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CORRIGENDA: BIMATRIX GAMES HAVE A QUASI-STRICT EQUILIBRIUM: AN ALTERNATIVE PROOF THROUGH A HEURISTIC APPROACH

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Abstract. Our paper published in this journal [Bimatrix games have a quasi-strict equilibrium: an alternative proof through a heuristic approach, The Teaching of Mathematics, 21, 2, 97–108] contains at least two errors, and the proof is too sloppy in dealing with a really subtle problem. The purpose of this note is to correct those errors and to give more explanation, and accordingly to present slightly different economic interpretations of our method as well as to show calculated results, suitable to our corrected proof, in the numerical example.

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1. Our proof corrected

We use the same notation as in our paper [4]. The following assertion in Section 4 of [4, p. 104], i.e., the proof of the main result, is wrong.

Since $p^\dagger \gg 0$ and $q^\dagger \gg 0$, the above inequality should hold as the equalities, i.e.,

$$p^{\dagger\prime} \cdot (A^\circ(p^*, q^*), B^\circ(p^*, q^*)) = 0 \text{ and}$$
$$\begin{pmatrix} A^\circ(p^*, q^*) \\ B^\circ(p^*, q^*) \end{pmatrix} \cdot q^\dagger = 0.$$

To sidestep this error, we had better allow $\overline{p}$ and $\overline{q}$ to have negative elements. Then we require only the original Stiemke's theorem in place of its generalization given as Theorem 2 in Section 2 of [4, p. 99]. So, we replace this by the following.

THEOREM 2S. (Stiemke's theorem [7]) *For a given real $m \times n$ matrix A, the equality system $Aq = 0$ has a strictly positive solution $q \gg 0$ if and only if the inequality system $p' \cdot A < 0$ has no solution p.*

We also abandon our *unnatural* juxtaposition of two matrices $(A^\circ(p^*, q^*), B^\circ(p^*, q^*))$, thus our proof will get a little simpler. Now our corrected proof goes like this.

We again first state our proposition.

Proposition. *For a bimatrix game (A, B)*$\in \mathbb{M}^2_{m\times n}$, *there exists at least one quasi-strict equilibrium.*

Proof. For a pair of vectors $(p, q) \in D \equiv S^m \times S^n$, we define two more symbols,

$$\mathbf{Z}(p;q) \equiv \{i \mid p_i = 0\} \cap \mathbf{B}_1(q); \quad \mathbf{Z}(q;p) \equiv \{j \mid q_j = 0\} \cap \mathbf{B}_2(p).$$

All we have to show is that there exists a Nash equilibrium $(p^*, q^*) \in D$ in which these sets $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$ are empty. As a routine procedure, we consider a multivalued mapping f from D into itself as follows:
for $(p^\circ, q^\circ) \in D$,

$$f(p^\circ, q^\circ) \equiv (\{p \in S^m \mid pAq^\circ = \max_{u\in S^m} uAq^\circ\}, \quad \{q \in S^n \mid p^\circ Bq = \max_{v\in S^n} p^\circ Bv\}).$$

By Berge's maximum theorem [1, (1963, p. 116)], [8, p. 235], this map f is upper semicontinuous, and the image set $f(p^\circ, q^\circ)$ is convex. Therefore, Kakutani's fixed point theorem [5] assures us of the existence of Nash equilibrium $EQ(A, B)$.

Although there may be infinitely many Nash equilibria, we have only a finite number of types $\mathbf{C}(p^*)$, $\mathbf{C}(q^*)$, $\mathbf{B}_1(q^*)$, and $\mathbf{B}_2(p^*)$ for $(p^*, q^*) \in EQ(A, B)$. Further we define, for a Nash equilibrium (p^*, q^*), the following symbols:

$$m' \equiv \#(\mathbf{B}_1(q^*)), \quad n' \equiv \#(\mathbf{B}_2(p^*)), \quad u^* \equiv p^*Aq^*, \quad \text{and} \quad v^* \equiv p^*Bq^*,$$

where #() signifies the number of elements in a given set. Now we take up a particular pair $\mathbf{B}_1(q^*)$ and $\mathbf{B}_2(p^*)$, corresponding to a Nash equilibrium (p^*, q^*), together with two matrices $A(p^*, q^*)$ and $B(p^*, q^*)$, which are composed of the rows of A and B in $\mathbf{B}_1(q^*)$ and the columns in $\mathbf{B}_2(p^*)$, respectively. Further we form the new matrices as

$$A^\circ(p^*, q^*) \equiv A(p^*, q^*) - u^*E \text{ and } B^\circ(p^*, q^*) \equiv B(p^*, q^*) - v^*E,$$

where E is the $m' \times n'$ matrix whose elements are all unity. That is, we subtract u^* from each entry of the matrix $A(p^*, q^*)$, and v^* from $B(p^*, q^*)$. Note that

$$\overline{p^*} \cdot B^\circ(p^*, q^*) = 0 \text{ and } A^\circ(p^*, q^*) \cdot \overline{q^*} = 0,$$

where $\overline{p^*}$ and $\overline{q^*}$ are m'-row vector and n'-column vector whose indices are included in $\mathbf{B}_1(q^*)$ and $\mathbf{B}_2(p^*)$, respectively.

Now we examine whether there exists a $\overline{q} \in \mathbb{R}^{n'}$ (now $\overline{q}$ is allowed to have negative elements) such that

$$B^\circ(p^*, q^*) \cdot \overline{q} < 0. \tag{1}$$

It is evident that an elementwise strict inequality in (1) can take place only in an index which is in $\mathbf{Z}(p^*; q^*)$. Therefore when $\mathbf{Z}(p^*; q^*)$ is empty, there can be no $\overline{q}$ which satisfies the inequality (1).

Let us now suppose that there exists no quasi-strict equilibrium, thus the sets $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$ cannot be empty at the same time, and we are able to find out a Nash equilibrium (p^*, q^*), which has the *minimum* total number of zero-indices in the sets $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$. We suppose our Nash equilibrium is such one. Further suppose there is only one index $i \in \mathbf{Z}(p^*; q^*)$, i.e., $(B^\circ(p^*, q^*) \cdot \overline{q})_i < v^*$. In this case, we modify the map f globally by reducing the codomain D to $D' \equiv \{S^m \cap \{\{p \in \mathbb{R}^m \mid p_i \geq \varepsilon\}\} \times S^n$, where $(1/m) > \varepsilon > 0$. As the codomain shrinks, we modify the multivalued map f as follows. Let a point in an image of $f(p^\circ, q^\circ)$ be (p', q'). If $(p', q') \in D'$, then no change is required; otherwise we set $p'_i = \varepsilon$ with the other entries of p decreased *proportionately* so that the modified vector p' can stay in D'. When there are two or more strict inequalities involved, we modify f in a similar manner, i.e., those entries are unchanged or set at ε depending upon whether they are not less than or less than ε, and the remaining ones decreased proportionately.

Next we do the same examination on the existence of a $\overline{p} \in \mathbb{R}^{m'}$ (now $\overline{p}$ is also allowed to have negative elements) such that

$$\overline{p}' \cdot (A^\circ(p^*, q^*)) < 0. \tag{2}$$

If there is one $\overline{p}$ with the maximum number of inequalities in (2), we reduce, this time, the codomain on the S^n in a similar way as is explained for $\overline{q}$, i.e., when the j-th elementwise comparison shows a strict inequality in (2), we set $q_j \geq \delta$, where $(1/n) > \delta > 0$, thus reducing the codomain smaller. (It should be noted that the reduction of the codomain will not produce a new Nash equilibrium. This is because a new equilibrium on the shrunken boundary should actually be an equilibrium before the transformation of the map due to the dual structure of our problem. And this new one would carry *fewer* zero-indices than in the sets $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$. A contradiction. Thus, no new equilibrium will be created.)

Since the above modification on f is conducted globally, i.e., not piece by piece, or neighbourhood by neighbourhood, the map after modification is still upper semicontinuous, and image sets always remain convex thanks to linearity involved; thus we can use Kakutani's fixed point theorem to secure the existence of at least one fixed point (p^*, q^*) even after the modification of the map.

Now consider a fixed point (p^*, q^*) of the transformed map. If $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$ are both empty, this pair (p^*, q^*) gives a quasi-strict equilibrium. Suppose that neither $\mathbf{Z}(p^*; q^*)$ nor $\mathbf{Z}(q^*; p^*)$ is empty. Because of our modification of the map f, there exists no $\overline{q} \in S^{n'}$ nor $\overline{p} \in S^{m'}$ for which the above inequalities (1) and (2) hold respectively. By Theorem 2S above (now changed to the original Stiemke's theorem), there exist $p^\dagger \gg 0$ and $q^\dagger \gg 0$ such that $p^\dagger \in S^{m'}$ and $q^\dagger \in S^{n'}$,

$$p^{\dagger\prime} \cdot (B^\circ(p^*, q^*)) = 0 \text{ and}$$
$$(A^\circ(p^*, q^*)) \cdot q^\dagger = 0.$$

Now we create a vector $p^{\dagger *} \in S^m$ from $p^\dagger$ by setting zero to its entries outside of $\mathbf{B}_1(q^*)$, and create a vector $q^{\dagger *} \in S^m$ from $q^\dagger$ in the same manner by setting zero to its entries outside of $\mathbf{B}_2(p^*)$. It is easy then to recognize that a pair of vectors

$$((p^* + \eta \cdot p^{\dagger *}) / \left\| p^* + \eta \cdot p^{\dagger *} \right\|, (q^* + \eta \cdot q^{\dagger *}) / \left\| q^* + \eta \cdot q^{\dagger *} \right\|),$$

where η is a positive scalar so small that no disturbance is made to the best choice set $\mathbf{B}_1(q^*)$ and $\mathbf{B}_2(p^*)$ and $\|\cdot\|$ stands for the sum norm, is indeed a quasi-strict equilibrium, showing a contradiction to our supposition that there exists none.

The other cases in which either of the two, $\mathbf{Z}(p^*;q^*)$ and $\mathbf{Z}(q^*;p^*)$, is empty, can similarly be dealt with. ■

2. Economic interpretation and an numerical example

In the numerical example below, we actually obtain nonnegative $\overline{p}$ and $\overline{q}$ for which the inequalities (1) and (2) holds. In such cases, there is no need to change our economic interpretation in section 5 of our paper [4]. Once $\overline{p}$ and $\overline{q}$ are allowed to contain negative entries, we divide the set of production processes as well as commodities into two groups; those having positive or zero entries and those with negative entries. When all the entries are either nonnegative or negative, the two groups are the entire set and the empty set. Note that the inequalities (1) and (2) have the same implications even if the direction of inequality sign is reversed, simply because $\overline{p}$ and $\overline{q}$ are to be replaced by $-\overline{p}$ and $-\overline{q}$. Now an economic interpretation of (1) is that one group of production processes can produce the output in all the commodities not less than the other group can, and in at least one commodity a strictly greater output. And our modification of map f is to set positive the price of such a commodity as having a strict inequality. This interpretation is not so easily swallowable, but is understandable as an analogy to the case where $\overline{p}$ and $\overline{q}$ are nonnegative. The reader may refer to [2] and [3] for further economic discussion of the matter.

We have to present calculated examples concerning the inequalities (1) and (2), which are appropriate after our corrections of the proof. Two given matrices are:

$$A \equiv \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B \equiv \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

A pair of vectors, $p^* = (1/2, 1/2, 0)'$ and $q^* = (1/2, 1/2, 0)'$, are a Nash equilibrium, but not quasi-strict, because $\mathbf{C}(p^*) = \mathbf{C}(q^*) = \{1, 2\}$, while $\mathbf{B}_1(q^*) = \mathbf{B}_2(p^*) = \{1, 2, 3\}$, as one can easily calculate. We can also calculate to have $u^* = v^* = 1/2$. In this Nash equilibrium, we can find a vector $\overline{q} = (3, 1, 0)'/4$, which will yield

$$B^\circ(p^*, q^*) \cdot 4 \cdot \overline{q} = \begin{pmatrix} 1/2 & -3/2 & -1/2 \\ -1/2 & 3/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Thus, p_3^* should be made positive. On the other hand, we can find a vector $\overline{p}$ $= (1, 3, 0)/4$ such that:

$$4 \cdot \overline{p}' \cdot A^\circ(p^*, q^*) = (\,1, \quad 3, \quad 0\,) \begin{pmatrix} -3/2 & 3/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} = (\,0, \quad 0, \quad -1\,).$$

Hence, q_3^* should be made positive. Actually we know, from Fig. 1 of Norde [6, p. 37], a quasi-strict equilibrium is given, e.g., as $p^{\dagger *} = (1/3, 0, 2/3)'$ and $q^{\dagger *} = (0, 0, 1)'$.

REFERENCES

[1] Berge, C., *Espaces topologiques. Fonctions multivoques*, Paris: Dunod, 1959. [English translation: *Topological spaces including a treatment of multi-valued functions, vector spaces and convexity*, Edinburgh: Oliver & Boyd, 1963].

[2] Filippini, C., Filippini, L., *Two theorems on joint production*, Economic Journal, **92** (1982), 386–390.

[3] Fujimoto, T., Krause, U., *More theorems on joint production*, Zeitschrift für Nationalökonomie, **48** (2) (1988), 189–196.

[4] Fujimoto, T., Karunathilake, N. G. A., Ranade, R. R., *Bimatrix games have a quasi-strict equilibrium: an alternative proof through a heuristic approach.* The Teaching of Mathematics, **21** (2) (2018), 97-108.

[5] Kakutani, S., *A generalization of Brouwer's fixed point theorem*, Duke Math Journal, **8** (3) (1941), 457–459.

[6] Norde, H., *Bimatrix games have quasi-strict equilibria*, Math. Prog., **85** (1999), 35–49.

[7] Stiemke, E., *Über positive Lösungen homogener linearer Gleichungen*, Math. Ann., **76** (1915), 340–342.

[8] Sundaram, R. K., *A First Course in Optimization Theory*, Cambridge: Cambridge University Press, 1996.

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