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PROOFS FOR OLD AND NEW TRIANGLE INEQUALITIES

D. M. Bătineţu-Giurgiu, Neculai Stanciu

Abstract. In this paper, we give new elementary proofs for some old triangle inequalities and we also present proofs for new inequalities in triangle.

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Why proofs? Proofs are the guts of mathematics. Producing a proof of a statement is the basic methodology by which we can ascertain that the statement is true. Anyone who wants to know what mathematics is about, have to learn how to write down a proof or at least to understand what a proof is. Sciences also use the same methodology to deduce complex conclusions from the first principles. Thus, all who want to study science would benefit from learning about proofs as well. For the others who are outside of science, learning how to prove theorems is an excellent way to sharpen one's mind. In a larger context, if anyone has any wish at all to find out how human beings can distinguish right from wrong or true from false, he or she would find in mathematical proofs the purest form of how this is done.

Let ABC be a triangle with usual notations: $a = BC$, $b = CA$, $c = AB$, R – circumradius, r – inradius, r_a, r_b, r_c – exradii, s – semiperimeter and F – the area of triangle.

Proposition 1 (Old). In any triangle ABC the following inequality is true

$$8(s-a)(s-b)(s-c) \leqslant abc. \qquad \text{A. Padoa, Problem 1.3 ([1], p. 12).}$$

New proof. $8(s-a)(s-b)(s-c) \leqslant abc \iff 8s(s-a)(s-b)(s-c) \leqslant sabc \iff 8F^2 \leqslant sabc \iff 8F^2 \leqslant s \cdot 4RF \iff 2F \leqslant sR \iff 2sr \leqslant sR \iff 2r \leqslant R$, i.e. Euler's inequality.

Proposition 2 (Old). In any triangle ABC the following inequality is true

$$64s^3(s-a)(s-b)(s-c) \leqslant 27a^2b^2c^2. \qquad \text{A. Padoa, Problem 1.12 ([1], p. 12).}$$

New proof. $64s^3(s-a)(s-b)(s-c) \leqslant 27a^2b^2c^2 \iff 64s^2F^2 \leqslant 27(4RF)^2 \iff 64s^2 \leqslant 27 \cdot 16 \cdot R^2 \iff 4s^2 \leqslant 27R^2 \iff R \geqslant \dfrac{2s}{3\sqrt{3}}$, i.e. Mitrinović's inequality.

Proposition 3 (New). If $u, v, w > 0$, then in any triangle ABC the following inequality is true

$$\frac{v+w}{u}\cdot a+\frac{w+u}{v}\cdot b+\frac{u+v}{w}\cdot c\geqslant 4\sqrt[4]{27}\cdot\sqrt{F}. \tag{1}$$

Proof. $\sum_{\text{cyc}}\frac{v+w}{u}\cdot a\overset{AM-GM}{\geqslant}2\sum_{\text{cyc}}\frac{\sqrt{vw}}{u}\cdot a\overset{AM-GM}{\geqslant}2\cdot 3\sqrt[3]{\prod_{\text{cyc}}\left(\frac{\sqrt{vw}}{u}\,a\right)}=$
$6\sqrt[3]{abc}=6\sqrt[3]{4RF}=6\sqrt[3]{8\cdot\frac{R}{2}\cdot F}=12\sqrt[3]{\sqrt{\frac{R}{2}}\sqrt{\frac{R}{2}}\cdot F}\overset{\text{Euler}}{\geqslant}12\sqrt[3]{\sqrt{r}\sqrt{\frac{R}{2}}\cdot F}$
$\overset{\text{Mitrinović}}{\geqslant}12\sqrt[3]{\sqrt{r}\sqrt{\frac{s}{3\sqrt{3}}}\cdot F}=12\sqrt[3]{\sqrt{r}\sqrt{s}\sqrt{\frac{1}{3\sqrt{3}}}\cdot F}=12\sqrt[3]{\left(\frac{1}{\sqrt[4]{3}}\right)^3\cdot F\sqrt{F}}$
$=\frac{12\sqrt{F}}{\sqrt[4]{3}}=4\sqrt[4]{27}\sqrt{F}.$

Proposition 4 (New). If $u, v, w > 0$, then in any triangle ABC the following inequality is true

$$\frac{v+w}{u}(b+c)+\frac{w+u}{v}(c+a)+\frac{u+v}{w}(a+b)\geqslant 8\sqrt[4]{27}\sqrt{F}. \tag{2}$$

Proof. $\sum_{\text{cyc}}\frac{v+w}{u}(b+c)=\sum_{\text{cyc}}\frac{v+w}{u}b+\sum_{\text{cyc}}\frac{v+w}{u}c\overset{(1)}{\geqslant}4\sqrt[4]{27}\sqrt{F}+4\sqrt[4]{27}\sqrt{F}\geqslant$
$8\sqrt[4]{27}\sqrt{F}.$

Proposition 5 (New). If $u, v, w > 0$, then in any triangle ABC the following inequality is true

$$\frac{v+w}{u}\cdot bc+\frac{w+u}{v}\cdot ca+\frac{u+v}{w}\cdot ab\geqslant 8\sqrt{3}\cdot F. \tag{3}$$

Proof. $\sum_{\text{cyc}}\frac{v+w}{u}\cdot bc\overset{AM-GM}{\geqslant}2\sum_{\text{cyc}}\frac{\sqrt{vw}}{u}\cdot bc\overset{AM-GM}{\geqslant}2\cdot 3\sqrt[3]{\prod_{\text{cyc}}\left(\frac{\sqrt{vw}}{u}\,bc\right)}=$
$6\sqrt[3]{(abc)^2}\overset{\text{Carliz}}{\geqslant}6\cdot\frac{4}{\sqrt{3}}\cdot F=8\sqrt{3}\cdot F.$

Proposition 6 (New). If $u, v, w > 0$, then in any triangle ABC the following inequality is true

$$\frac{v+w}{u}\cdot r_a+\frac{w+u}{v}\cdot r_b+\frac{u+v}{w}\cdot r_c\geqslant 6\sqrt[3]{sF}. \tag{4}$$

Proof. $\sum_{\text{cyc}}\frac{v+w}{u}r_a\overset{AM-GM}{\geqslant}2\sum_{\text{cyc}}\frac{\sqrt{vw}}{u}r_a\overset{AM-GM}{\geqslant}2\cdot 3\sqrt[3]{\prod_{\text{cyc}}\left(\frac{\sqrt{vw}}{u}\,r_a\right)}=$
$6\sqrt[3]{r_ar_br_c}=6\sqrt[3]{sF}.$

Proposition 7 (Old). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}\cdot a^2+\frac{z+x}{y}\cdot b^2+\frac{x+y}{z}\cdot c^2\geqslant 8\sqrt{3}\cdot F. \qquad \text{(B-G.1) (see, e.g., [2]).}$$

Proof. Denote $u=\sqrt{x}$, $v=\sqrt{y}$, $w=\sqrt{z}$. Then $\sum_{\text{cyc}}\frac{y+z}{x}a^2=\sum_{\text{cyc}}\frac{v^2+w^2}{u^2}a^2$ $\overset{\text{Bergstrom}}{\geqslant}\frac{1}{2}\sum_{\text{cyc}}\left(\frac{v+w}{u}a\right)^2\overset{\text{Bergstrom}}{\geqslant}\frac{1}{2}\cdot\frac{1}{3}\left(\sum_{\text{cyc}}\frac{v+w}{u}a\right)^2=\frac{1}{6}\left(\sum_{\text{cyc}}\frac{v+w}{u}a\right)^2$ $\overset{(1)}{\geqslant}\frac{1}{6}(4\sqrt[4]{27}\sqrt{F})^2=\frac{16}{6}\sqrt{27}\cdot F=8\sqrt{3}\cdot F.$

Proposition 8 (New). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}\cdot a^3+\frac{z+x}{y}\cdot b^3+\frac{x+y}{z}\cdot c^3\geqslant 16\sqrt[4]{3}\cdot F\sqrt{F}. \tag{5}$$

Proof. Denote $u^3=x$, $v^3=y$, $w^3=z$. Then $\sum_{\text{cyc}}\frac{y+z}{x}a^3=\sum_{\text{cyc}}\frac{v^3+w^3}{u^3}a^3$ $\overset{\text{Radon}}{\geqslant}\frac{1}{2^2}\sum_{\text{cyc}}\left(\frac{v+w}{u}a\right)^3\overset{\text{Radon}}{\geqslant}\frac{1}{4}\cdot\frac{1}{3^2}\left(\sum_{\text{cyc}}\frac{v+w}{u}a\right)^3\overset{(1)}{\geqslant}\frac{1}{36}(4\sqrt[4]{27}\sqrt{F})^3$ $=\frac{1}{36}\cdot 4^3\sqrt[4]{3^9}(\sqrt{F})^3=16\sqrt[4]{3}\cdot F\sqrt{F}.$

Proposition 9 (Old). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}\cdot a^4+\frac{z+x}{y}\cdot b^4+\frac{x+y}{z}\cdot c^4\geqslant 32F^2. \qquad \text{(B-G.2) (see, e.g., [2]).}$$

Proof. Denote $u^4=x$, $v^4=y$, $w^4=z$. Then $\sum_{\text{cyc}}\frac{y+z}{x}a^4=\sum_{\text{cyc}}\frac{v^4+w^4}{u^4}a^4$ $\overset{\text{Radon}}{\geqslant}\frac{1}{2^3}\sum_{\text{cyc}}\left(\frac{v+w}{u}a\right)^4\overset{\text{Radon}}{\geqslant}\frac{1}{8}\cdot\frac{1}{3^3}\left(\sum_{\text{cyc}}\frac{v+w}{u}a\right)^4\overset{(1)}{\geqslant}\frac{1}{2^3\cdot 3^3}(4\sqrt[4]{27}\sqrt{F})^4=32F^2.$

Proposition 10 (New). If $x, y, z > 0$ and $m\geqslant 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}\cdot a^{m+1}+\frac{z+x}{y}\cdot b^{m+1}+\frac{x+y}{z}\cdot c^{m+1}\geqslant 2^{m+2}\cdot 3^{\frac{3-m}{4}}\cdot F^{\frac{m+1}{2}}.$$

Proof. Denote $u^{m+1}=x$, $v^{m+1}=y$, $w^{m+1}=z$. Then $\sum_{\text{cyc}}\frac{y+z}{x}a^{m+1}=$ $\sum_{\text{cyc}}\frac{v^{m+1}+w^{m+1}}{u^{m+1}}a^{m+1}\overset{\text{Radon}}{\geqslant}\frac{1}{2^m}\sum_{\text{cyc}}\left(\frac{v+w}{u}a\right)^{m+1}\overset{\text{Radon}}{\geqslant}\frac{1}{2^m}\cdot\frac{1}{3^m}\left(\sum_{\text{cyc}}\frac{v+w}{u}a\right)^{m+1}$ $\overset{(1)}{\geqslant}\frac{1}{2^m\cdot 3^m}(4\sqrt[4]{27}\sqrt{F})^{m+1}=2^{m+2}\cdot 3^{\frac{3-m}{4}}\cdot F^{\frac{m+1}{2}}.$

Proposition 11 (New). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}(b+c)^2+\frac{z+x}{y}(c+a)^2+\frac{x+y}{z}(a+b)^2 \geqslant 32\sqrt{3}\cdot F.$$

Proof. Denote $u^2=x$, $v^2=y$, $w^2=z$. Then

$$\sum_{\text{cyc}}\frac{y+z}{x}(b+c)^2=\sum_{\text{cyc}}\frac{v^2+w^2}{u^2}(b+c)^2 \overset{\text{Bergstrom}}{\geqslant} \frac{1}{2}\sum_{\text{cyc}}\left(\frac{v+w}{u}(b+c)\right)^2 \overset{\text{Bergstrom}}{\geqslant} \frac{1}{2}\cdot\frac{1}{3}\left(\sum_{\text{cyc}}\frac{v+w}{u}(b+c)\right)^2 \overset{(2)}{\geqslant} \frac{1}{6}(8\sqrt[4]{27}\sqrt{F})^2=32\sqrt{3}\cdot F.$$

Proposition 12 (New). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}(b+c)^4+\frac{z+x}{y}(c+a)^4+\frac{x+y}{z}(a+b)^4 \geqslant 512F^2.$$

Proof. Denote $u^4=x$, $v^4=y$, $w^4=z$. Then

$$\sum_{\text{cyc}}\frac{y+z}{x}(b+c)^4 = \sum_{\text{cyc}}\frac{v^4+w^4}{u^4}(b+c)^4 \overset{\text{Radon}}{\geqslant} \frac{1}{2^3}\sum_{\text{cyc}}\left(\frac{v+w}{u}(b+c)\right)^4 \overset{\text{Radon}}{\geqslant} \frac{1}{2^3}\cdot\frac{1}{3^3}\left(\sum_{\text{cyc}}\frac{v+w}{u}(b+c)\right)^4 \overset{(2)}{\geqslant} \frac{1}{2^3\cdot 3^3}(8\sqrt[4]{27}\sqrt{F})^4=512F^2.$$

Proposition 13 (New). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}b^2c^2+\frac{z+x}{y}c^2a^2+\frac{x+y}{z}a^2b^2 \geqslant 32F^2.$$

Proof. Denote $u^2=x$, $v^2=y$, $w^2=z$, then $\sum_{\text{cyc}}\frac{y+z}{x}b^2c^2=\sum_{\text{cyc}}\frac{v^2+w^2}{u^2}b^2c^2 \overset{\text{Bergstrom}}{\geqslant} \frac{1}{2}\sum_{\text{cyc}}\left(\frac{v+w}{u}bc\right)^2 \overset{\text{Bergstrom}}{\geqslant} \frac{1}{2}\cdot\frac{1}{3}\left(\sum_{\text{cyc}}\frac{v+w}{u}bc\right)^2 \overset{(3)}{\geqslant} \frac{1}{6}(8\sqrt{3}\cdot F)^2=32F^2.$

Proposition 14 (New). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x}b^4c^4+\frac{z+x}{y}c^4a^4+\frac{x+y}{z}a^4b^4 \geqslant \frac{512}{3}F^4.$$

Proof. Denote $u^4=x$, $v^4=y$, $w^4=z$, then $\sum_{\text{cyc}}\frac{y+z}{x}b^4c^4=\sum_{\text{cyc}}\frac{v^4+w^4}{u^4}b^4c^4 \overset{\text{Radon}}{\geqslant} \frac{1}{2^3}\sum_{\text{cyc}}\left(\frac{v+w}{u}bc\right)^4 \overset{\text{Radon}}{\geqslant} \frac{1}{2^3}\cdot\frac{1}{3^3}\left(\sum_{\text{cyc}}\frac{v+w}{u}bc\right)^4 \overset{(3)}{\geqslant} \frac{1}{2^3\cdot 3^3}(8\sqrt{3}\cdot F)^4 = \frac{512}{3}F^4.$

Proposition 15 (New). If $x, y, z > 0$, then in any triangle ABC the following inequality is true

$$\frac{y+z}{x} \cdot r_a^2 + \frac{z+x}{y} \cdot r_b^2 + \frac{x+y}{z} \cdot r_c^2 \geqslant 6\sqrt{3} \cdot F.$$

Proof. Denote $u^2 = x$, $v^2 = y$, $w^2 = z$. Then $\sum_{\text{cyc}} \frac{y+z}{x} r_a^2 = \sum_{\text{cyc}} \frac{v^2+w^2}{u^2} r_a^2$ $\overset{\text{Bergstrom}}{\geqslant} \frac{1}{2} \sum_{\text{cyc}} \left(\frac{v+w}{u} r_a\right)^2 \overset{\text{Bergstrom}}{\geqslant} \frac{1}{2} \cdot \frac{1}{3} \left(\sum_{\text{cyc}} \frac{v+w}{u} r_a\right)^2 \overset{(4)}{\geqslant} \frac{1}{6}(6\sqrt[3]{sF})^2 =$ $6\sqrt[3]{s^2F^2} = 6\sqrt[3]{s \cdot s \cdot F^2} \overset{\text{Mitrinović}}{\geqslant} 6\sqrt[3]{s \cdot (3\sqrt{3}\, r)F^2} = 6\sqrt{3} \cdot F.$

Finally, mathematics without proofs is still a nice and useful subject for certain purposes, just as opera without human voice could be a relaxing and enjoyable art form to some people.

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D.M.B-G.: "Matei Basarab", National College, Bucharest, Romania

E-mail: dmb_g@yahoo.com

N.S.: "George Emil Palade" School, Buzău, Romania

E-mail: stanciuneculai@yahoo.com