indeterminates of all kinds and the concept of polynomial. This has to be built upon a good knowledge of operations with explicit numbers – also a procedural task, belonging to arithmetics, which precedes algebra. A good knowledge of arithmetic procedures is essential for understanding arithmetics concepts (the rules for operations with numbers), and then the algebraic rules and finally algebraic concepts. The knowledge of algebra is very often built upon procedural knowledge using the recognition and active use of various algebraic patterns. These recognised and repeating patterns then become new concepts. Patterns appear from the very beginning, such as $2+3 = 3+2$, the pattern of commutativity of number addition (which obviously has to come after the addition procedure $3 + 2 = 5$, and not before it). There are also deeper patterns, such as $36 + 7 = (36 + 4) + 3$ or $12 \cdot 9 = 12 \cdot 10 - 12 \cdot 1$, which involve new intellectual efforts, all very important for the general mathematical proficiency. Patterns which are mastered in arithmetics make it much easier to pass to algebraic concepts and then new patterns and concepts in algebra, such as completing the square. Learning by patterns is actually the use of the process of induction (not mathematical induction, which is a deduction process).

Hence, the knowledge of algebra is the second important step in the development of pupil's mathematical abilities, the first being arithmetics. Already from this first stage – the knowledge of arithmetics, there is a watershed in mathematical knowledge: the flow of algebra and the flow of geometry. This is the main point which has determined the development of mathematics from ancient Greek times. When people started to understand numbers not (only) as quantity cardinals, but as segment lengths, this marked the beginning of the new era. Of course, this didn't come immediately, but to the time of ancient Greeks it was already completed. The Greeks got afraid of the infinity hidden behind the incommensurability – today, we would say, the real numbers, and they stickled to geometry instead of expanding arithmetics. This has determined further development of mathematics in the following 15 centuries.

Al-Khwarizmi's work was not known in European mathematics (actually, there was no European mathematics at all, other than ancient Greek), until 1145, when two Latin translations inspired by Al-Khwarizmi appeared: Plato of Tivoli translated the book of Abraham bar Hiyya, entitled Hibbur ha-Meshihah ve-ha-Tishboret, written in Hebrew after Al-Khwarizmi's book some years before, and Robert of Chester translated Al-Khwarizmi's book.

This gap between two mainstreams in mathematical knowledge, algebra and geometry, continued until de Fermat's and Descartes' introduction of the method of coordinates in the 17th century. Geometrical knowledge from Euclid's times is based on deduction reasoning, which is prevailingly conceptual, and algebraic knowledge on calculation procedures, which are clearly procedural. However, the gap is still present, and in many educational systems in the world today, teaching of algebra is separated from teaching of geometry, either as two different curricular subjects, or interwoven within one mathematical subject, but with different curricular streams. In some educational paradigms, geometrical deductive concepts are replaced completely by procedures of analytical geometry. This, however, leads to huge misconceptions in the process of learning mathematics.

One has to point out that there is a very common but dangerous misconception in restricting ourselves to the pragmatic point of view: 'teach only what is used later'. In mathematics, knowledge has to be built upon a sound fundament, and it has to be much wider than it appears. And if some bricks in the lower layer are missing, the upper layers would collapse. Do not let it collapse!

3. Procedural and conceptual knowledge in mathematics

Learning mathematics is a very complex and challenging activity. It relies on coordinated processes of acquisition and connection of two essential types of mathematical knowledge, conceptual and procedural knowledge. For a long time, research in mathematics was focused on procedural knowledge and its assessment. However, in recent years special attention has been focused on conceptual knowledge and evaluation methods. Also, efforts are being made to shed light on the connection between these types of knowledge. See for example [8], [9] and [15]. Although these two types of mathematical knowledge have been studied since the 70s of the last century, it is impossible to define them simply and precisely. One of the reasons is that it may be challenging to distinguish conceptual from procedural knowledge at some points in development because the two forms of knowledge are deeply intertwined [2]. In the same paper, the authors analysed the existing literature on conceptual knowledge. They classified the definitions of conceptual knowledge into six types: connection knowledge, general principle knowledge, knowledge of principles underlying procedures, category knowledge, symbol knowledge and domain structure knowledge. The fact that there are so many types of definitions of a term indicates a lack of consistency in the meaning of conceptual knowledge and its measurement methods.

We will single out two definitions of procedural and conceptual knowledge in mathematics that are present and accepted in the literature.

- According to [5], procedural knowledge denotes the dynamic and successful utilisation of practical rules, algorithms or procedures within relevant representation form(s). This usually requires not only the knowledge of the objects being used but also the knowledge of the format and syntax for the representational system(s) expressing them. The same authors define conceptual knowledge as knowledge of and skilful drive along particular networks, the elements of which can be concepts, rules and even problems given in various representation forms.
- In [12], procedural knowledge is defined as the ability to execute action sequences to solve problems. This type of knowledge is tied to specific problem types and therefore needs to be more widely generalizable. Conceptual knowledge is defined as implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain. This type of knowledge is flexible and not tied to specific problem types and is therefore generalizable.

Although there are many definitions of conceptual knowledge, we will offer a new one slightly different from the existing ones, giving us a framework for measuring conceptual knowledge. We define conceptual knowledge in mathematics as the understanding of mathematical ideas, their properties and the connections between these ideas within the observed field of mathematics, and connections with concepts from other fields of mathematics.

Let us take a closer look at this definition in the example of the concept of an equation. The equation appears for the first time in mathematics in the lower grades. It is constantly present in mathematics classes in primary and secondary schools and all mathematics courses taught at universities. Students in schools traditionally do not learn what an equation is, but the focus is on 'how to solve a given equation'. Thus, it is expected that over time the students become aware of the concept of the equation as a mathematical expression in which equality and an unknown appear and understand its characteristics, which influence the choice of procedure for solving it.

When learning to solve different equations during class, students are expected to notice the properties of equations such as the number of unknowns, the nature of the solution, the degree of the polynomial equation, the presence of a parameter and its relationship with the solution. Furthermore, students are expected to understand the concept of a system of equations and its solution, inseparable from the concept of the equation itself. Through analytical geometry, the concept of an equation relates to different geometric objects and their mutual relations, which are reflected in the equations that describe these objects. This gives the equation as an algebraic concept a new expanded meaning in geometry. The concept of differential, difference and integral equations is also developed based on the algebraic equation, and they represent the fundamental concepts of important mathematical fields.

Haapasalo & Kadijevich [5] suggested the existence of different levels of conceptual knowledge, which denote *concept understanding on the identification level* and *concept understanding on the verbalisation level*. We have specified these levels and refined them.

Our definition of conceptual knowledge in mathematics suggests the five levels of development of this type of knowledge.

(I) The first level of conceptual knowledge is determined by recognising the observed concept. This level of conceptual knowledge does not imply that students can verbalise an idea but that they understand the fundamental feature that sets an idea apart from others.

(II) The next level is a recognition of the specific characteristics of the observed idea, which are usually related to the implementation of procedures in which the given concept is included.

(III) The third level in the development of conceptual knowledge is the level of verbalisation. A student who understands a mathematical idea at this level can describe its fundamental properties and explain it using a common language and not necessarily strict mathematical terminology.

(IV) At the fourth level of conceptual knowledge development, the student understands the importance of the idea itself in the field of mathematics, in which it is observed and the connection to other ideas within the same field.

(V) Finally, at the fifth level of conceptual knowledge, the student is aware of the different forms a mathematical idea can have in different areas of mathematics. The student understands the similarities and differences between different forms of the observed idea and its importance in each of the observed areas.

Building conceptual mathematical knowledge takes a lot of time and patience. Knowledge of these ideas is developed gradually by doing tasks and applying various forms of procedural knowledge. Of course, the role of the teacher and the textbook should be to direct the student's focus on understanding the concept and not only on the search for a solution to the task. Conceptual knowledge of mathematics in the population of children in primary school can range within the first two levels. Rarely are young children able to verbalise a mathematical idea. During primary and secondary school, students needed the opportunity (at least most) to develop their conceptual knowledge of basic mathematical ideas at the fourth or fifth level. Still, they should have mastered the knowledge up to the third level.

How far first-year students at mathematics faculties in the Western Balkan have mastered the conceptual knowledge of basic algebraic ideas is the subject of research presented in the paper.

4. Study design

4.1. Test design

Data for the study are collected using specially designed test with the aim to determine the level of conceptual knowledge of some fundamental algebraic concepts. Four basic algebraic concepts are selected: relations and operations (R&O), equations (E), parameters in equations (PiE) and polynomials (P).

The selected concepts are the subject of mandatory topics in primary and high school mathematics curricula. Also, since these concepts are fundamental not only in algebra, but generally in mathematics, they are deeply related to many other topics covered during primary and high school classes in mathematics. Moreover, they are widely used in other primary and high school subjects such as physics and chemistry.

The test consists of four parts corresponding to selected concepts. Each part consists of six tasks. These tasks are design to identify level of conceptual knowledge of participants. Our focus are the first three levels, defined in Section 3, so in the sequel, we categorize tasks according to these levels. Corresponding groups of tasks are denoted by I, II and III, respectively.

Tasks are design to include commonly used examples and expression from the widely used literature in primary and highs schools in Western Balkan. Some tasks are motivated by the examples given in [1] and [10].

4.1.1. Relations and operations. For the category I task students are asked to identify operation symbols among different 16 classical mathematical symbols. Symbols for arithmetic, set, and logical operations are included. Also, some classical relation symbols, as well as universal quantifier symbol, are included in the given list. Two tasks from category II are related to the notion of associativity, commutativity, and distributivity property. In the other two tasks (R&O_4 and R&O_5) from category II we ask students whether we are able to replace symbol ? in $P?Q$ if $P = a+3$, $Q = 3+a$ and a is an arbitrary integer, with some mathematical operation/relation symbol. Students are asked to justify their answers. The category III task, i.e. the task to identify conceptual knowledge on verbalization level, is devoted to *equality* relation. Students are asked to derive some conclusions from a few given expressions, based on their basic properties (namely, symmetry and transitivity), to explain the reasoning behind the conclusion, and to name properties they used.

4.1.2. Equations. For the category I task, students are asked to identify equations through multiple-choice question. Four tasks from category II are related to the property of the order of polynomial equation, equivalent equations, the notion of solution of the equation, and the notion of mathematical identity. Students are asked to identify polynomial equation of the third order, then to identify equations equivalent to a given equation, and mathematical identity among given expressions through multiple-choice questions. Then they were asked to check whether a given number is a solution to a given equation and to justify their answers. In the last task, category III, students are asked to explain the notion of the equation and its relation to the notion of mathematical identity.

4.1.3. Parameters in equations. For the category I task, students are asked to identify equations with parameters through multiple-choice question. Two category II tasks are designed to check whether students understand that the number of solutions, and solutions themselves, for a given equation with a parameter, may depend on the value of the parameter. The following task asks students to identify what is a parameter from a given textual description of the problem and its corresponding equation. In the last task in this category, students are asked to construct an equation with given (in terms of a parameter) solutions. For the category III task, students are asked to give a verbalization of a problem that corresponds to a given equation with a parameter.

4.1.4. Polynomials. Identification of polynomials through multiple-choice question is subject of the category I task in this part of the test. Tasks in category II are related to the notion of the degree of a polynomial, its coefficients, and exponents. Additionally, students are asked to find a neutral element for addition operation in a set of all polynomials in one variable with real coefficients. Also, they were asked to find the inverse element under the addition of the given polynomial. In the introduction of these tasks notion of a neutral and inverse element under addition in the set of real numbers is recalled. Thus, correct answers may be derived using a process of generalization. For the verbalization level, i.e. category III task, students are asked to explain the notion of polynomials.

4.2. Participants and data collection

Participants in the research (N=180) were freshmen (students who enrolled in the first year of study for the first time) from the Faculty of Mathematics, University of Belgrade (86 students) and the Department of Mathematical and Computer Sciences, University of Sarajevo (94 students). According to the number of enrolled students, these are the two largest faculties for studying mathematics in the Western Balkans.

The mathematics achievement standards for students who attended these faculties are above the national average. At both faculties, students are offered the study of theoretical mathematics and mathematics in education and computer science, making these faculties very attractive.

Ranking of the students in the process of enrollment there is based on high-school grades (overall average combined with grades from selected subjects, including mathematics). In addition at Faculty of Mathematics, University of Belgrade the entrance exam in mathematics is applied.

Testing was performed in October 2022, during the first weeks of classes and before the students started learning content that could affect their previous knowledge of algebra. Students took part in the study during their regular mathematics lessons. Participants were given 60 min to complete the test.

5. Results and discussion

5.1. Evaluation

In this section, we provide details regarding the evaluation of the tests. For each task, responses are categorized in the following way:

(CC) Correct and complete answer

(PC) Partially correct answer (for multiple-choice questions majority of the correct answers are selected, but no wrong answer is selected, for the open questions not all, but a majority of crucial information is included in the given answer)

(NC) Not correct answer (for multiple choice questions at least one of the incorrect answers is selected, for the open questions no or very little of crucial information is included in the given answer)

(NA) No answer or 'I do not know' is written.

In addition for each task, the most common mistakes are identified and recorded in the process of test evaluation. Some of them are listed and elaborated in the following subsection.

5.2. Results

In the following subsections we give an overview, per selected concepts, of the evaluation of the test under consideration. Complete statistical data for all tasks per introduced categories are given, combined with some specific data induced by frequent mistakes.

5.2.1. Relations and operations. Percentages of the student answers that correspond to categories explained in 5.1 for the first part of the test are given in Table 1 and presented using graph in Figure 1.

Category	Task	CC (%)	PC (%)	NC (%)	NA (%)
I	Task R&O$_1$	8.38	69.27	21.23	1.12
II	Task R&O$_2$	23.46	32.40	33.52	10.61
	Task R&O$_3$	66.48	3.35	28.49	1.68
	Task R&O$_4$	31.28	28.49	18.44	21.79
	Task R&O$_5$	15.64	12.85	32.96	38.55
III	Task R&O$_6$	12.29	28.49	52.51	6.70

Table 1. Responses for the part *Relations and operations* of the test

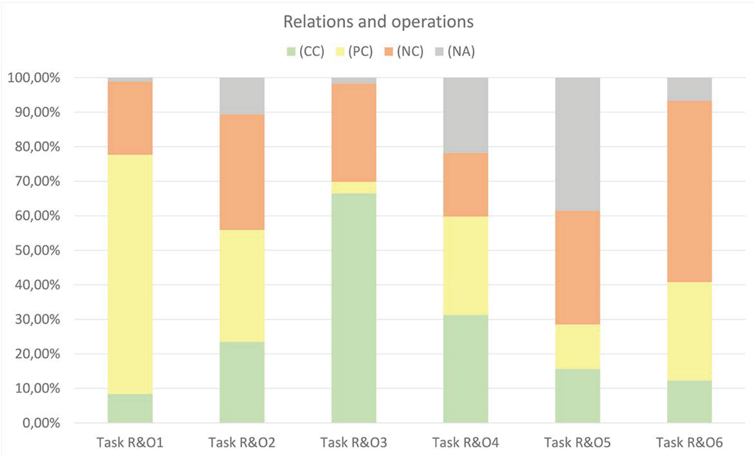

Figure 1. Responses for the part *Relations and operations* of the test

The given table indicates that most of the participants can only partially identify operation symbols. Namely, correct answers in the given list of symbols in R&O_1 include arithmetic, set, and logical operation symbols. Only, 8.38 % of students gave complete and correct answers. Most commonly (51.4%) participants recognize only symbols of arithmetic operations, but not set and logic operations.

A level of recognition of the specific characteristics of the concepts of relations and operation differs across the tasks from category II. Most of the students (66.48%) correctly identify that associative and commutative laws are related to mathematical operations.

It is interesting to compare results for tasks R&O_4 and R&O_5.

Task R&O_4: *Let $P = a + 3$ and $Q = 3 + a$, where a is positive integer. Let $P?Q$. Is it possible to replace the symbol ? with a symbol of mathematical relation? If yes, explain what mathematical expression is obtained. If no, justify your answer.*

Percentage of correct answers in R&O_4 is nearly double the corresponding percentage for question R&O_5, indicating that it is easier for students to recognize that they can compare $a + 3$ and $3 + a$ than that they can do some operations (for example addition, subtraction or multiplication) on these expressions. A large number of participants (38.55%) even have not tried to solve question R&O_5. Also, 15.08% of respondents indicate that there is no difference between questions R&O_4 and R&O_5, showing that they do not distinguish concepts of operations and relations at all.

Task R&O_6: *If $b = a$, $c = d$, $b = d$ and $e = c$ then $a = 5$.*
Justify your answer. Name the properties of the relations used to obtain result.

In order to assess students' performance on verbalization level, in R&O_6, students are asked to explain the reasoning behind their conclusion and to name properties they used. Only when for one of these two parts the correct answer is given, we assume the answer is partially correct. The percentage of 64.80% correct conclusion but without any or with an incorrect explanation or names of properties indicates that there is a large gap between the number of students that are able to apply some procedure and the number of those that understand the concept behind it and can verbalize it. Additionally, the number of completely correct answers is equal to the number of responses with an acceptable explanation of the reasoning behind the conclusion. Thus, partially correct answers are those with correct conclusion and correct names of the properties. This indicates that it is most difficult for students to explain their reasoning, which is the essence of conceptual knowledge at the verbalization level.

5.2.2. Equations. Percentages of the student answers that correspond to evaluation procedure given in 5.1 for the second part of the test are given in Table 2 and Figure 2.

Category	Task	CC (%)	PC (%)	NC (%)	NA (%)
I	Task E_1	31.11	40.00	28.33	0.56
II	Task E_2	50.00	22.22	26.67	1.11
	Task E_3	61.67	22.22	13.89	2.22
	Task E_4	52.22	21.67	19.44	6.67
	Task E_5	25.56	34.44	27.78	12.22
III	Task E_6	6.67	20.56	51.11	21.67

Table 2. Responses for the part *Equations* of the test

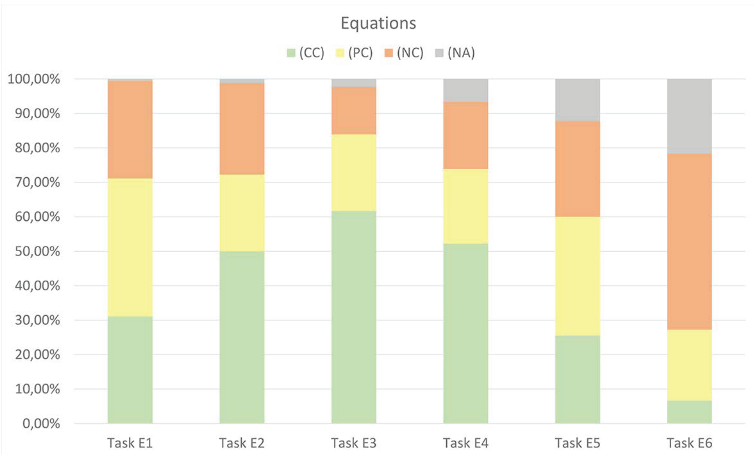

Figure 2. Responses for the part *Equations* of the test

Task E$_1$: *Find the equation(s). Encircle letter(s) in front of correct answer(s).*

(a) $x^2 - 3x - 2 = 0$ (c) $x - y + 2ax$ (e) $\frac{a^2x-2aby+c}{3x-2y}$

(b) $3a - 0.75 = b$ (d) $(x-y)^2 = x^2 - 2xy + y^2$ (f) $3 - 0.75 = 2.25$

In task E_1 given answers are chosen in such a way to assess students' ability to identify two main elements (the existence of the equality sign and the existence of one or more variables in order to have an equation). Our results show that the second one is more difficult to capture by our respondents. The most common mistake was identifying (f) as an equation. A large portion (18.89%) of partially correct answers (40.00%) corresponds to those where mathematical identity (d) is not recognized as an equation valid for all values of the variables.

A relatively large percentage of correct or at least partially correct answers (compared to the rest of the test) to tasks E_2 to E_4 indicates that participants better identify and understand concept attributes than the concept of the equation itself. We understand that it is closely related to the fact that attributes (equivalent equations and solution to the equation) are extensively used when solving procedural tasks.

The question E_5 is related to the identification of mathematical identities among given expressions. The most commonly selected wrong answer (25.00%) is the one where the domain of the sides of the equation is not taken into consideration (precisely, $\frac{(x+1)^3}{x+1} = (x+1)^2$ is recognized as an identity despite the fact that domains of the expression of the sides of equality sign are different). A complete misunderstanding of the identity concept is demonstrated in the 13.33% of responses where a polynomial $x^3 + 3x^2 + 3x + 1$ is marked as an identity. We assume that is closely related to the fact that the given polynomial is a part of the classical formula for the cube of the binomial.

Task E_6: *Explain the term 'equation'.*

Explain the relation between terms 'mathematical identity' and 'equation', if you assume that there is such a relation.

From percentages that correspond to E_6, we are able to conclude that there is a very low level of understanding of the concept of the mathematical equation on the verbalization level. 51.11% of given answers do not include any of the key elements necessary to form an equation, and in the case of 21.67%, there is no given answer to this question at all.

The most frequent incorrect answers include the one where the equation is described as a 'mathematical task', while the identity is incorrectly described as 'a part of the equation'. We believe that these answers come from a predominant focus on procedural tasks during mathematical classes. Namely, the most familiar (to participants) form of tasks that include equations reads 'Solve the equation'. Also, mathematical identities are commonly used in the procedures of solving equations, thus they are perceived as a part of the mathematical task of solving equations.

5.2.3. Parameters in equations. Percentages of the student answers that correspond to introduced evaluation categories for the third part of the test are given in Table 3 and presented by Figure 3.

Given results, for item PiE_1, indicate that nearly half of the students are able to identify properly (and completely) an equation with a parameter. A bit more than an additional quarter can partially identify it. Most of these partially correct answers come from the fact that students missed labeling an equation with two parameters as a correct answer.

Category	Task	CC (%)	PC (%)	NC (%)	NA (%)
I	Task PiE_1	49.44	28.33	19.44	2.78
II	Task PiE_2	30.56	24.44	38.33	6.67
	Task PiE_3	26.67	43.89	25.00	4.44
	Task PiE_4	62.22		33.33	4.44
	Task PiE_5	26.67		28.89	44.44
III	Task PiE_6	11.11	3.89	37.78	47.22

Table 3. Responses for the part *Parameters in equations* of the test

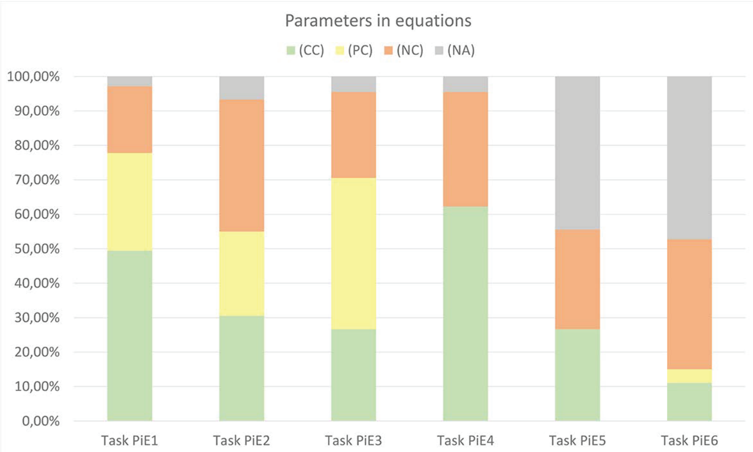

Figure 3. Responses for the part *Parameters in equations* of the test

Questions PiE_2 and PiE_3 are designed to assess the ability of students to recognize that the number of solutions and solutions of the equations itself, in general, depend on the values of a parameter. A relatively large percentage of partially correct answers in PiE_3 indicates that respondents are missing one of these two facts.

The form of the questions PiE_4 and PiE_5 is such that there is no option for a partially correct answer. Basically, in PiE_4 students are asked to identify a parameter from a given textual description of the problem and its corresponding equation. The problem contains two quantities and the equation in question is a simple linear one. Note that there was a chance that some of the correct answers may be obtained by chance.

Task PiE_5: *Form the equation whose solutions are 1, a and a^2.*

Note that in PiE_5 there is no any special request for the type of equation or its form, so there is no single correct answer. However, we expected the equation written in terms of simple third order monic polynomial written as a product of linear factors. Most of the correct answers were in such a form. A very high percentage of those who have not even tried to solve this task is noticeable. Some of the incorrect answers that include expression in terms only of parameter a (i.e. without any variable) indicate that the fact that solutions are given in terms of parameters caused a problem for a non-negligible number of participants.

Results for PiE_6 indicate that almost half of the students even have not tried to answer a question designed to assess their understanding of a concept of a parameter in the equation on a verbalization level.

5.2.4. Polynomials. Percentages of the student answers that correspond to introduced evaluation categories for the fourth part of the test are given in Table 4 and presented in Figure 4.

Category	Task	CC (%)	PC (%)	NC (%)	NA (%)
I	Task P_1	6.70	5.59	85.47	2.23
II	Task P_2	27.93	14.53	50.28	7.26
	Task P_3	49.16		45.25	5.59
	Task P_4	70.95		25.14	3.91
	Task P_5	24.20	23.46	17.32	35.20
III	Task P_6	7.26	13.41	48.60	30.73

Table 4. Responses for the part *Polynomials* of the test

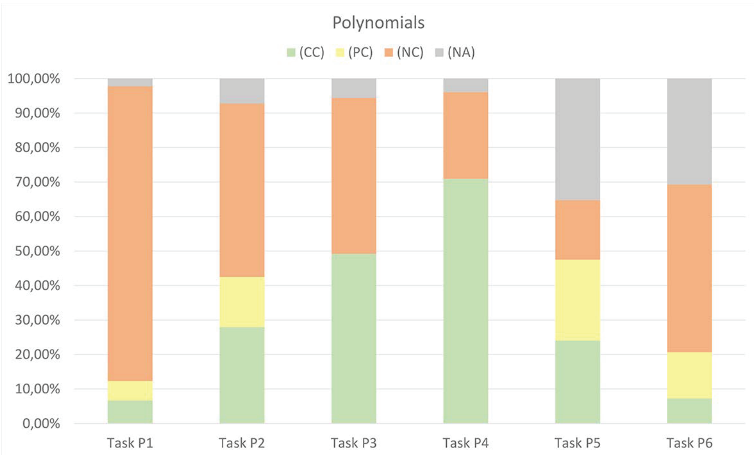

Figure 4. Responses for the part *Polynomials* of the test

Task P_1: *Find the polynomial(s). Encircle letter(s) in front of correct answer(s).*

(a) $2x^2 + 5x = 0$
(b) $3x^{100} - 2$
(c) 5
(d) $x^{-3} + 2x - 1$

Percentages given in Table 4 corresponding to task P_1 indicate a very low level of ability of participants to identify polynomials. We noted 70.39% of students' answers where (d), the expression with a negative power of the variable, is denoted as a polynomial. Additionally, we noticed that most of the partially correct answers are due to not recognizing a constant as a special case of polynomials, i.e. (c) is not labeled as a correct answer. We found these results very unexpected. This can be understood as an example of an inappropriate way of generalization of objects familiar to students. Basically, commonly used examples of polynomials are those with very small degrees (usually 1, 2, 3) and some students found -3 very close to these numbers, while 100 seems very far from them. This produces incorrect conclusions. Such kind of reasoning implies a complete absence of understanding of the concept of polynomials.

Results for tasks P_3 and P_4 indicate a relatively good (compare to the rest of the test) understanding of the notion of the degree of a polynomial, its coefficients, and exponents.

Task $\mathbf{P}_5$: *In the set of real numbers $\mathbb{R}$ there is an element 0 such that for all $a \in \mathbb{R}$: $a + 0 = 0 + a = a$. 0 is the neutral element for addition in $\mathbb{R}$. If P is the set of all polynomials in variable x with coefficients in $\mathbb{R}$, is there neutral element for addition in P? If yes, what is it?*

In the set of real numbers $\mathbb{R}$ for all $a \in \mathbb{R}$ there is $-a \in \mathbb{R}$ such that $a+(-a) = -a + a = 0$. $-a$ is the additive inverse of element a in $\mathbb{R}$. If P is the set of all polynomials in variable x with coefficients in $\mathbb{R}$, is there an additive inverse for $x^4 - 3x^2 + 2$ in P? If yes, what is it?

Results for the question P_5 indicate a relatively low ability of respondents to generalize the notion of neutral element and additive inverse for polynomials from the given definition of these terms for real numbers. This is in accordance with the findings that students commonly do not recognize constants as polynomials.

Interestingly, there are those (6.15%) claiming that the neutral element exists, but the additive inverse does not, but also those (7.26%) claiming that the neutral element does not exist, while the additive inverse exists (despite the fact that the given definition of the additive inverse element includes the existence of neutral element for addition).

Results for P_6 indicate a very low level of understanding of the concept of a polynomial on the verbalization level.

5.3. General discussion

Here we give overall discussion of our research results, bearing in mind that the participants are the Department of Mathematics students. It is known that the COVID-19 pandemic impacted the knowledge of mathematics. Still, other causes of such results include old textbooks and teaching methods and the constant tendency to reduce the number of mathematics classes in schools. However, the most significant influence was the teaching traditionally focused on acquiring procedural knowledge. The paradigm that conceptual knowledge of mathematics can be built on its own, based on acquired procedural knowledge and after a certain number of repeated procedures, needs to be revised. The most apparent examples supporting this claim are the tasks E_1, E_6, and P_1 results.

An equation is a mathematical concept that students are introduced to in the third grade of elementary school, and then equations are worked on in detail in the fourth and sixth grades. A significant part of the mathematics curriculum in the last grade of primary school is dedicated to solving linear equations, inequalities and systems of equations. According to the curriculum for secondary schools, in addition to linear equations, methods for solving quadratic, irrational, algebraic equations of a higher degree and exponential, logarithmic and trigonometric equations are taught. Thus, more than 100 mathematics lessons in high school are devoted to various types of equations. We are confident that the examined group of students knows how to solve, at least, simple examples of the mentioned types of equations. However, according to the results of our research, only 31% of students correctly recognised the equation (task E_1), and 7.31% could explain the meaning of the term equation. This means that a tiny number of students are even at the first level, the level of recognition of conceptual understanding of the concept of equation and an even smaller number at the level of verbalisation. We can imagine what these results would look like if the research sample were from the general population.

We see similar results in the case of the conceptual understanding of the term polynomial. Almost all students participating in the research know how to add, multiply and factor polynomials, at least in simple cases. But still, most of the students, 86%, incorrectly recognised the polynomial, and only 6.7% explained the definition of the term polynomial. Analysing the available textbooks from both countries that deal with polynomials, a rough introduction with a correct definition of polynomials is noticeable. Still, it needs to be followed by appropriate examples and counterexamples. Instead, it immediately switches to operations with polynomials. Thus, the polynomials present in textbooks do not have a degree higher than 10, and as a counterexample, an expression with a negative exponent does not appear. This led the students to conclude that $x^{-3}+2x-1$ is a polynomial, but $3x^{100} - 2$ is not, which is wrong. We expected that the concept image of polynomial is formed and controlled by the definition of the concept, but we see that this is not the case. As we can conclude from [3], understanding mathematical definitions requires special attention.

Both examples clearly show the harmful consequences of insisting on developing only one type of mathematical knowledge. In the middle of the last century, when mathematics was taught in schools as a prerequisite for other subjects such as physics and when numerous calculations were done manually, it was understandable to insist on acquiring procedural knowledge. However, nowadays, when technological resources enable fast, accurate and precise mathematical calculations, teaching mathematics must be adapted. Engineers agree that a conceptual approach to mathematics in engineering work is essential and that a predominantly conceptual approach to mathematics is preferable to a procedural approach [4].

Conceptual knowledge of mathematics enables us to use technology to perform mathematical calculations, check the obtained results and apply mathematics. It is unnecessary to say how crucial conceptual knowledge of mathematical concepts is for the development of mathematics itself. However, there is evidence [5, 12, 13, 14] that these two types of knowledge should be taught simultaneously because they intertwine and support each other. The research presented here shows the results that arise when procedural knowledge is insisted upon, and we believe that the other extreme would have a similar result. Therefore, the balance between these two types of mathematical knowledge is crucial.

At the end, a short overview of the teaching of algebra within the mathematics curriculum. More than 30 years ago, it was written in [17]:

> *But history reminds us that algebra is much more than generalised arithmetic, and we should be more realistic in our approach to algebra. We should not assume that the transition from arithmetic to algebra is obvious and clear sailing.*

The presented results indicate that we did not take the recommendation seriously enough. Algebra represents an inevitable passage into the world of mathematics. Regardless of the intention of acquiring mathematical knowledge (application of mathematics or development of mathematical theories), using all new knowledge on how to learn mathematics would be recommended because:

> *We need to change the approach to teaching algebra for at last two reasons: first, to make essential concepts more understandable to students; second, to give students access to topics that otherwise would be delayed until late in the curriculum.* [11]

Acknowledgements. The authors are grateful to colleagues and students who made this study possible.

The authors would like to thank the referee for the comments and suggestions, which helped to improve the manuscript.

REFERENCES

[1] A. Arcavi, P. Drijvers, K. Stacey, *The Learning and Teaching of Algebra*, Routledge, 2017.

[2] N. M. Crooks, M. W. Alibali, *Defining and measuring conceptual knowledge in mathematics*, Developmental Rev., **34** (2014), 344–377.

[3] B. S. Edwards, M. B. Ward, *Surprises from mathematics education research student (Mis) use of mathematical definitions*, Amer. Math. Monthly, **111**, 5 (2004), 411–424.

[4] J. Engelbrecht, C. Bergsten, O. Kägesten, *Conceptual and procedural approaches to mathematics in the engineering curriculum: Views of qualified engineers*, Inter. J. Math. Educ. Sci. Techn., **46** (2015), 979–990.

[5] L. Haapasalo, Dj. Kadijevich, *Two types of mathematical knowledge and their relation*, J. für Mathematik-Didaktik, **21**, 2 (2000), 139–157.

[6] L. Hakim, B. Yasmadi, *Conceptual and procedural knowledge in mathematics education*, Design Engineering, **9** (2021), 1271–1280.

[7] N. Herscovics, *Cognitive obstacles encountered in the learning of algebra*, In: S. Wagner, C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*, Lawrence Erlbaum associates National Council of Teachers of Mathematics, 1989, pp. 60–86.

[8] D. P. Hurrell, *Conceptual knowledge OR procedural knowledge OR Conceptual knowledge AND Procedural knowledge: why the conjunction is important for teachers*, Australian J. Teacher Educ., **46**, 2 (2021), 57–71.

[9] Dj. M. Kadijevich, *Relating procedural and conceptual knowledge*, The Teaching Math., **II**, 1 (1999), 59–64.

[10] Dj. M. Kadijevich, *Conceptual tasks in mathematics education*, The Teaching Math., **XXI**, 1 (2018), 15–28.

[11] J. R. Leitzel, *A reaction to: 'Algebra: What should we teach and how should we teach it'*, In: S. Wagner, C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*, Lawrence Erlbaum associates National Council of Teachers of Mathematics, 1989, pp. 25–32.

[12] B. Rittle-Johnson, R. S. Siegler, M. W. Alibali, *Developing conceptual understanding and procedural skill in mathematics: An iterative process*, J. Educ. Psychology, **93**, 2 (2001), 346–362.

[13] B. Rittle-Johnson, J. R. Star, *Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations*, J. Educ. Psychology, **99**, 3 (2007), 561–574.

[14] B. Rittle-Johnson, M. Schneider, J. R. Star, *Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics*, Educ. Psychol. Rev., DOI 10.1007/s10648-015-9302-x, 587–597.

[15] J. A. Thorpe, *Algebra: What should we teach and how should we teach it*, In: S. Wagner, C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*, Lawrence Erlbaum associates National Council of Teachers of Mathematics, 1989, pp. 11–24.

[16] H. E. Zuya, *Prospective teachers' conceptual and procedural knowledge in mathematics: The case of algebra*, Amer. J. Educ. Res., **5**, 3 (2017), 310–315.

[17] D. Wheeler, *Context for research on the teaching and learning of algebra*, In: S. Wagner, C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*, Lawrence Erlbaum associates National Council of Teachers of Mathematics, 1989, pp. 278–287.

A.L.: Faculty of Mathematics, University of Belgrade, Serbia

ORCID: 0000-0002-7267-1490

E-mail: `aleksandar.lipkovski@matf.bg.ac.rs`

M.M.D.: Faculty of Natural Sciences and Mathematics, University of Sarajevo, Bosnia and Herzegovina

ORCID: 0000-0002-0384-1994

E-mail: `manuela.md@pmf.unsa.ba`

A.O.: Faculty of Natural Sciences and Mathematics, University of Sarajevo, Bosnia and Herzegovina

ORCID: 0000-0001-6269-9759

E-mail: `almasa.odzak@pmf.unsa.ba`

Received: 16.02.2024, in revised form 25.04.2024

Accepted: 26.04.2024