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A GENERAL METHOD OF PROVING SOME CLASSICAL INEQUALITIES USING AM-GM INEQUALITY

Xu Yanhui

Abstract. In this paper, a general method is presented of proving some inequalities of Cauchy-Hölder-Carlson type using AM-GM inequality.

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1. Introduction

In the field of classical analysis, the most well-known and frequently used inequality is Arithmetic mean-Geometric mean inequality, widely known as AM-GM inequality. Cauchy-Bunyakovsky-Schwarz (CBS) inequality is a very powerful inequality. It is very useful in proving both cyclic and symmetric inequalities. In addition, Hölder inequality found by Leonard James Rogers (1888) and discovered independently by Otto Hölder (1889) is a basic and indispensable inequality for studying integrals and L_p spaces, and is also an extension of CBS inequality [1, 2]. Carlson inequality, as one of important inequalities in higher mathematics, has deep roots in many classical algebraic inequalities and famous geometric inequalities [5].

There are already a few articles on the relationship between these classical inequalities. For example, Lin [4] provides a very short proof of equivalence of AM-GM inequality and CBS inequality. Li Yongtao, Gu Xian-Ming and Zhao Jianxing [3] give proofs of mathematical equivalence among the weighted AM-GM inequality, the weighted power-mean inequality and Hölder inequality. In this paper, a general method is presented of proving inequalities of Cauchy-Hölder-Carlson type using AM-GM inequality. Also, some examples of application of these classical inequalities are given.

Lemma (AM-GM inequality). *Assume that n is a natural number. Then for arbitrary nonnegative numbers $x_1, x_2, \dots, x_n$, the following inequality holds*

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geqslant \sqrt[n]{\prod_{i=1}^{n} x_i}. \tag{1}$$

More generally, if $0 \leqslant x_i \in \mathbb{R}$ *and* $0 \leqslant \alpha_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ *are such that* $\sum_{i=1}^{n} \alpha_i = 1$*, then*

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leqslant \sum_{i=1}^{n} \alpha_i x_i. \tag{2}$$

2. Main results

THEOREM 1. (CBS INEQUALITY) *Let* $x_1, x_2, \ldots, x_n$ *and* $y_1, y_2, \ldots, y_n$ *be real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geqslant \left(\sum_{i=1}^{n} x_i y_i\right)^2. \tag{3}$$

The equality holds if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2} = \cdots = \dfrac{x_n}{y_n}$.

Proof. The inequality (3) is equivalent to

$$\frac{\left|\sum\limits_{i=1}^{n} x_i y_i\right|}{\sqrt{\sum\limits_{i=1}^{n} x_i^2}\sqrt{\sum\limits_{i=1}^{n} y_i^2}} \leqslant 1.$$

Because of $\left|\sum_{i=1}^{n} x_i y_i\right| \leqslant |x_1 y_1| + |x_2 y_2| + \cdots + |x_n y_n|$, we only need to prove that

$$\sqrt{\frac{x_1^2}{\sum\limits_{i=1}^{n} x_i^2} \cdot \frac{y_1^2}{\sum\limits_{i=1}^{n} y_i^2}} + \sqrt{\frac{x_2^2}{\sum\limits_{i=1}^{n} x_i^2} \cdot \frac{y_2^2}{\sum\limits_{i=1}^{n} y_i^2}} + \cdots + \sqrt{\frac{x_n^2}{\sum\limits_{i=1}^{n} x_i^2} \cdot \frac{y_n^2}{\sum\limits_{i=1}^{n} y_i^2}} \leqslant 1.$$

By using (1), we get

$$\sqrt{\frac{x_1^2}{\sum\limits_{i=1}^{n} x_i^2} \cdot \frac{y_1^2}{\sum\limits_{i=1}^{n} y_i^2}} + \sqrt{\frac{x_2^2}{\sum\limits_{i=1}^{n} x_i^2} \cdot \frac{y_2^2}{\sum\limits_{i=1}^{n} y_i^2}} + \cdots + \sqrt{\frac{x_n^2}{\sum\limits_{i=1}^{n} x_i^2} \cdot \frac{y_n^2}{\sum\limits_{i=1}^{n} y_i^2}}$$

$$\leqslant \frac{1}{2}\left(\frac{x_1^2}{\sum\limits_{i=1}^{n} x_i^2} + \frac{y_1^2}{\sum\limits_{i=1}^{n} y_i^2}\right) + \frac{1}{2}\left(\frac{x_2^2}{\sum\limits_{i=1}^{n} x_i^2} + \frac{y_2^2}{\sum\limits_{i=1}^{n} y_i^2}\right) + \cdots + \frac{1}{2}\left(\frac{x_n^2}{\sum\limits_{i=1}^{n} x_i^2} + \frac{y_n^2}{\sum\limits_{i=1}^{n} y_i^2}\right) = 1.$$

THEOREM 2. (CAUCHY INEQUALITY FOR INTEGRALS) *If* $f, g\colon [a, b] \to \mathbb{R}$ *are integrable then*

$$\left|\int_a^b f(x) g(x)\, dx\right| \leqslant \left[\int_a^b f^2(x)\, dx\right]^{\frac{1}{2}} \left[\int_a^b g^2(x)\, dx\right]^{\frac{1}{2}}, \tag{4}$$

with equality if and only if f *and* g *are proportional a.e.*

More generally, let $a < b$ and let $f_j(x) \neq 0$ ($j = 1, 2, \ldots, m$) be real and integrable functions defined on $[a, b]$. Then

$$\left| \int_a^b \prod_{j=1}^{m} |f_j(x)|\, dx \right|^m \leqslant \prod_{j=1}^{m} \int_a^b |f_j(x)|^m\, dx. \tag{5}$$

Proof. The inequality (4) is equivalent to

$$\left| \int_a^b \left[\frac{f^2(x)}{\int_a^b f^2(x)\, dx} \right]^{\frac{1}{2}} \cdot \left[\frac{g^2(x)}{\int_a^b g^2(x)\, dx} \right]^{\frac{1}{2}} dx \right| \leqslant 1.$$

By applying (2), we get

$$\left| \int_a^b \left[\frac{f^2(x)}{\int_a^b f^2(x)\, dx} \right]^{\frac{1}{2}} \cdot \left[\frac{g^2(x)}{\int_a^b g^2(x)\, dx} \right]^{\frac{1}{2}} dx \right| \leqslant \left| \int_a^b \left[\frac{1}{2} \cdot \frac{f^2(x)}{\int_a^b f^2(x)\, dx} + \frac{1}{2} \cdot \frac{g^2(x)}{\int_a^b g^2(x)\, dx} \right] dx \right| = 1.$$

That is, we have proved the inequality (4).

Similarly, the inequality (5) can be proved. ■

Theorem 3. (Hölder inequality) *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\sum_{i=1}^{n} x_i y_i \leqslant \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}}. \tag{6}$$

Let $x_{ij} > 0$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and $\sum_{j=1}^{m} p_j^{-1} = 1$, with $p_j > 1$ for $j = 1, 2, \ldots, m$. Then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} x_{ij} \leqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij}^{p_j} \right)^{\frac{1}{p_j}}. \tag{7}$$

More generally, let $x_{ij} \geqslant 0$, $\alpha_i, \beta_j > 0$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$), $\sum_{j=1}^{m} \beta_j = 1$. Then,

$$\prod_{j=1}^{m} \left(\sum_{i=1}^{n} \alpha_i x_{ij} \right)^{\beta_j} \geqslant \sum_{i=1}^{n} \alpha_i \left(\prod_{j=1}^{m} x_{ij}^{\beta_j} \right). \tag{8}$$

The equality holds if and only if $x_{ij_0} = 0$ for some $j_0 \in \{1, 2, \ldots, m\}$ and all $i \in \{1, 2, \ldots, n\}$, or

$$\frac{x_{i1}}{\sum\limits_{i=1}^{n} \alpha_i x_{i1}} = \frac{x_{i2}}{\sum\limits_{i=1}^{n} \alpha_i x_{i2}} = \cdots = \frac{x_{im}}{\sum\limits_{i=1}^{n} \alpha_i x_{im}}, \quad i \in \{1, 2, \ldots, n\}.$$

Proof. The given inequality (6) is equivalent to

$$\left(\frac{x_1^p}{\sum\limits_{i=1}^n x_i^p}\right)^{\frac{1}{p}}\left(\frac{y_1^q}{\sum\limits_{i=1}^n y_i^q}\right)^{\frac{1}{q}}+\left(\frac{x_2^p}{\sum\limits_{i=1}^n x_i^p}\right)^{\frac{1}{p}}\left(\frac{y_2^q}{\sum\limits_{i=1}^n y_i^q}\right)^{\frac{1}{q}}+ +\cdots+\left(\frac{x_n^p}{\sum\limits_{i=1}^n x_i^p}\right)^{\frac{1}{p}}\left(\frac{y_n^q}{\sum\limits_{i=1}^n y_i^q}\right)^{\frac{1}{q}}\leqslant 1.$$

By using (2), we get

$$\left(\frac{x_1^p}{\sum\limits_{i=1}^n x_i^p}\right)^{\frac{1}{p}}\left(\frac{y_1^q}{\sum\limits_{i=1}^n y_i^q}\right)^{\frac{1}{q}}+\cdots+\left(\frac{x_n^p}{\sum\limits_{i=1}^n x_i^p}\right)^{\frac{1}{p}}\left(\frac{y_n^q}{\sum\limits_{i=1}^n y_i^q}\right)^{\frac{1}{q}} \leqslant \frac{1}{p}\frac{x_1^p}{\sum\limits_{i=1}^n x_i^p}+\frac{1}{q}\frac{y_1^q}{\sum\limits_{i=1}^n y_i^q}+\cdots+\frac{1}{p}\frac{x_n^p}{\sum\limits_{i=1}^n x_i^p}+\frac{1}{q}\frac{y_n^q}{\sum\limits_{i=1}^n y_i^q}=1.$$

We have proved the inequality (6).

The given inequality (7) is equivalent to

$$\left(\frac{x_{11}^{p_1}}{\sum\limits_{i=1}^n x_{i1}^{p_1}}\right)^{\frac{1}{p_1}}\left(\frac{x_{12}^{p_2}}{\sum\limits_{i=1}^n x_{i2}^{p_2}}\right)^{\frac{1}{p_2}}\cdots\left(\frac{x_{1m}^{p_m}}{\sum\limits_{i=1}^n x_{im}^{p_m}}\right)^{\frac{1}{p_m}}+ +\cdots+\left(\frac{x_{n1}^{p_1}}{\sum\limits_{i=1}^n x_{i1}^{p_1}}\right)^{\frac{1}{p_1}}\left(\frac{x_{n2}^{p_2}}{\sum\limits_{i=1}^n x_{i2}^{p_2}}\right)^{\frac{1}{p_2}}\cdots\left(\frac{x_{nm}^{p_m}}{\sum\limits_{i=1}^n x_{im}^{p_m}}\right)^{\frac{1}{p_m}}\leqslant 1.$$

By using (2), we get

$$\left(\frac{x_{11}^{p_1}}{\sum\limits_{i=1}^n x_{i1}^{p_1}}\right)^{\frac{1}{p_1}}\left(\frac{x_{12}^{p_2}}{\sum\limits_{i=1}^n x_{i2}^{p_2}}\right)^{\frac{1}{p_2}}\cdots\left(\frac{x_{1m}^{p_m}}{\sum\limits_{i=1}^n x_{im}^{p_m}}\right)^{\frac{1}{p_m}}+ +\cdots+\left(\frac{x_{n1}^{p_1}}{\sum\limits_{i=1}^n x_{i1}^{p_1}}\right)^{\frac{1}{p_1}}\left(\frac{x_{n2}^{p_2}}{\sum\limits_{i=1}^n x_{i2}^{p_2}}\right)^{\frac{1}{p_2}}\cdots\left(\frac{x_{nm}^{p_m}}{\sum\limits_{i=1}^n x_{im}^{p_m}}\right)^{\frac{1}{p_m}}$$

$$\leqslant \frac{1}{p_1}\frac{x_{11}^{p_1}}{\sum\limits_{i=1}^n x_{i1}^{p_1}}+\frac{1}{p_2}\frac{x_{12}^{p_2}}{\sum\limits_{i=1}^n x_{i1}^{p_2}}+\cdots+\frac{1}{p_m}\frac{x_{1m}^{p_m}}{\sum\limits_{i=1}^n x_{im}^{p_m}}+ +\cdots+\frac{1}{p_1}\frac{x_{n1}^{p_1}}{\sum\limits_{i=1}^n x_{i1}^{p_1}}+\frac{1}{p_2}\frac{x_{n2}^{p_2}}{\sum\limits_{i=1}^n x_{i1}^{p_2}}+\cdots+\frac{1}{p_m}\frac{x_{nm}^{p_m}}{\sum\limits_{i=1}^n x_{im}^{p_m}}=1.$$

We have proved the inequality (7).

When $\prod\limits_{j=1}^{m}\Big(\sum\limits_{i=1}^{n}\alpha_i x_{ij}\Big)^{\beta_j} = 0$, then there must exist $j_0 \in \{1,2,\dots,m\}$ with $x_{ij_0} = 0$, for $i = 1,2,\dots,n$. In this case the equality in (8) obviously holds.

When $\prod\limits_{j=1}^{m}\Big(\sum\limits_{i=1}^{n}\alpha_i x_{ij}\Big)^{\beta_j} > 0$, by applying (2), we get

$$\frac{\sum\limits_{i=1}^{n}\alpha_i\Big(\prod\limits_{j=1}^{m}x_{ij}^{\beta_j}\Big)}{\prod\limits_{j=1}^{m}\Big(\sum\limits_{i=1}^{n}\alpha_i x_{ij}\Big)^{\beta_j}} = \sum_{i=1}^{n}\alpha_i\prod_{j=1}^{m}\Bigg(\frac{x_{ij}}{\sum\limits_{i=1}^{n}\alpha_i x_{ij}}\Bigg)^{\beta_j}$$

$$\leqslant \sum_{i=1}^{n}\alpha_i\sum_{j=1}^{m}\frac{\beta_j x_{ij}}{\sum\limits_{i=1}^{n}\alpha_i x_{ij}} = \sum_{j=1}^{m}\beta_j\frac{\sum\limits_{i=1}^{n}\alpha_i x_{ij}}{\sum\limits_{i=1}^{n}\alpha_i x_{ij}} = 1.$$

In this case, the equality holds if an only if

$$\frac{x_{i1}}{\sum\limits_{i=1}^{n}\alpha_i x_{i1}} = \frac{x_{i2}}{\sum\limits_{i=1}^{n}\alpha_i x_{i2}} = \cdots = \frac{x_{im}}{\sum\limits_{i=1}^{n}\alpha_i x_{im}}, \quad i \in \{1,2,\dots,n\}.$$

Thus, we have proved the inequality (8). ∎

THEOREM 4. (HÖLDER INEQUALITY FOR INTEGRALS) *Let $w, f, g\colon D \to \mathbb{R}$ be integrable functions on a finite interval $D \subset \mathbb{R}$. Then,*

$$\int_D f(x)g(x)w(x)\,dx \leqslant \Big[\int_D |f(x)|^p w(x)\,dx\Big]^{\frac{1}{p}}\Big[\int_D |g(x)|^q w(x)\,dx\Big]^{\frac{1}{q}}, \tag{9}$$

where $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

More generally, suppose that $\alpha, \beta, \dots, \lambda$ are positive real numbers satisfying $\alpha + \beta + \cdots + \lambda = 1$. Then,

$$\int_{x_1}^{x_2} f^\alpha g^\beta \cdots h^\lambda\,dx \leqslant \Big(\int_{x_1}^{x_2} f\,dx\Big)^\alpha\Big(\int_{x_1}^{x_2} g\,dx\Big)^\beta\Big(\int_{x_1}^{x_2} h\,dx\Big)^\lambda, \tag{10}$$

under the condition that $f, g, \dots, h$ are positive real integrable functions defined on the finite interval $[x_1, x_2]$.

Proof. The given inequality (9) is equivalent to

$$\int_D\Big[\frac{f^p(x)w(x)}{\int_D |f(x)|^p w(x)\,dx}\Big]^{\frac{1}{p}}\Big[\frac{g^q(x)w(x)}{\int_D |g(x)|^q w(x)\,dx}\Big]^{\frac{1}{q}}dx \leqslant 1.$$

Since

$$\int_D\Big[\frac{f^p(x)w(x)}{\int_D |f(x)|^p w(x)\,dx}\Big]^{\frac{1}{p}}\Big[\frac{g^q(x)w(x)}{\int_D |g(x)|^q w(x)\,dx}\Big]^{\frac{1}{q}}dx$$

$$\leqslant \int_D\Big[\frac{|f(x)|^p w(x)}{\int_D |f(x)|^p w(x)\,dx}\Big]^{\frac{1}{p}}\Big[\frac{|g(x)|^q w(x)}{\int_D |g(x)|^q w(x)\,dx}\Big]^{\frac{1}{q}}dx,$$

we only need to prove that

$$\int_D \left[\frac{|f(x)|^p w(x)}{\int_D |f(x)|^p w(x)\,dx}\right]^{\frac{1}{p}} \left[\frac{|g(x)|^q w(x)}{\int_D |g(x)|^q w(x)\,dx}\right]^{\frac{1}{q}} dx \leqslant 1.$$

By using (2), we get

$$\begin{aligned}\int_D &\left[\frac{|f(x)|^p w(x)}{\int_D |f(x)|^p w(x)\,dx}\right]^{\frac{1}{p}} \left[\frac{|g(x)|^q w(x)}{\int_D |g(x)|^q w(x)\,dx}\right]^{\frac{1}{q}} dx \\ &\leqslant \int_d \left[\frac{1}{p}\cdot\frac{|f(x)|^p w(x)}{\int_D |f(x)|^p w(x)\,dx} + \frac{1}{q}\cdot\frac{|g(x)|^q w(x)}{\int_D |g(x)|^q w(x)\,dx}\right] dx \\ &= \frac{1}{p}\frac{\int_D |f(x)|^p w(x)\,dx}{\int_D |f(x)|^p w(x)\,dx} + \frac{1}{q}\cdot\frac{\int_D |g(x)|^q w(x)\,dx}{\int_D |g(x)|^q w(x)\,dx} = \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

Thus, we have proved the inequality (9).

The inequality (10) can be proved in a similar way. ■

Theorem 5. (Carlson inequality) *Let* $x_{ij} > 0$ *for* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ *and* $A_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$, $G_i = \Big(\prod_{j=1}^{n} x_{ij}\Big)^{\frac{1}{n}}$. *Then,*

$$\sqrt[n]{A_1 A_1 \cdots A_n} \geqslant \frac{G_1 + G_2 + \cdots + G_n}{n}. \tag{11}$$

More generally, if $x_{ij} \geqslant 0$, $p_j, q_j > 0$ *for* $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *and* $\big(\sum_{j=1}^{m} p_j - 1\big)\big(\sum_{j=1}^{m} q_j - 1\big) > 0$, *then,*

$$\prod_{j=1}^{m}\Big(\sum_{i=1}^{n} q_i x_{ij}\Big)^{p_j} \geqslant \sum_{i=1}^{n} q_i \prod_{j=1}^{m} x_{ij}^{p_j}. \tag{12}$$

The equality holds if and only if $x_{ij_0} = 0$ *for some* $j_0 \in \{1, 2, \ldots, m\}$ *and all* $i = 1, 2, \ldots, n$, *or* $\sum_{j=1}^{m} p_j = 1$ *and* $\frac{x_{i_1 1}}{x_{i_2 1}} = \frac{x_{i_1 2}}{x_{i_2 2}} = \cdots = \frac{x_{i_1 m}}{x_{i_2 m}}$, $i_1, i_2 \in \{1, 2, \ldots, n\}$.

Proof. The given inequality (11) is equivalent to

$$\frac{G_1 + G_2 + \cdots + G_n}{\sqrt[n]{A_1 A_2 \cdots A_n}} \leqslant n.$$

Hence. we only need to prove the following inequality

$$\begin{aligned}&\sqrt[n]{\frac{\prod_{j=1}^{n} x_{1j}}{A_1 A_2 \cdots A_n}} + \sqrt[n]{\frac{\prod_{j=1}^{n} x_{2j}}{A_1 A_2 \cdots A_n}} + \cdots + \sqrt[n]{\frac{\prod_{j=1}^{n} x_{nj}}{A_1 A_2 \cdots A_n}} \\ &\quad = \sqrt[n]{\frac{x_{11}}{A_1}\cdot\frac{x_{12}}{A_2}\cdots\frac{x_{1n}}{A_n}} + \sqrt[n]{\frac{x_{21}}{A_1}\cdot\frac{x_{22}}{A_2}\cdots\frac{x_{2n}}{A_n}} + \sqrt[n]{\frac{x_{n1}}{A_1}\cdot\frac{x_{n2}}{A_2}\cdots\frac{x_{nn}}{A_n}} \leqslant n.\end{aligned}$$

By using (1), we get

$$\sqrt[n]{\frac{x_{11}}{A_1}\cdot\frac{x_{12}}{A_2}\cdots\frac{x_{1n}}{A_n}}+\sqrt[n]{\frac{x_{21}}{A_1}\cdot\frac{x_{22}}{A_2}\cdots\frac{x_{2n}}{A_n}}+\sqrt[n]{\frac{x_{n1}}{A_1}\cdot\frac{x_{n2}}{A_2}\cdots\frac{x_{nn}}{A_n}}$$
$$\leqslant\frac{1}{n}\Big(\frac{x_{11}}{A_1}+\frac{x_{12}}{A_2}+\cdots+\frac{x_{1n}}{A_n}\Big)+\frac{1}{n}\Big(\frac{x_{21}}{A_1}+\frac{x_{22}}{A_2}+\cdots+\frac{x_{2n}}{A_n}\Big)+\frac{1}{n}\Big(\frac{x_{n1}}{A_1}+\frac{x_{n2}}{A_2}+\cdots+\frac{x_{nn}}{A_n}\Big)$$
$$=n.$$

Thus, we have proved the inequality (11).

When $\prod_{j=1}^{m}\Big(\sum_{i=1}^{n}q_ix_{ij}\Big)^{p_j}=0$, then there must exist $j_0\in\{1,2,\ldots,m\}$ such that $x_{ij_0}=0$ for all $i=1,2,\ldots,n$. In this case, equality in (12) obviously holds.

When $\prod_{j=1}^{m}\Big(\sum_{i=1}^{n}q_ix_{ij}\Big)^{p_j}>0$, by applying (2) and the condition $\big(\sum_{j=1}^{m}p_j-1\big)\big(\sum_{j=1}^{m}q_j-1\big)>0$, we get

$$\frac{\sum_{i=1}^{n}q_i\prod_{j=1}^{m}x_{ij}^{p_j}}{\prod_{j=1}^{m}\Big(\sum_{i=1}^{n}q_ix_{ij}\Big)^{p_j}}=\sum_{i=1}^{n}q_i\prod_{j=1}^{m}\Big(\frac{x_{ij}}{\sum_{i=1}^{n}q_ix_{ij}}\Big)^{p_j}=\sum_{i=1}^{n}q_i\Bigg[\prod_{j=1}^{m}\Big(\frac{x_{ij}}{\sum_{i=1}^{n}q_ix_{ij}}\Big)^{\frac{p_j}{\sum_{i=1}^{n}p_j}}\Bigg]^{\sum_{j=1}^{m}p_j}$$

$$\leqslant\sum_{i=1}^{n}q_i\left[\frac{\sum_{j=1}^{m}\frac{p_jx_{ij}}{\sum_{i=1}^{n}q_ix_{ij}}}{\sum_{j=1}^{m}p_j}\right]^{\sum_{j=1}^{m}p_j}\leqslant\left[\sum_{i=1}^{n}q_i\frac{\sum_{j=1}^{m}\frac{p_jx_{ij}}{\sum_{i=1}^{n}q_ix_{ij}}}{\sum_{j=1}^{m}p_j}\right]^{\sum_{j=1}^{m}p_j}=\left[\sum_{j=1}^{m}p_j\cdot\frac{\frac{\sum_{i=1}^{n}q_ix_{ij}}{\sum_{i=1}^{n}q_ix_{ij}}}{\sum_{j=1}^{m}p_j}\right]^{\sum_{j=1}^{m}p_j}$$
$$=1.$$

Thus, we have proved the inequality (12). ∎

THEOREM 6. (CARLSON INEQUALITY FOR INTEGRALS) *If $q(x)$ and $f_j(x)$, $j=1,2,\ldots,n$, are integrable functions on $[a,b]$, $\int_a^b|q(x)\,dx<1$, $p_j>0$, $\sum_{j=1}^{m}p_j\leqslant 1$, then,*

$$\prod_{j=1}^{m}\Big(\int_a^b|q(x)f_j(x)|\,dx\Big)^{p_j}\geqslant\int_a^b|q(x)|\prod_{j=1}^{m}|f_j(x)|^{p_j}\,dx. \tag{13}$$

Proof. The given inequality (13) is equivalent to

$$\int_a^b\frac{|q(x)|\prod_{j=1}^{m}|f_j(x)|^{p_j}}{\prod_{j=1}^{m}\Big(\int_a^b|q(x)f_j(x)|\,dx\Big)^{p_j}}\,dx$$
$$=\int_a^b\left[\frac{|q(x)|\,|f_1(x)|}{\int_a^b|q(x)f_1(x)|\,dx}\right]^{p_1}\left[\frac{|q(x)|\,|f_2(x)|}{\int_a^b|q(x)f_2(x)|\,dx}\right]^{p_2}\cdots\left[\frac{|q(x)|\,|f_m(x)|}{\int_a^b|q(x)f_m(x)|\,dx}\right]^{p_m}dx$$
$$\leqslant 1.$$

By using (2), we get

$$\int_a^b \left[\frac{|q(x)|\,|f_1(x)|}{\int_a^b |q(x)f_1(x)|\,dx}\right]^{p_1} \left[\frac{|q(x)|\,|f_2(x)|}{\int_a^b |q(x)f_2(x)|\,dx}\right]^{p_2} \cdots \left[\frac{|q(x)|\,|f_m(x)|}{\int_a^b |q(x)f_m(x)|\,dx}\right]^{p_m} dx$$
$$\leqslant \int_a^b \left[p_1 \frac{|q(x)|\,|f_1(x)|}{\int_a^b |q(x)f_1(x)|\,dx} + p_2 \frac{|q(x)|\,|f_2(x)|}{\int_a^b |q(x)f_2(x)|\,dx} + \cdots + p_m \frac{|q(x)|\,|f_m(x)|}{\int_a^b |q(x)f_m(x)|\,dx}\right] dx$$
$$\leqslant 1.$$

Thus, we have proved the inequality (13). ■

3. Examples and applications

Example 1. Let $a_1, \dots, a_n$; $b_1, \dots, b_n$; $c_1, \dots, c_n$; $d_1, \dots, d_n$ be four sequences of positive real numbers. Prove the inequality.

$$(a_1^4 + \cdots + a_n^4)(b_1^4 + \cdots + b_n^4)(c_1^4 + \cdots + c_n^4)(d_1^4 + \cdots + d_n^4)$$
$$\geqslant (a_1b_1c_1d_1 + a_2b_2c_2d_2 + a_3b_3c_3d_3 + a_4b_4c_4d_4)^4.$$

Proof. Put $m = n = 4$, $p_1 = p_2 = p_3 = p_4 - 4$ in inequality (7).

Example 2. Let $a_1, \dots, a_n$; $b_1, \dots, b_n$; $c_1, \dots, c_n$ be three sequences of positive real numbers. Prove the inequality

$$\left(\sum_{k=1}^n a_k b_k c_k\right)^4 \leqslant \left(\sum_{k=1}^n a_k^2\right)^2 \left(\sum_{k=1}^n b_k^4\right)\left(\sum_{k=1}^n c_k^4\right). \tag{14}$$

Proof. The inequality (14) is equivalent to

$$\sum_{k=1}^n a_k b_k c_k \leqslant \left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^n b_k^4\right)^{\frac{1}{4}} \left(\sum_{k=1}^n c_k^4\right)^{\frac{1}{4}}.$$

That is, we only need to prove that

$$\sum_{k=1}^n \frac{a_k b_k c_k}{\left(\sum\limits_{k=1}^n a_k^2\right)^{\frac{1}{2}} \left(\sum\limits_{k=1}^n b_k^4\right)^{\frac{1}{4}} \left(\sum\limits_{k=1}^n c_k^4\right)^{\frac{1}{4}}} = \sum_{k=1}^n \left(\frac{a_k^2}{\sum\limits_{k=1}^n a_k^2}\right)^{\frac{1}{2}} \left(\frac{b_k^4}{\sum\limits_{k=1}^n b_k^4}\right)^{\frac{1}{4}} \left(\frac{c_k^4}{\sum\limits_{k=1}^n c_k^4}\right)^{\frac{1}{4}} \leqslant 1.$$

Using (2), we get

$$\sum_{k=1}^n \left(\frac{a_k^2}{\sum\limits_{k=1}^n a_k^2}\right)^{\frac{1}{2}} \left(\frac{b_k^4}{\sum\limits_{k=1}^n b_k^4}\right)^{\frac{1}{4}} \left(\frac{c_k^4}{\sum\limits_{k=1}^n c_k^4}\right)^{\frac{1}{4}} \leqslant \sum_{k=1}^n \left[\frac{1}{2} \cdot \frac{a_k^2}{\sum\limits_{k=1}^n a_k^2} + \frac{1}{4} \cdot \frac{b_k^4}{\sum\limits_{k=1}^n b_k^4} + \frac{1}{4} \cdot \frac{c_k^4}{\sum\limits_{k=1}^n c_k^4}\right] = 1.$$

Thus, we have proved the inequality (14).

EXAMPLE 3. Let α, β be positive rational numbers, where $\alpha + \beta = 1$. Prove that for any positive numbers $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$, the following inequality holds:

$$a_1^\alpha b_1^\beta + a_2^\alpha + \cdots + a_n^\alpha b_n^\beta \leqslant (a_1 + a_2 + \cdots + a_n)^\alpha (b_1 + b_2 + \cdots + b_n)^\beta.$$

Proof. We can prove this inequality by putting $x_i^p = a_i$, $y + i^q = b_i$, $i = 1, 2, \ldots, n$. $\dfrac{1}{p} = \alpha$, $\dfrac{1}{q} = \beta$ in inequality (6).

EXAMPLE 4. Let $a_1, a_2, \ldots, a_m$ be n positive numbers, and let g be their geometric mean. Prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geqslant (1 + g)^n. \tag{15}$$

Proof. The inequality (15) is equivalent to

$$1 + \sqrt[n]{a_1 a_2 \cdots a_n} \leqslant \sqrt[n]{(1 + a_1)(1 + a_2) \cdots (1 + a_n)}.$$

Hence, we only need to prove that

$$\sqrt[n]{\frac{1}{1 + a_1} \cdot \frac{1}{1 + a_2} \cdots \frac{1}{1 + a_n}} + \sqrt[n]{\frac{a_1}{1 + a_1} \cdot \frac{a_2}{1 + a_2} \cdots \frac{a_n}{1 + a_n}} \leqslant 1.$$

Using (1), we get

$$\begin{aligned} &\sqrt[n]{\frac{1}{1 + a_1} \cdot \frac{1}{1 + a_2} \cdots \frac{1}{1 + a_n}} + \sqrt[n]{\frac{a_1}{1 + a_1} \cdot \frac{a_2}{1 + a_2} \cdots \frac{a_n}{1 + a_n}} \\ &\leqslant \frac{1}{n}\Big(\frac{1}{1 + a_1} + \frac{1}{1 + a_2} + \cdots + \frac{1}{1 + a_n}\Big) + \frac{1}{n}\Big(\frac{a_1}{1 + a_1} + \frac{a_2}{1 + a_2} + \cdots + \frac{a_n}{1 + a_n}\Big) = 1. \end{aligned}$$

Thus, we have proved the inequality (15). ■

EXAMPLE 5. Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$; $\ldots$; $l_1, l_2, \ldots, l_n$ be k sequences of positive numbers. Then the following inequality holds:

$$\begin{aligned} &\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} + \cdots + \sqrt[n]{l_1 l_2 \cdots l_n} \\ &\qquad \leqslant \sqrt[n]{(a_1 + b_1 + \cdots + l_1)(a_2 + b_2 + \cdots + l_2) \cdots (a_n + b_n + \cdots + l_n)}. \end{aligned}$$

Proof. We can prove this inequality by putting the following in inequality (7): $m = n$, $p_j = n$, $j = 1, 2, \ldots, n$; $a_1 = x_{11}^n$, $b_1 = x_{21}^n$, $\ldots$ $l_1 = x_{n1}^n$; $a_2 = x_{12}^n$, $b_2 = x_{22}^n$, $\ldots$ $l_2 = x_{n2}^n$; $a_n = x_{1n}^n$, $b_n = x_{2n}^n$, $\ldots$ $l_n = x_{nn}^n$.

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Department of Mathematics, Wenzhou University, Zhejiang 325035, P. R. China

E-mail: `wzxuyanhui@126.com`