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FORMALISING THE USE OF THE CENTRE OF MASS METHOD IN MATHEMATICAL PROBLEMS

Meirav Amram, Miriam Dagan, Sagi Levi, Artour Mouftakhov

Abstract. In this paper, we adopt from physics the concept of centre of mass and use it in the solution of classical geometric problems. Such uses include finding both the incentre and orthocentre of a triangle. We define the concept of centre of mass mathematically and show how to formalise the intuitive solution from physics into a formal mathematical solution. This concept is easy to teach because it requires only basic knowledge of vectors; it can help in solving many complicated geometrical problems.

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1. Scientific background

Do different fields of science have common topics and methods, or are they independent of one another? If you ask students, they will probably answer that different fields are independent and have nothing in common. In fact, it is relatively common for one field to use methods developed in other fields.

Treeby [6] gives good examples of using the centre of mass method when solving combinatorial problems. The centre of mass method was used by Hodgson & Shultz [5] to solve the soda can problem. In Hausner [3], the author used the centre of mass method to solve several geometrical problems. In Dana-Picard [1] the author used barycentric coordinates to solve problems in geometry and calculus. One of those problems is the incentre problem that we solve, although we use the centre of mass method.

In this article we focus on the centre of mass method from physics and use it to easily solve problems from geometry and analytic geometry. The centre of mass method is used as an intuition for problem solving, and we formalise this intuition to a mathematically accepted solution. We use a table to show how we formalise the physical intuition, which is mathematically unacceptable, to an accepted mathematical solution. Our method can be taught in high school because we only use the mathematical concept of vectors, which is taught at that level.

The centre of mass concept that we use helps us solve the incentre problem very easily. This problem is partially taught in high school in the following way: The incentre of a triangle is the point where all its bisectors cross. However, the division ratio is only taught in the case of a triangle's medians (a 1 : 2 division ratio). This is because all other cases are too difficult to prove using the geometry tools taught in most high schools.

In addition to the incentre problem, we solve the orthocentre problem and other problems from analytic geometry. We also give the reader other problems for exercise.

The importance of our paper comes from the fact that we show how to adapt the centre of mass method for use in most high schools, which enables teachers and students to explain and solve geometrical problems that are too complicated to solve using customary methods.

Our paper is divided as follows: In Section 2, we show how to use vectors to solve problems in geometry. In Section 3, a formal mathematical definition of the centre of mass method is given. In Section 4 we solve the incentre and orthocentre problems using the centre of mass method; then we solve other problems. In Section 5 we offer our conclusions.

2. Using vectors to solve problems in geometry

In this section we show how to use vectors to solve some problems in geometry.

2.1 Problem 1. *In triangle ABC, points D, E are the midpoints of AB, AC respectively, as shown in Fig. 1. If we connect points D, E, then the length of DE is half of the length of BC, and $DE \parallel BC$.*

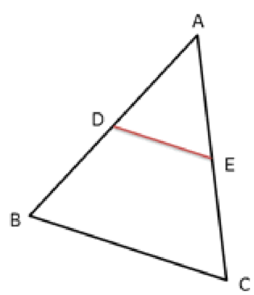

Fig. 1

Proof. Let $\overrightarrow{BA} = \vec{u}$ and $\overrightarrow{AC} = \vec{v}$; then $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \vec{u} + \vec{v}$. Points D, E are the midpoints of AB, AC, respectively, hence $\overrightarrow{DA} = \frac{1}{2}\vec{u}$ and $\overrightarrow{AE} = \frac{1}{2}\vec{v}$. Now we can write

$$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}\vec{u} + \frac{1}{2}\vec{v} = \frac{1}{2}(\vec{u} + \vec{v}) = \frac{1}{2}\overrightarrow{BC},$$

hence $\overrightarrow{DE} \upuparrows \overrightarrow{BC}$ and $|\overrightarrow{DE}| = \frac{1}{2}|\overrightarrow{BC}|$. ■

2.2. Problem 2. *In a quadrilateral $ABCD$, if diagonals AC and BD halve each other (as shown in Fig. 2), then $ABCD$ is a parallelogram.*

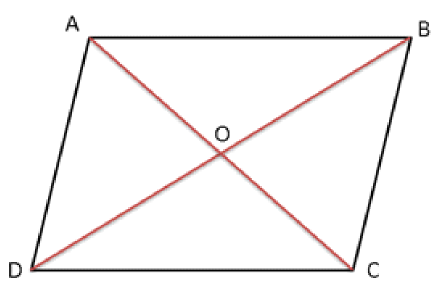

Fig. 2

Proof. Let O be the intersections of the diagonals and let $\overrightarrow{AO} = \vec{u}$ and $\overrightarrow{OB} = \vec{v}$; then $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \vec{u} + \vec{v}$. Since diagonals AC and BD halve each other, then $\overrightarrow{OC} = \overrightarrow{AO} = \vec{u}$ and $\overrightarrow{DO} = \overrightarrow{OB} = \vec{v}$, implying that $\overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC} = \vec{u} + \vec{v} = \overrightarrow{AB}$. It follows that $DC \parallel AB$ and their lengths are equal, hence $ABCD$ is a parallelogram. ■

2.3. Problem 3. *In the parallelogram ABCD (see Fig. 2), the diagonals halve each other.*

Proof. Let $\overrightarrow{AO} = \vec{u}$ and $\overrightarrow{OB} = \vec{v}$. Since $OC \upuparrows AO$ and $DO \upuparrows OB$, then $\overrightarrow{OC} = a\,\vec{u}$ and $\overrightarrow{DO} = b\,\vec{v}$ for some $a, b \in \mathbb{R}$. We now use the fact that $\overrightarrow{AB} = \overrightarrow{DC}$:

$$\left.\begin{array}{l} \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \vec{u} + \vec{v} \\ \overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC} = b\,\vec{v} + a\,\vec{u} \end{array}\right\} \Rightarrow\ \vec{u} + \vec{v} = b\,\vec{v} + a\,\vec{u}\ \Rightarrow\ (1-a)\vec{u} + (1-b)\vec{v} = \vec{0}.$$

Since $\vec{u} \not\parallel \vec{v}$, these vectors are linearly independent, meaning that $(1-a)\vec{u}+(1-b)\vec{v} = \vec{0}$ has only the trivial solution $1-a = 1-b = 0$, implying that $a = b = 1$ and hence $\overrightarrow{OC} = \overrightarrow{AO} = \vec{u}$ and $\overrightarrow{DO} = \overrightarrow{OB} = \vec{v}$. ∎

3. Centre of mass – formal mathematical definition

In this section, we formalise the physics-based notion of centre of mass. For this, we need the following definitions and theorems.

3.1. Definition 1. A pair (P, m), where $P \in \mathbb{R}^3$ represents a position in space and $m \in \mathbb{R}^+$ represents P's mass, is called a "mass point". The mass point (P, m) can be also denoted as mP.

3.2. Definition 2. Given two mass points $\{m_1P_1, m_2P_2\}$ (see Fig. 3 in Theorem 1, below), their centre of mass is a point Z such that:

(1) $Z \in P_1P_2$, i.e., Z is on the line between the points P_1, P_2.

(2) "Lever Rule": $\dfrac{|\overrightarrow{P_1Z}|}{|\overrightarrow{ZP_2}|} = \dfrac{m_2}{m_1}$.

3.2.1. Theorem 1. *For any two mass points, their centre of mass exists and is unique, as illustrated in Fig. 3.*

Proof. It is known that any segment can be divided in a given ratio. Let point Z divide the segment P_1P_2 in the ratio of $m_2 : m_1$. From Definition 1, it follows that point Z is the centre of mass of the mass points $\{m_1P_1, m_2P_2\}$. Thus, centre of mass exists.

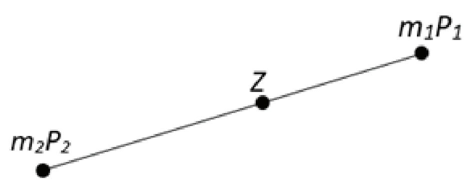

Fig. 3

Assume now that there are two centres of mass, Z_1 and Z_2, such that $Z_1 \neq Z_2$. Then from (2) of Definition 2 we get

$$\frac{|\overrightarrow{P_1Z_1}|}{|\overrightarrow{Z_1P_2}|} = \frac{m_2}{m_1} = \frac{|\overrightarrow{P_1Z_2}|}{|\overrightarrow{Z_2P_2}|}.$$

From (1) of Definition 2 we get $|\overrightarrow{P_1Z_i}|+|\overrightarrow{Z_iP_2}| = |\overrightarrow{P_1P_2}|$, for $i = 1,2$, implying that $\dfrac{|\overrightarrow{P_1Z_i}|}{|\overrightarrow{Z_iP_2}|} = \dfrac{|\overrightarrow{P_1P_2}| - |\overrightarrow{Z_iP_2}|}{|\overrightarrow{Z_iP_2}|} = \dfrac{|\overrightarrow{P_1P_2}|}{|\overrightarrow{Z_iP_2}|} - 1$, and using that $\dfrac{|\overrightarrow{P_1Z_1}|}{|\overrightarrow{Z_1P_2}|} = \dfrac{|\overrightarrow{P_1Z_2}|}{|\overrightarrow{Z_2P_2}|}$, we get $\dfrac{|\overrightarrow{P_1P_2}|}{|\overrightarrow{Z_1P_2}|} - 1 = \dfrac{|\overrightarrow{P_1P_2}|}{|\overrightarrow{Z_2P_2}|} - 1$, wherefrom $|\overrightarrow{Z_1P_2}| = |\overrightarrow{Z_2P_2}|$ and hence $Z_1 = Z_2$, which contradicts our assumption. ■

3.2.2 Theorem 2. *Given two mass points $\{m_1P_1, m_2P_2\}$, a point Z is their centre of mass if and only if $m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} = \vec{0}$.*

Proof. $\Leftarrow$: Assume that $m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} = \vec{0}$. Then

$$\overrightarrow{P_1Z} = \frac{m_2}{m_1} \cdot \overrightarrow{ZP_2}, \text{ implying that } \begin{cases} (1) & \overrightarrow{P_1Z} \upuparrows \overrightarrow{ZP_2} \\ (2) & \dfrac{|\overrightarrow{P_1Z}|}{|\overrightarrow{ZP_2}|} = \dfrac{m_2}{m_1}, \end{cases} \text{ hence } Z \in P_1P_2.$$

$\Rightarrow$: Assume Z is the centre of mass of $\{m_1P_1, m_2P_2\}$. Then $Z \in P_1P_2$, hence $\overrightarrow{P_1Z} \upuparrows \overrightarrow{ZP_2}$, i.e., $\overrightarrow{P_1Z} = \gamma \cdot \overrightarrow{ZP_2}$, for some $\gamma > 0$. According to the "Lever Rule" of Definition 2, we also have $\dfrac{|\overrightarrow{P_1Z}|}{|\overrightarrow{ZP_2}|} = \dfrac{m_2}{m_1}$, leading to $\dfrac{m_2}{m_1} = \gamma$. Therefore, $\overrightarrow{P_1Z} = \gamma \cdot \overrightarrow{ZP_2}$, hence $\overrightarrow{P_1Z} = \dfrac{m_2}{m_1} \cdot \overrightarrow{ZP_2}$ and $m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} = \vec{0}$. ■

We now generalize the case of a centre of mass of two mass points to a case of a centre of mass of three mass points.

3.3. Definition 3. Given three mass points $\{m_1P_1, m_2P_2, m_3P_3\}$, their centre of mass is a point Z such that

$$m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} + m_3 \cdot \overrightarrow{ZP_3} = \vec{0}. \tag{3}$$

3.3.3. Theorem 3. *For any three mass points $\{m_1P_1, m_2P_2, m_3P_3\}$, their centre of mass exists and is unique, as illustrated in Fig. 4.*

Proof. We show that the equation $m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} + m_3 \cdot \overrightarrow{ZP_3} = \vec{0}$ has a unique solution. Since $\overrightarrow{ZP_2} = \overrightarrow{ZP_1} + \overrightarrow{P_1P_2}$ and $\overrightarrow{ZP_3} = \overrightarrow{ZP_1} + \overrightarrow{P_1P_3}$, it is equivalent to each of the following equations:

$$\begin{gathered} m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot (\overrightarrow{ZP_1} + \overrightarrow{P_1P_2}) + m_3 \cdot (\overrightarrow{ZP_1} + \overrightarrow{P_1P_3}) = \vec{0}, \\ (m_1 + m_2 + m_3) \cdot \overrightarrow{ZP_1} = m_2 \cdot \overrightarrow{P_2P_1} + m_3 \cdot \overrightarrow{P_3P_1}, \\ \overrightarrow{ZP_1} = \frac{m_2}{m_1 + m_2 + m_3} \cdot \overrightarrow{P_2P_1} + \frac{m_3}{m_1 + m_2 + m_3} \cdot \overrightarrow{P_3P_1}. \end{gathered}$$

The right-hand side of the last equality exists and is uniquely determined and independent of a point Z, and hence the left-hand side exists and is also unique.

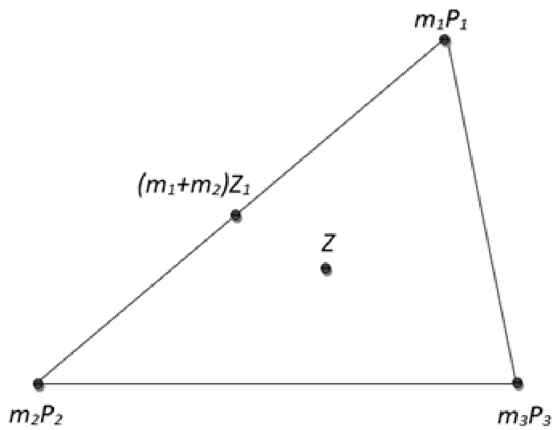

Fig. 4

That is, the vector $\overrightarrow{ZP_1}$ exists and is unique. It follows that the point Z exists and is unique. This means that the centre of mass of $\{m_1P_1, m_2P_2, m_3P_3\}$ exists and is unique. ■

3.3.4. Theorem 4. *For any three mass points $\{m_1P_1, m_2P_2, m_3P_3\}$ (see Fig. 4 in Theorem 3), if Z is their centre of mass and Z_1 is the centre of mass of $\{m_1P_1, m_2P_2\}$, Then Z is the centre of mass of $\{(m_1+m_2)Z_1, m_3P_3\}$.*

Proof. If point Z is the centre of mass of $\{m_1P_1, m_2P_2, m_3P_3\}$, then $m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} + m_3 \cdot \overrightarrow{ZP_3} = \vec{0}$. If point Z_1 is the centre of mass of $\{m_1P_1, m_2P_2\}$, then $m_1 \cdot \overrightarrow{Z_1P_1} + m_2 \cdot \overrightarrow{Z_1P_2} = \vec{0}$. We now prove that Z is the centre of mass of $\{(m_1+m_2)Z_1, m_3P_3\}$:

$$\begin{aligned}
\vec{0} &= m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} + m_3 \cdot \overrightarrow{ZP_3} \\
&= m_1 \cdot (\overrightarrow{ZZ_1} + \overrightarrow{Z_1P_1}) + m_2 \cdot (\overrightarrow{ZZ_1} + \overrightarrow{Z_1P_2}) + m_3 \cdot \overrightarrow{ZP_3} \\
&= (m_1+m_2) \cdot \overrightarrow{ZZ_1} + m_3 \cdot \overrightarrow{ZP_3} + \underbrace{m_1 \cdot \overrightarrow{Z_1P_1} + m_2 \cdot \overrightarrow{Z_1P_2}}_{\vec{0}} \\
&= (m_1+m_2) \cdot \overrightarrow{ZZ_1} + m_3 \cdot \overrightarrow{ZP_3}. \quad ■
\end{aligned}$$

We now generalize the case of a centre of mass of three mass points to the case of a centre of mass of any number of mass points.

3.4. Definition 4. Given n mass points $\{m_1P_1, m_2P_2, \ldots, m_nP_n\}$, their centre of mass is a point Z such that $m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} + \cdots + m_n \cdot \overrightarrow{ZP_n} = \vec{0}$.

3.4.1. Theorem 5. *For any n mass points $\{m_1P_1, m_2P_2, \ldots, m_nP_n\}$, their centre of mass exists and is unique.*

Proof. We have that $\overrightarrow{ZP_k} = \overrightarrow{ZP_1} + \overrightarrow{P_1P_k}$ for $k = 2, 3, \dots, n$. Hence, the equation $m_1 \cdot \overrightarrow{ZP_1} + m_2 \cdot \overrightarrow{ZP_2} + \cdots + m_n \cdot \overrightarrow{ZP_n} = \vec{0}$ is equivalent to each of the following:

$$m_1 \cdot \overrightarrow{ZP_1} + \sum_{k=2}^{n} m_k \cdot \overrightarrow{ZP_k} = \vec{0},$$

$$m_1 \cdot \overrightarrow{ZP_1} + \sum_{k=2}^{n} m_k \cdot (\overrightarrow{ZP_1} + \overrightarrow{P_1P_k}) = \vec{0},$$

$$(m_1 + \cdots + m_k) \cdot \overrightarrow{ZP_1} = \sum_{k=2}^{n} m_k \cdot \overrightarrow{P_kP_1},$$

$$\overrightarrow{ZP_1} = \frac{m_2}{m_1 + \cdots + m_n} \cdot \overrightarrow{P_2P_1} + \cdots + \frac{m_n}{m_1 + \cdots + m_n} \cdot \overrightarrow{P_nP_1}.$$

This means that the centre of mass of $\{m_1P_1, m_2P_2, \dots, m_nP_n\}$ exists and is unique. ∎

3.4.2. Theorem 6. *For any n mass points $\{m_1P_1, m_2P_2, \dots, m_nP_n\}$ let Z be their centre of mass, and for any k of them, say $\{m_1P_1, m_2P_2, \dots, m_kP_k\}$, let Z_1 be their centre of mass. Then Z is the centre of mass of*

$$\{(m_1 + \cdots + m_k)Z_1, m_{k+1}P_{k+1}, \dots, m_nP_n\}.$$

Proof is done by induction on the number of points. Theorem 2 proves the case of $n = 2$, and Theorem 3 proves the case of $n = 3$. We now assume Theorem 6 is true for any set of l points such that $l < n$, and prove for any set of size n. We denote $m_1 + m_2 + \cdots + m_n = M_n$ and $m_1 + m_2 + \cdots + m_k = M_k$. Point Z is the center of mass of $\{m_1P_1, m_2P_2, \dots, m_nP_n\}$. Using Theorem 5 it follows that $\overrightarrow{ZP_1} = \frac{\sum_{j=2}^{n} m_j \cdot \overrightarrow{P_jP_1}}{M_n}$. Similarly, $\overrightarrow{Z_1P_1} = \frac{\sum_{j=2}^{n} m_j \cdot \overrightarrow{P_jP_1}}{M_k}$. Using Theorem 5 on $\{(m_1 + \cdots + m_k)Z_1, m_{k+1}P_{k+1}, \dots, m_nP_n\}$ it follows that the centre of mass O satisfies the equation $\overrightarrow{OP_1} = \frac{M_k}{M_n} \cdot \overrightarrow{Z_1P_1} + \frac{\sum_{j=k+1}^{n} m_j \cdot \overrightarrow{P_jP_1}}{M_n}$. Now we prove that $O = Z$. Indeed,

$$\begin{aligned}
\overrightarrow{OP_1} &= \frac{M_k}{M_n} \cdot \overrightarrow{Z_1P_1} + \frac{\sum_{j=k+1}^{n} m_j \cdot \overrightarrow{P_jP_1}}{M_n} \\
&= \frac{M_k}{M_n} \cdot \frac{\sum_{j=2}^{k} m_j \cdot \overrightarrow{P_jP_1}}{M_k} + \frac{\sum_{j=k+1}^{n} m_j \cdot \overrightarrow{P_jP_1}}{M_n} \\
&= \frac{\sum_{j=2}^{k} m_j \cdot \overrightarrow{P_jP_1}}{M_n} + \frac{\sum_{j=k+1}^{n} m_j \cdot \overrightarrow{P_jP_1}}{M_n} \\
&= \frac{\sum_{j=2}^{n} m_j \cdot \overrightarrow{P_jP_1}}{M_n} = \overrightarrow{ZP_1}. \quad \blacksquare
\end{aligned}$$

4. Using the centre of mass method to solve problems in geometry

In this section we choose some problems that are difficult to solve using standard high school geometry tools; the centre of mass method makes their solution much easier.

4.1. PROBLEM 1. Given a triangle ABC, we denote by A_1 (B_1, C_1 respectively) the point where A's (B's, C's respectively) angle bisector intersects with BC (AC, AB respectively). We also denote $a = |BC|$, $b = |AC|$, $c = |AB|$, as Fig. 5 shows. Then, all three bisectors intersect at a single point O (the incentre of triangle ABC) that satisfies

$$\frac{|AO|}{|OA_1|} = \frac{b+c}{a}, \quad \frac{|BO|}{|OB_1|} = \frac{a+c}{b}, \quad \frac{|CO|}{|OC_1|} = \frac{a+b}{c}.$$

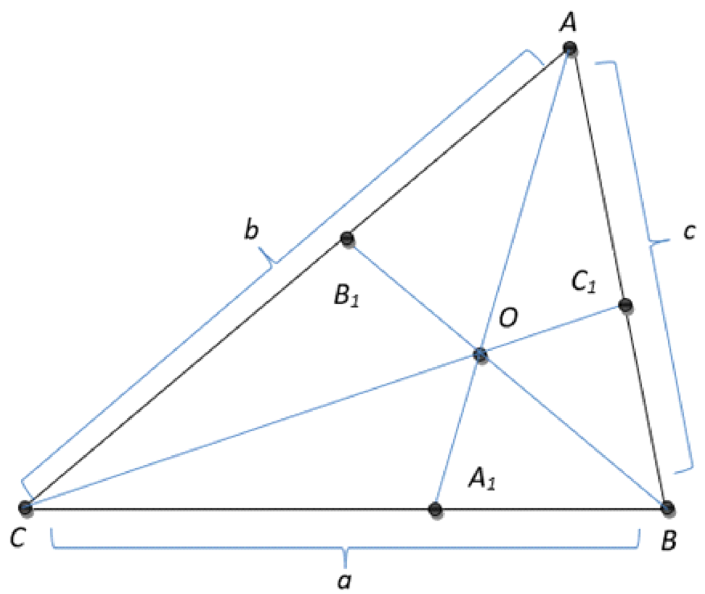

Fig. 5

Proof.

Intuition from physics

We give to points A, B, C, the masses a, b, c, respectively, such that: $a = |BC|$, $b = |AC|$, $c = |AB|$. We now have a system of three mass points $\{aA, bB, cC\}$ with a centre of mass, which is Z (Fig. 6).

Formal mathematical solution

According to Theorem 3, there is a unique Z such that $a \cdot \overrightarrow{ZA} + b \cdot \overrightarrow{ZB} + c \cdot \overrightarrow{ZC} = \vec{0}$.

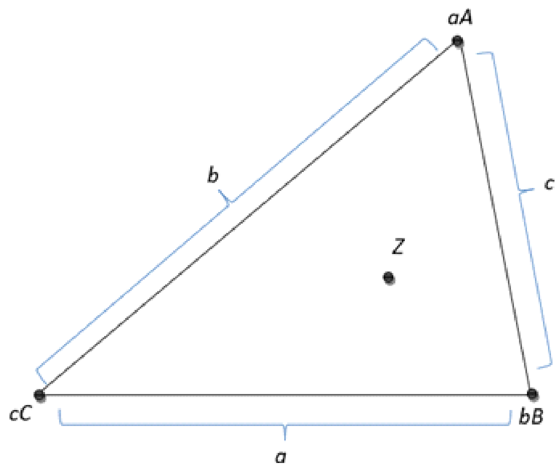

Fig. 6

The centre of mass of $\{aA, bB\}$ is C_1. We move masses from points A and B to point C_1, which becomes a mass point $(a+b)C_1$ (Fig. 7).

CC_1 is the bisector if and only if $\frac{|\overrightarrow{AC}|}{|\overrightarrow{BC}|} = \frac{|\overrightarrow{C_1A}|}{|\overrightarrow{BC_1}|}$ and $C_1 \in AB$ implies that $a \cdot \overrightarrow{C_1A} = b \cdot \overrightarrow{BC_1}$, i.e., $a \cdot \overrightarrow{C_1A} + b \cdot \overrightarrow{C_1B} = \vec{0}$. Now, $\vec{0} = a \cdot \overrightarrow{ZA} + b \cdot \overrightarrow{ZB} + c \cdot \overrightarrow{ZC} = a \cdot (\overrightarrow{ZC_1} + \overrightarrow{C_1A}) + b \cdot (\overrightarrow{ZC_1} + \overrightarrow{C_1B}) + c \cdot \overrightarrow{ZC} = (a+b) \cdot \overrightarrow{ZC_1} + c \cdot \overrightarrow{ZC} + \underbrace{a \cdot \overrightarrow{C_1A} + b \cdot \overrightarrow{C_1B}}_{\vec{0}} = (a+b) \cdot \overrightarrow{ZC_1} + c \cdot \overrightarrow{ZC}$.

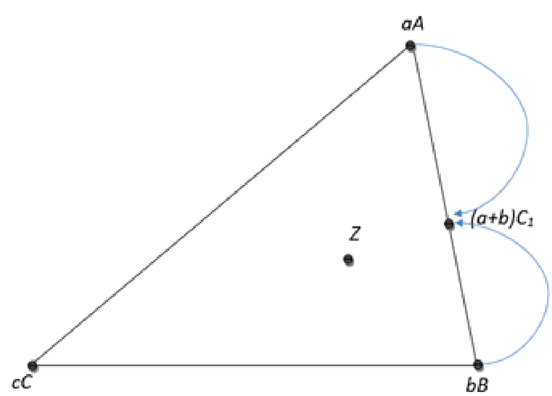

Fig. 7

Producing the system $\{(a+b)C_1, cC\}$ with Z as the centre of mass, thus $\frac{|CZ|}{|ZC_1|} = \frac{a+b}{c}$ (Fig. 8).

Resulting in $(a+b)\cdot\overrightarrow{ZC_1} + c\cdot\overrightarrow{ZC} = \vec{0}$. By Theorem 1 we have $Z \in CC_1$, and $\frac{|CZ|}{|ZC_1|} = \frac{a+b}{c}$.

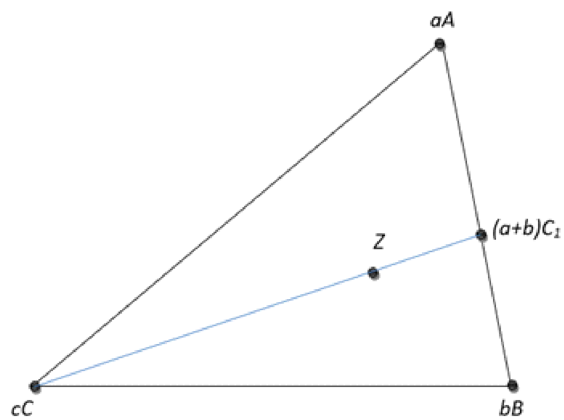

Fig. 8

From this proof we have $Z \in CC_1$. In a similar way we can prove that $Z \in BB_1$ and $Z \in AA_1$; thus Z is the intersection point of all bisectors. This means that $Z = O$ (as illustrated in Fig. 5), and

$$\frac{|AO|}{|OA_1|} = \frac{b+c}{a}, \quad \frac{|BO|}{|OB_1|} = \frac{a+c}{b}, \quad \frac{|CO|}{|OC_1|} = \frac{a+b}{c}. \quad \blacksquare$$

4.2. Problem 2. Given an acute-angled triangle ABC, denote by A_1 (B_1, C_1, respectively) the intersection point of the altitude to the edge BC (AC, AB, respectively), as the diagram on Fig. 9 shows. Also denote $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$. Then, all the altitudes in the triangle intersect in a single point O (orthocentre) and

$$\frac{|AO|}{|OA_1|} = \frac{\tan\beta + \tan\gamma}{\tan\alpha}, \quad \frac{|BO|}{|OB_1|} = \frac{\tan\alpha + \tan\gamma}{\tan\beta}, \quad \frac{|CO|}{|OC_1|} = \frac{\tan\alpha + \tan\beta}{\tan\gamma}.$$

Proof.

Intuition from physics

We endow points A, B, C with the masses m_1, m_2, m_3, respectively, such that: $m_1 = \tan\alpha$, $m_2 = \tan\beta$, $m_3 = \tan\gamma$. We now have a system of three mass points $\{m_1A, m_2B, m_3C\}$ with the centre of mass which is Z (Fig. 10).

Formal mathematical solution

According to Theorem 3, there is a unique Z such that $m_1\cdot\overrightarrow{ZA} + m_2\cdot\overrightarrow{ZB} + m_3\cdot\overrightarrow{ZC} = \vec{0}$.

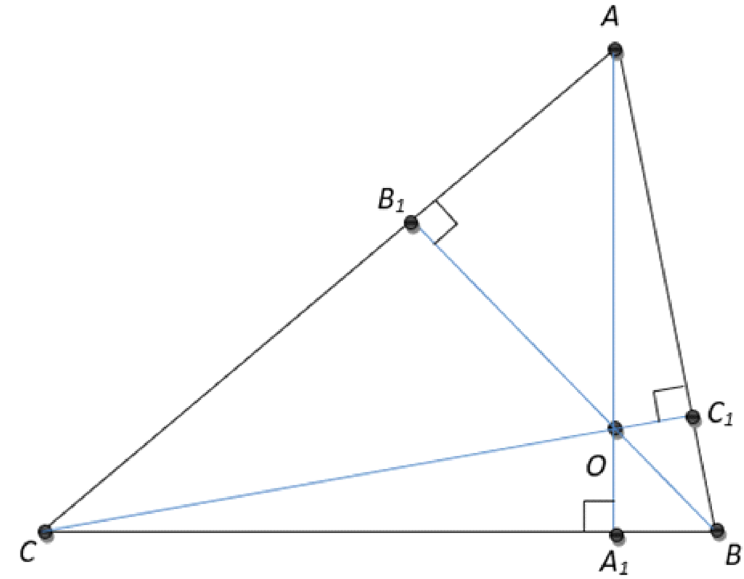

Fig. 9

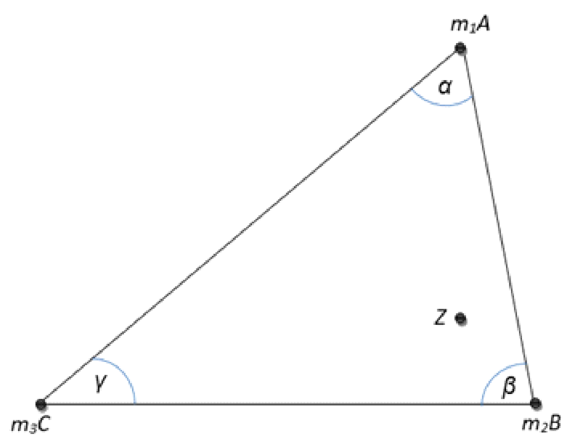

Fig. 10

Let the centre of mass of the system $\{m_2B, m_3C\}$ be A_1. We move masses from points B and C to point A_1, which becomes a mass point $(m_2+m_3)A_1$ (Fig. 11).

AA_1 is an altitude if and only if $\tan\beta\cdot|\overrightarrow{A_1B}| = \tan\gamma\cdot|\overrightarrow{CA_1}|$ and $A_1 \in BC$. It follows that $\tan\beta\cdot\overrightarrow{A_1B} = \tan\gamma\cdot\overrightarrow{CA_1}$, hence $m_2\cdot\overrightarrow{A_1B} + m_2\cdot\overrightarrow{A_1C} = \vec{0}$. Now, $\vec{0} = m_1\cdot\overrightarrow{ZA} + m-2\cdot\overrightarrow{ZB} + m_3\cdot\overrightarrow{ZC} = m_1\cdot\overrightarrow{ZA} + m_2\cdot(\overrightarrow{ZA_1} + \overrightarrow{A_1B}) + m_3\cdot(\overrightarrow{ZA_1} + \overrightarrow{A_1C}) = (m_2+m_3)\cdot\overrightarrow{ZA_1} + m_1\cdot\overrightarrow{ZA} + \underbrace{m_2\cdot\overrightarrow{A_1B} + m_3\cdot\overrightarrow{A_1C}}_{\vec{0}} = (m_2+m_3)\cdot\overrightarrow{ZA_1} + m_1\cdot\overrightarrow{ZA}$.

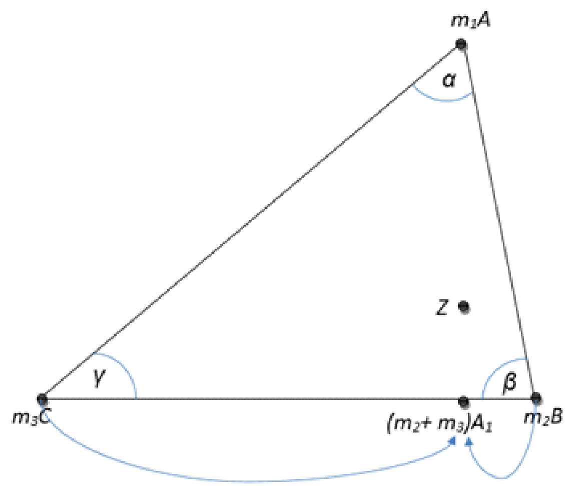

Fig. 11

Producing the system $\{(m_2+m_3)A_1, m_1A\}$ with Z as the center of mass, thus $\frac{|AZ|}{|ZA_1|} = \frac{m_2+m_3}{m_1}$, i.e., $\frac{|AZ|}{|ZA_1|} = \frac{\tan\beta+\tan\gamma}{\tan\alpha}$ (Fig. 12).

Resulting in $(m_2+m_3)\cdot\overrightarrow{ZA_1} + m_1\cdot\overrightarrow{ZA} = \vec{0}$. By Theorem 1 we have $Z \in AA_1$ and $\frac{|AZ|}{|ZA_1|} = \frac{m_2+m_3}{m_1}$, implying that $\frac{|AZ|}{|ZA_1|} = \frac{\tan\beta+\tan\gamma}{\tan\alpha}$.

From this proof we have $Z \in AA_1$. In a similar way we can prove that $Z \in BB_1$ and $Z \in CC_1$, thus Z is the intersection point of all altitudes. This means that $Z = O$ (as illustrated in Fig. 9), and

$$\frac{|AO|}{|OA_1|} = \frac{\tan\beta+\tan\gamma}{\tan\alpha}, \quad \frac{|BO|}{|OB_1|} = \frac{\tan\alpha+\tan\gamma}{\tan\beta}, \quad \frac{|CO|}{|OC_1|} = \frac{\tan\alpha+\tan\beta}{\tan\gamma}. \quad \blacksquare$$

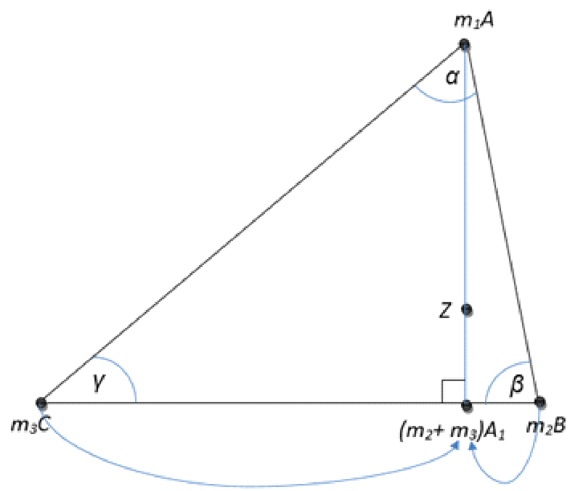

Fig. 12

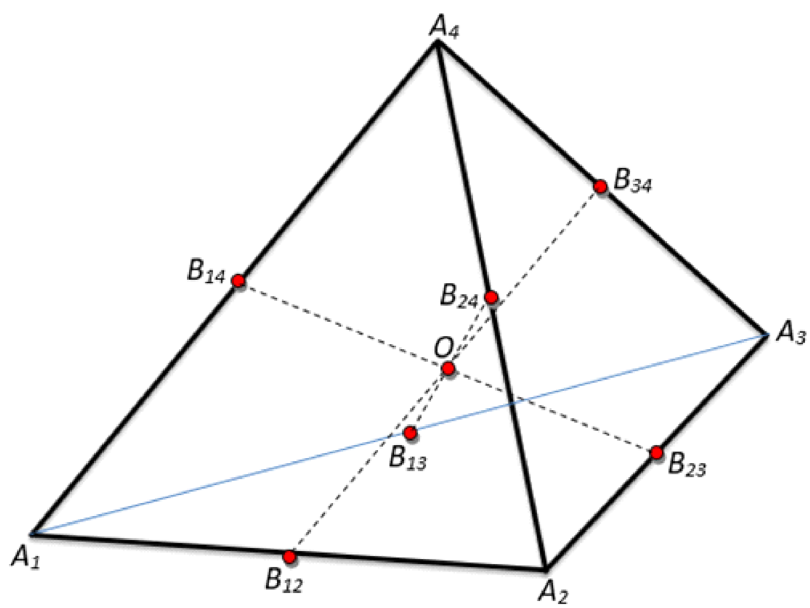

Fig. 13

4.3. Problem 3. Given a triangle-based pyramid $A_1A_2A_3A_4$, we denote by B_{ij} the middle-point of edge A_iA_j, for every $1 \leqslant i < j \leqslant 4$, as the diagram on Fig. 13 shows. Then, all line segments $B_{12}B_{34}$, $B_{13}B_{24}$, $B_{14}B_{23}$ intersect in point O that satisfies

$$\frac{|B_{12}O|}{|OB_{34}|} = \frac{|B_{13}O|}{|OB_{24}|} = \frac{|B_{14}O|}{|OB_{23}|} = 1.$$

Proof.

Intuition from physics

We endow points A_1, A_2, A_3, A_4, with the masses $1, 1, 1, 1$. We now have a system of four mass points $\{1A_1, 1A_2, 1A_3, 1A_4\}$ with the centre of mass which is Z (Fig. 14).

Formal mathematical solution

According to Theorem 5, there is a unique Z such that $\overrightarrow{ZA_1}+\overrightarrow{ZA_2}+\overrightarrow{ZA_3}+\overrightarrow{ZA_4}=\vec{0}$.

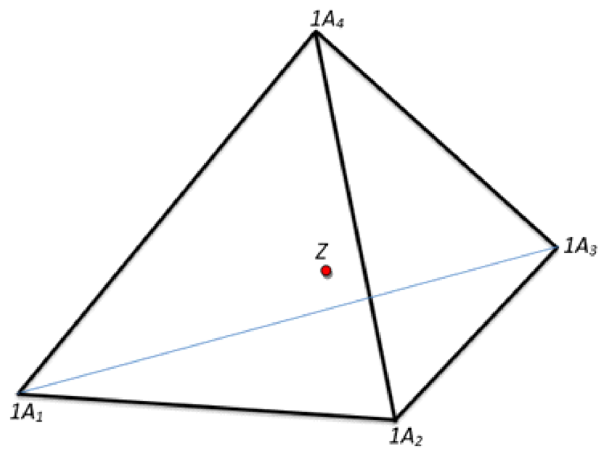

Fig. 14

For every $1 \leqslant i < j \leqslant 4$, the centre of mass of the system $\{1A_i, 1A_j\}$ is B_{ij}. We move masses from points A_1 and A_2 to point B_{12}, which becomes a mass point $2B_{12}$, and from points A_3 and A_4 to point B_{34}, which becomes a mass point $2B_{34}$ (Fig. 15).

For every $1 \leqslant i < j \leqslant 4$, $B_{ij} \in A_iA_j$ and $|B_{ij}A_i| = |B_{ij}A_j|$, implying that $\overrightarrow{B_{ij}A_i} = \overrightarrow{A_jB_{ij}}$, i.e., $\overrightarrow{B_{ij}A_i} + \overrightarrow{B_{ij}A_j} = \vec{0}$. It follows that $\vec{0} = (\overrightarrow{ZA_1} + \overrightarrow{ZA_2}) + (\overrightarrow{ZA_3} + \overrightarrow{ZA_4}) = ((\overrightarrow{ZB_{12}} + \overrightarrow{B_{12}A_1}) + (\overrightarrow{ZB_{12}} + \overrightarrow{B_{12}A_2})) + ((\overrightarrow{ZB_{34}} + \overrightarrow{B_{34}A_3}) + (\overrightarrow{ZB_{34}} + \overrightarrow{B_{34}A_4})) = 2\overrightarrow{ZB_{12}} + 2\overrightarrow{ZB_{34}} + \underbrace{\overrightarrow{B_{12}A_1} + \overrightarrow{B_{12}A_2}}_{\vec{0}} + \underbrace{\overrightarrow{B_{34}A_3} + \overrightarrow{B_{34}A_4}}_{\vec{0}} = 2\overrightarrow{ZB_{12}} + 2\overrightarrow{ZB_{34}}$.

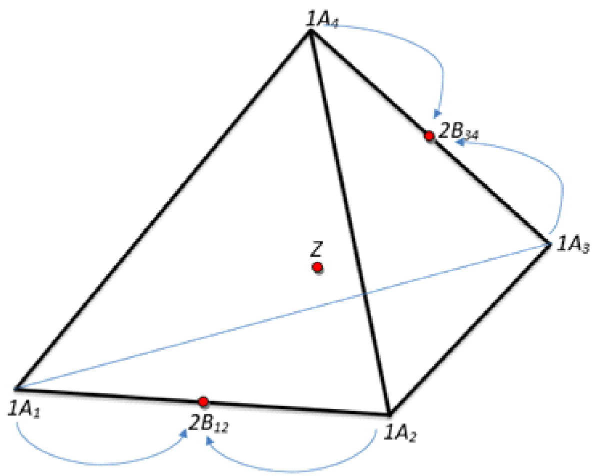

Fig. 15

Producing the system $\{2B_{12}, 2B_{34}\}$ with Z as the centre of mass, thus $\dfrac{|B_{12}Z|}{|ZB_{34}|} = \dfrac{2}{2} = 1$ (Fig. 16).

Resulting in $2\overrightarrow{ZB_{12}} + 2\overrightarrow{ZB_{34}} = \vec{0}$. By Theorem 1 we have $Z \in B_{12}B_{34}$, and $\dfrac{|B_{12}Z}{|ZB_{34}|} = \dfrac{2}{2} = 1$.

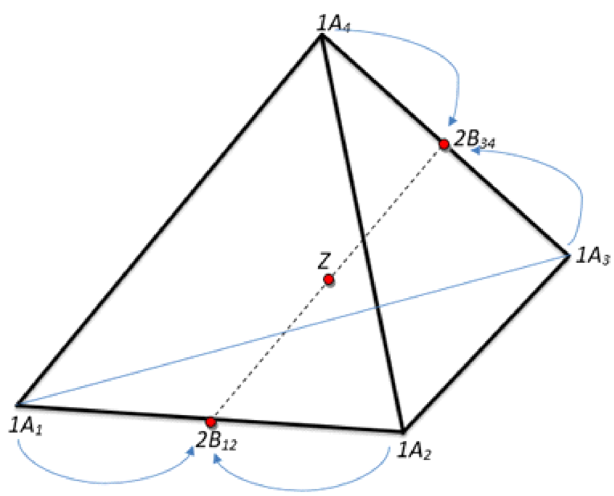

Fig. 16

From this proof we have $Z \in B_{12}B_{34}$. In a similar way we can prove that $Z \in B_{13}B_{24}$ and $Z \in B_{14}B_{23}$, thus Z is the intersection point of all line segments

$B_{12}B_{34}$, $B_{13}B_{24}$, $B_{14}B_{23}$ (as illustrated in Fig. 13). This means that $Z = O$, and

$$\frac{|B_{12}O|}{|OB_{34}|} = \frac{|B_{13}O|}{|OB_{24}|} = \frac{|B_{14}O|}{|OB_{23}|} = 1. \quad \blacksquare$$

4.4. The following generalized centroid problem is easy to solve using the centre of mass method, making it a good exercise. We only provide the problem and its answer, leaving the solution to the student.

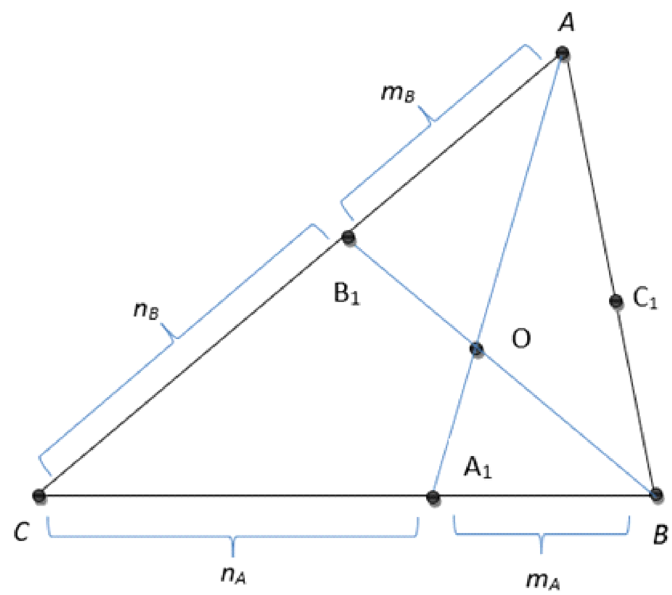

Fig. 17

Problem 4. Given a triangle ABC, we denote by A_1 the point on BC such that $\frac{|BA_1|}{|CA_1|} = \frac{m_A}{n_A}$, by B_1 the point on AC such that $\frac{|AB_1|}{|CB_1|} = \frac{m_B}{n_B}$ and by O the point where AA_1 and BB_1 intersect, as the diagram on Fig. 17 shows. Then

$$\frac{|AO|}{|A_1O|} = \frac{m_B \cdot (m_A + n_A)}{n_B \cdot m_A}.$$

Remark: This is a generalization of the centroid problem, which states that the triangle's medians divide each other in a 1 : 2 proportion.

5. Conclusions

In modern school geometry, the axiomatic Euclidean method is still in use. This method was modernized over the years by many mathematicians, e.g., by David Hilbert [4]. With this approach, teachers do not pay enough attention to the study of vectors and the use of vectors for solving geometric problems. Hermann Weyl proposed the "vector" axiomatics of geometry in 1917. One of the most zealous supporters of teaching geometry was Jean Dieudonne [2]. Most pupils are familiar with the concept of vector, mainly from their lessons in physics. However, physics teachers do not usually familiarize their pupils with vector methods for solving geometric problems. We think that the study of vectors for solving geometric problems should be given much more time, because it will help pupils look at problems from a different angle and makes some problems solution much easier than regular geometry solution.

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A.M.: Shamoon College of Engineering, 84 Jabotinski St., Ashdod 77245, Israel.

E-mail: `meiravt@sce.ac.il`

D.M.: Shamoon College of Engineering, Beer-Sheva 84100, Israel.

E-mail: `Dagan@sce.ac.il`

L.S.: Shamoon College of Engineering, 84 Jabotinski St., Ashdod 77245, Israel.

E-mail: `sagile@sce.ac.il`

M.A.: Shamoon College of Engineering, 84 Jabotinski St., Ashdod 77245, Israel.

E-mail: `artourm@sce.ac.il`