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AN APPLICATION OF THE QUADRILATERAL'S GEOMETRY IN SOLVING COMPETITIVE PLANIMETRIC PROBLEMS

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Abstract. In the present publication, which can be considered as a continuation of the paper V. Nenkov, St. Stefanov, H. Haimov, *An application of quadrilateral's geometry in solving competitive mathematical problems*, Synergetics and reflection in mathematics education, Proceedings of the anniversary international scientific conference, Pamporovo, October 16-18, pp. 121–128, 2020, the application of the geometry of quadrilateral to the solution of exams is considered. Three examples given in the magazine "Mathematics and Informatics" have been selected, the solutions of which illustrate well the benefit of studying the recently discovered properties of convex quadrilaterals. Two solutions to the tasks are presented for comparison. The first, proposed by participants in the competition, are relatively complex and longer, and the second—based precisely on elements of the geometry of quadrilateral, are significantly simpler and shorter. These solutions are based on properties of quadrilaterals associated with some of their remarkable points.

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1. Introduction

Unlike the well-studied triangle, an arbitrary convex quadrilateral has remained an understudied geometric figure until now. In a series of papers, significant progress was made in the study of convex quadrilaterals. Numerous remarkable points, lines and circles, and two universal transformations were found. Using their properties, the classical theorems of Mickel, Gauss, Ober and Steiner on complete quadrilaterals [8] and a theorem of Carnot on triangles [3] were generalized. A generalization of Steiner's trapezoid theorem was also found [1], as well as an addition to Brocard's inscribed quadrilateral theorem [2]. Two popular theorems were also transferred from triangles to quadrilaterals: the cosine theorem and the so-called cotangent theorem [3]. The obtained results proved to be useful in solving non-standard tasks. We will focus on three tasks from the competition of the "Mathematics and Informatics" magazine. We will see that knowing even only a part of the notable points of a quadrilateral can contribute to simplifying and shortening of solutions of similar problems. We will first introduce these properties of two remarkable points in a convex quadrilateral, which will help us find short solutions to the three competition problems in question.

2. Incenter of an arbitrary quadrilateral

It was proved in [7], that an arbitrary convex quadrilateral $ABCD$ has a unique point J, for which (Fig. 1):

$$\measuredangle JAD + \measuredangle JCD = \frac{1}{2}(\measuredangle A + \measuredangle C), \quad \measuredangle JBA + \measuredangle JDA = \frac{1}{2}(\measuredangle B + \measuredangle D).$$

As it is easy to check, in a circumscribed quadrilateral this point (called an incenter) coincides with the center of its inscribed circle.

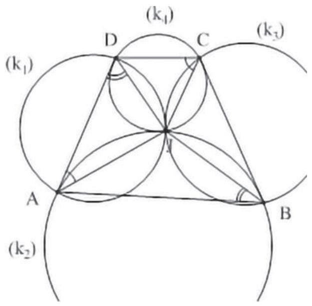

Fig. 1. The incenter

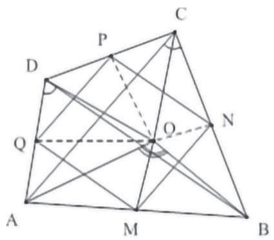

Fig. 2. Property 3

The incenter has the following properties, that will used here:

PROPERTY 1. The following equations hold [7]:

$$\frac{AJ}{CJ} = \frac{\sqrt{AB \cdot AD}}{\sqrt{BC \cdot CD}}, \quad \frac{BJ}{DJ} = \frac{\sqrt{AB \cdot BC}}{\sqrt{AD \cdot DC}}.$$

PROPERTY 2. The circumcircles of $\triangle AJD$ and $\triangle BJC$ touch each other, as well as those of $\triangle AJB$ and $\triangle CJD$ [7].

3. Pseudocenter of a quadrilateral

It was proven in [4] that there is a unique point O in the plane of any convex quadrilateral, which satisfies the equalities:

$$AO \cdot R_{BCD} = BO \cdot R_{CDA} = CO \cdot R_{DAB} = DO \cdot R_{ABC},$$

where $R_{\triangle}$ is the circumradius of triangle $\triangle$ (Fig. 2). This point is called the pseudocenter of the quadrilateral.

It is easy to check that the pseudocenter of an inscribed quadrilateral coincides with the center of its circumscribed circle. It has the following properties.

PROPERTY 3. The orthogonal projections of the pseudocenter O to the lines that the sides of the quadrilateral lie on, form a parallelogram [4] (Fig. 2).

Property 4. The following equalities hold [8]:

$$\measuredangle AOB = \measuredangle ADB + \measuredangle ACB, \quad \measuredangle DOC = \measuredangle DAC + \measuredangle DBC,$$
$$\measuredangle AOD = \measuredangle ABD + \measuredangle ACD, \quad \measuredangle BOC = \measuredangle BAC + \measuredangle BDC.$$

4. Inverse isogonality

Let $ABCD$ be an arbitrary convex quadrilateral, such that the extensions of each couple of its opposite sides intersect: $AD \cap BC = U$ and $AB \cap DC = V$. It is known that the circumcircles of $\triangle ABU$, $\triangle DCU$, $\triangle ABV$ and $\triangle BCV$ intersect at one point M, called Mikel's point (Fig. 3). Let us assume that the vertex C lies between the points U and B, and between D and V. The composition of the symmetry g with the bisector axis of $\measuredangle DMB$ and the inversion I with the pole M and degree $r^2 = BM \cdot DM$, is called the inverse isogonality with respect to $ABCD$ and it is denoted as $I \circ g(M; r^2)$ [5].

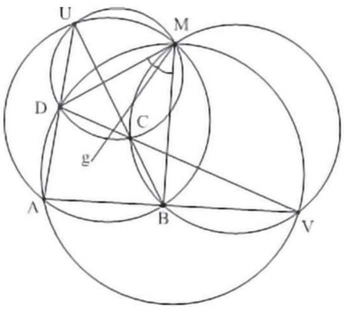

Fig. 3. Inverse isogonality

Property 5. The incenter of a convex quadrilateral is a fixed point of the inverse isogonality, i.e. $I \circ g(J) = J$ [7].

5. Competitive planimetric problems and different solutions of them – with and without the quadrilateral's geometry

Problem 1. Let $AB \cdot CD = AD \cdot BC$ hold in the convex quadrilateral $ABCD$. For its interior point O, the following equalities are satisfied: $\dfrac{AO}{CO} = \dfrac{AD}{CD}$ and $\dfrac{BO}{DO} = \dfrac{AB}{AD}$. If k_1, k_2, k_3 and k_4 are the circumcircles of $\triangle ADO$, $\triangle ABO$, $\triangle BCO$ and $\triangle CDO$, respectively, prove that k_1 touches k_3 and k_2 touches k_4 (Fig. 4).

Solution I. Denote the angles at A, B, C and D, respectively, by α, β, γ and δ. We will prove that the equalities $AB \cdot CD = AD \cdot BC$ and $\dfrac{AO}{CO} = \dfrac{AD}{CD}$ lead to the equality $\measuredangle ADO + \measuredangle ABO = \dfrac{1}{2}(\beta + \delta)$. Indeed, we have $\dfrac{AD}{CD} = \dfrac{AB}{CB}$ and $\dfrac{AD}{CD} = \dfrac{AO}{CO}$, and therefore $\dfrac{AD}{CD} = \dfrac{AB}{CB} = \dfrac{AO}{CO} = k$. Let the points M and N lie on the line AC, and $\dfrac{AM}{CM} = \dfrac{AN}{CN} = k$. The points D, B and O lie on the Apollonius circle c of AC with ratio k (MN is its diameter). As DM is the bisector of $\measuredangle ADC$, then

$$\measuredangle CDO - \measuredangle ADO = (\measuredangle CDM + \measuredangle ODM) - (\measuredangle ADM - \measuredangle ODM) = 2\measuredangle ODM,$$

i.e., $\measuredangle COD - \measuredangle ADO = 2\measuredangle ODM$.

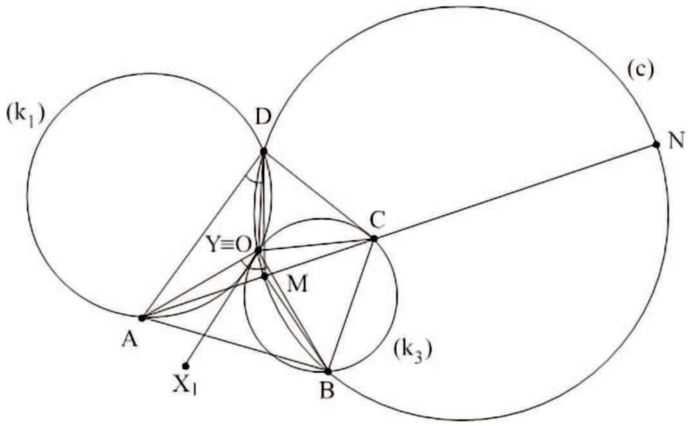

Fig. 4. Problem 1

Similarly, BM is the bisector of $\measuredangle ABC$ and $\measuredangle ABO - \measuredangle CBO = 2\measuredangle OBM$. But $\measuredangle ODM = \measuredangle OBM$ (as inscribed in the Apollonius circle c). Therefore $\measuredangle CDO - \measuredangle ADO = \measuredangle ABO - \measuredangle CBO$, and then

$$\begin{aligned}\measuredangle ADO + \measuredangle ABO &= \measuredangle CDO + \measuredangle CBO \\ &= \frac{1}{2}(\measuredangle ADO + \measuredangle ABO + \measuredangle CDO + \measuredangle CBO) = \frac{1}{2}(\beta + \delta).\end{aligned}$$

Analogously, from $AB{\cdot}CD = AD{\cdot}BC$ and $\dfrac{BO}{DO} = \dfrac{AB}{AD}$ it follows that $\measuredangle BAO + \measuredangle BCO = \dfrac{1}{2}(\alpha + \gamma)$. Using the resulting equalities we get

$$\begin{aligned}\measuredangle AOB &= 180^\circ - \measuredangle BAO - \measuredangle ABO \\ &= 180^\circ - \Big(\frac{\alpha+\gamma}{2} - \measuredangle BCO\Big) - \Big(\frac{\beta+\delta}{2} - \measuredangle ADO\Big) = \measuredangle BCO + \measuredangle ADO,\end{aligned}$$

i.e.,

$$(*) \qquad \measuredangle AOB = \measuredangle BCO + \measuredangle ADO.$$

Let X_1 be an internal point of $\measuredangle AOB$, such that $\measuredangle AOX_1 = \measuredangle ADO$. Then the line OX_1 touches the circumcircle k_1 of $\triangle ADO$ at the point O. Since from $(*)$ we have $\measuredangle BOX_1 = \measuredangle AOB - \measuredangle AOX_1 = \measuredangle AOB - \measuredangle ADO = \measuredangle BCO$, therefore OX_1 touches at O the circumcircle k_3 of $\triangle BOC$. Hence, k_1 and k_3 meet at one point. Analogously, the same is proved for k_2 and k_4.

Solution II. What we want to prove in the problem (that the circles k_1 and k_3 are tangent, and the circles k_2 and k_4 are also tangent) tells us that the point O mentioned in the condition must coincide with the incenter J of the quadrilateral $ABCD$ (according to property 2). It remains to prove that, for the incenter J the equality $\dfrac{AJ}{CJ} = \dfrac{\sqrt{AD \cdot AB}}{\sqrt{CD \cdot BC}}$ is fulfilled (from Property 1). But, given the condition

$AB \cdot CD = AD \cdot BC$, i.e. $\dfrac{AD}{CD} = \dfrac{AB}{BC}$, we obtain $\dfrac{\sqrt{AD \cdot AB}}{\sqrt{CD \cdot BC}} = \dfrac{AB}{BC} = \dfrac{AD}{BD}$ and therefore $\dfrac{AJ}{CJ} = \dfrac{AB}{BC} = \dfrac{AD}{CD}$. But we also have that $\dfrac{AO}{CO} = \dfrac{AD}{CD}$ (by the condition), therefore $\dfrac{AJ}{CJ} = \dfrac{AO}{CO}$. Therefore, both J and O lie on the same Apollonius circle for the segment AC. Analogously, from $\dfrac{BO}{DO} = \dfrac{AB}{AD}$ it follows that J and O lie on the same Apollonius circle for BD. Therefore $O \equiv J$, hence the problem is solved (according to Property 2).

PROBLEM 2. *I is the center of the inscribed circle in the circumscribed quadrilateral $ABCD$. The extensions of its sides AD and BC intersect at point U. If M is the second common point of the circumcircles of $\triangle ABU$ and $\triangle DCU$, prove that $MI = \sqrt{MB \cdot MD}$.*

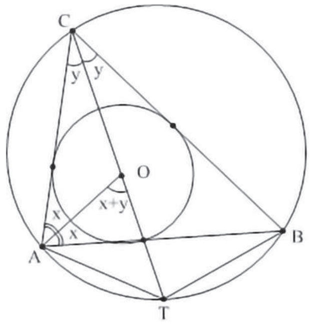

Fig. 5. Problem 2

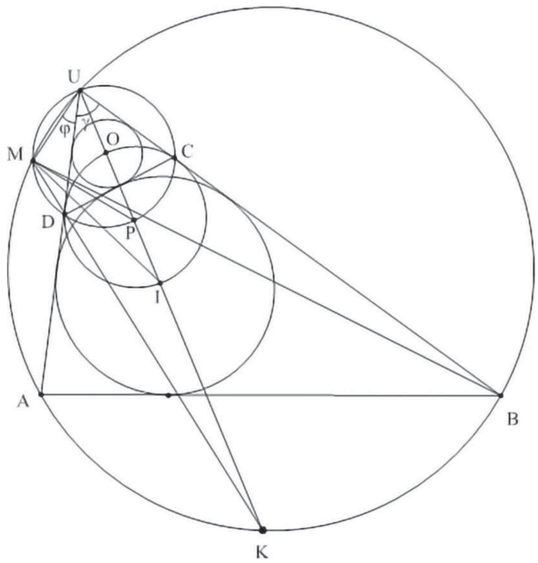

Fig. 6. Problem 2

Solution I. We will prove the following

LEMMA. *If k is the circumcircle of $\triangle ABC$, the point O is the center of the inscribed circle of $\triangle ABC$ and T the second common point of the line CO and k, then $TA = TB = TO$.*

Proof. CO is the angle bisector of $\measuredangle ACB$. Therefore, T is the midpoint of the arc AB (Fig. 5), and the chords TA and TB are equal. If $\measuredangle CAB = 2x$ and $\measuredangle ACB = 2y$, then we have

$$\measuredangle OAT = \measuredangle OAB + \measuredangle BAT = x + \measuredangle BCT = x + y.$$

Therefore, $\triangle OAT$ is isosceles, with $OT = AT$. Hence $TA = TB = TO$.

Let us go back to the problem's solution. Denote $\measuredangle AUB = \gamma$, $\measuredangle MUA = \varphi$ (Fig. 6), the circumcircles' radii of $\triangle ABU$ and $\triangle DCU$ with R and r, and with K and P, respectively, the second intersection points of these circles with UI. Let O be the the center of the inscribed circle of $\triangle DCU$. From the previous lemma it follows that $KI = KA$ and $PO = PC = PD$. Then P is the circumcenter of $\triangle OCD$. DI and DO are bisectors of adjacent angles, and therefore $\measuredangle ODI = 90°$.

Analogously, $\measuredangle OCI = 90°$. Hence the quadrilateral $ODIC$ is inscribed in a circle of center P. Hence, $PI = PD$. Now from the sine theorem for the triangles MPU, MKU, DPU, AKU and MBU, we obtain:

$$MP = 2r\sin\Big(\varphi + \frac{\gamma}{2}\Big), \quad MK = 2R\sin\Big(\varphi + \frac{\gamma}{2}\Big), \quad PI = DP = 2r\sin\frac{\gamma}{2},$$
$$KI = KA = 2R\sin\frac{\gamma}{2}, \quad MD = 2r\sin\varphi, \quad MB = 2R\sin(\varphi+\gamma).$$

From these equalities, it follows that $MP : MK = PI : KI = r : R$. Therefore, MI is an angle bisector in $\triangle MKP$. Now we express the bisector in the following way:

$$MI^2 = MP\cdot MK - PI\cdot KI = 4r\cdot R\sin^2\Big(\varphi+\frac{\gamma}{2}\Big) - 4r\cdot R\sin^2\frac{\gamma}{2}$$
$$= 4r\cdot R\sin\varphi\sin(\varphi+\gamma) = MD\cdot MB.$$

This proves the required statement.

Solution II. The center I of the inscribed circle in $ABCD$ is its incenter (Fig. 6), and the common point M of the circumcircles of $\triangle ABU$ and $\triangle DCU$ is its Mikel point. By definition, the pole of the inverse isogonality $I{\circ}g(M; r^2)$ with respect to the quadrilateral $ABCD$ is M. Its degree is $r^2 = MD\cdot MC$. The incenter I is a fixed point for $I{\circ}g(M; r^2)$, i.e. $I{\circ}g(I) = I$ (according to Property 5). Therefore, $r^2 = MI\cdot M(I{\circ}g(I)) = MI\cdot MI$, i.e., $r^2 = MI^2$. Thus we obtain the equality $MI^2 = MD\cdot MC$, that we had to prove.

Problem 3. H is the orthocenter and R is the radius of circumscribed circle about an acute-angled $\triangle ABC$, for which $\measuredangle CAB = \alpha$ and $\measuredangle ABC = \beta$. If D is the point in the half-plane with the border line AB, not containing the triangle, for which $\measuredangle ADC = 180° - 2\beta$ and $\measuredangle BDC = 180° - 2\alpha$, prove that $HD = R$.

Solution I. If $\measuredangle BCA = \gamma$, then $\measuredangle ADB = \measuredangle ADC + \measuredangle BDC = 2\gamma$ (Fig. 7). We obtain easily: $AH = 2R\cos\alpha$, $BH = 2R\cos\beta$ and $CH = 2R\cos\gamma$. Let us prove that the orthogonal projections M, N, P and Q of H, respectively, on the lines AD, DB, BC and CA form a parallelogram.

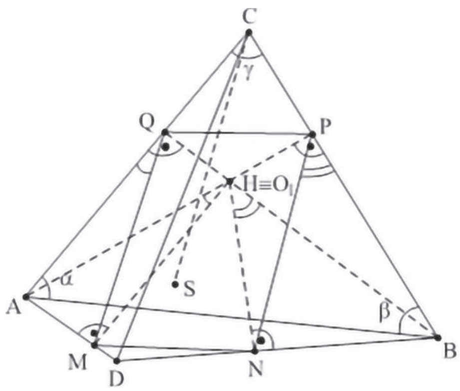

Fig. 7. Problem 3

We have

$$QM = AH \sin\measuredangle QAM = 2R\cos\alpha \cdot \frac{CD}{AC}\sin\measuredangle ADC$$
$$= \frac{CD}{\sin\beta}\cos\alpha\sin 2\beta = 2CD\cos\alpha\cos\beta.$$

Analogously, we obtain $PN = 2CD\cos\alpha\cos\beta$. Hence, $QM = PN$.

On the other hand, we have

$$\measuredangle AQM + \measuredangle BPN = \measuredangle AHM + \measuredangle BHN = \measuredangle AHB - \measuredangle MHN$$
$$= (180^\circ - \gamma) - (180^\circ - 2\gamma) = \gamma,$$

i.e., $\measuredangle ACB = \measuredangle AQM + \measuredangle BPN$.

Now let $CS \parallel QM$. From $\measuredangle PCS = \measuredangle ACB - \measuredangle ACS = \measuredangle ACB - \measuredangle AQM = \measuredangle BPN$, it follows that $CS \parallel PN$, and therefore $QM \parallel PN$, which proves that $MNPQ$ is a parallelogram and $MN = QP$. We get

$$HD = \frac{MN}{\sin\measuredangle MDN} = \frac{QP}{\sin 2\gamma} = \frac{CH\cdot\sin\gamma}{\sin 2\gamma} = \frac{2R\cos\gamma\sin\gamma}{2\sin\gamma\cos\gamma} = R.$$

With that, the problem is solved.

Solution II. We will prove that the orthocenter H of $\triangle ABC$ coincides with the pseudocenter O_1 of $ADBC$. From Property 4 of the pseudocenter, we have

$$\measuredangle CO_1B = \measuredangle CAB + \measuredangle CDB = \alpha + (180^\circ - 2\alpha) = 180^\circ - \alpha.$$

But also $\measuredangle CHB = 180^\circ - \alpha$, and so $\measuredangle CO_1B = \measuredangle CHB$. Therefore the pseudocenter O_1 and the orthocenter H lie on the same arc of a circle with endpoints B and C. Analogously, it is proved that they also lie on the same arc of a circle with endpoints A and C. Therefore $O_1 \equiv H$. From Property 3 of the pseudocenter, we can now conclude that the orthogonal projections M, N, P and Q of H, respectively on the lines AD, DB, BC and CA, form a parallelogram. Therefore $MN = QP$ and the proof can be finished as in the previous solution.

6. Conclusion

The geometry of the quadrilateral also helps to make easier solving problems given in various other competitions, including problems from International Mathematical Olympiads. However, they require an even more detailed knowledge of it, so we postpone the consideration of these tasks to some other paper.

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