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A SHORT AND ELEMENTARY PROOF OF BRAUER'S THEOREM

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Abstract. A short and elementary proof is given of a celebrated eigenvalue-perturbation result due to Alfred Brauer.

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In 1952, Alfred Brauer [1, Theorem 27] proved the following useful eigenvalue perturbation result, which is used in, e.g., *deflation techniques* when computing eigenvalues (see, e.g., Saad [6, Section 4.2]) and the longstanding *nonnegative inverse eigenvalue problem* (see, e.g., Julio and Soto [3] and references therein).

THEOREM 1. *Let A be an n-by-n matrix with complex entries and suppose that A has eigenvalues $\{\lambda, \lambda_2, \dots, \lambda_n\}$ (including multiplicities). If x is an eigenvector associated with λ and $y \in \mathbb{C}^n$, then the matrix $A + xy^*$ has eigenvalues $\{\lambda + y^*x, \lambda_2, \dots, \lambda_n\}$.*

Brauer's proof and a recent proof by Melman [4] rely on the *principle of biorthogonality*. While both proofs are elementary, they are somewhat lengthy. Meanwhile, Horn and Johnson gave two brief proofs, but the first relies on the *adjoint of a matrix* [2, p. 51] and the second requires a unitary matrix [2, p. 122] as opposed to just an invertible matrix (see details below). Another proof by Reams [5, p. 368] implicitly relies on *Schur triangularization*.

The proof below relies on the fact that similar matrices are cospectral, an elementary fact that is covered in a first-course in linear algebra.

Proof. Let Q be any invertible matrix whose first column is x—say $Q = [\, x \quad R \,]$. Notice that $Q^{-1}x = e_1$—the first canonical basis vector of $\mathbb{C}$—and that $Q^{-1}AQe_1 = \lambda e_1$. A simple calculation reveals that

$$\begin{aligned} Q^{-1}(A + xy^*)Q &= Q^{-1}AQ + Q^{-1}xy^*Q \\ &= \begin{bmatrix} \lambda & u^* \\ 0 & C \end{bmatrix} + e_1 y^* Q = \begin{bmatrix} \lambda + y^*x & u^* + y^*R \\ 0 & C \end{bmatrix}, \end{aligned}$$

i.e., $\sigma(A) = \{\lambda\} \cup \sigma(C)$ and $\sigma(A + xy^*) = \{\lambda + y^*x\} \cup \sigma(C)$. ∎

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