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BIMATRIX GAMES HAVE A QUASI-STRICT EQUILIBRIUM: AN ALTERNATIVE PROOF THROUGH A HEURISTIC APPROACH

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Abstract. We present an alternative proof for the existence of at least one quasi-strict equilibrium in every bimatrix game. While Norde [Bimatrix games have quasi-strict equilibria. Math Prog, 85, 35–49] uses Brouwer's fixed point theorem, we employ Kakutani's fixed point theorem for multivalued maps, and make our proof shorter, thus teachable in a couple of lecture talks. Besides our approach admits of natural economic interpretations of some technicalities used in the proof. We also explain how we get to our method of proof. In addition, it is remarked that it is possible to adopt a field more general than that of real numbers.

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1. Introduction

A quasi-strict equilibrium in noncooperative games is an equilibrium where all the best pure strategies of each player is actually given a positive probability. As such, quasi-strict equilibria, in a special form, originally appeared in zero-sum games as those associated with essential strategies in Shapley, Karlin and Bohnenblust [20] and Bohnenblust, Karlin and Shapley [4]. These results were summarized in Karlin [16, Theorem 3.1.1; pp. 64–67]. On the other hand Tucker [23, p. 11] noted that his Theorem 3 on linear inequalities would be powerful enough to derive the results of Bohnenblust, Karlin and Shapley as well as the minimax theorem of von Neumann and Morgenstern [25, p. 153 (1953)].

Harsany [11] defined quasi-strong equilibrium for general noncooperative n-person games. Later this has changed its name to quasi-strict equilibrium. Damme then gave an example of three-player game in which no quasi-strict equilibrium exists [7, p. 61]. Thus the problem was boiled down to whether every bimatrix game has at least one quasi-equilibrium or not. In view of the participation of only two players, it is curious that many years elapsed before Norde [18] finally succeeded in proving the existence of quasi-strict equilibrium for bimatrix games. In between there were contributions offering partial answers, notably by Jansen [14] and Borm [6]. Norde's paper was first publishes as a working paper in 1994, thus it took five years for the article to be published for a wider access. It seems that the referees of Norde's paper were not easily convinced about its publishability because his proof is full of technicalities and indeed tough with 7 lemmata needed to reach his theorem. Teachers of game theory have surely found it difficult to reproduce Norde's proof in their lecture course. And yet, since 1999, no simpler alternative proof has been supplied in any web sites or journals. Our task in this article is to provide a proof which is teachable to math students who gather to study game theory. In so doing, we describe how to obtain a leading light in groping for a way forward in our proof; that is, reviewing the proofs for the zero-sum case, which had been done with no recourse to fixed point theorems, but with the help of theorems on linear inequalities. We have realized it is better to use both Kakutani's fixed point theorem and Stiemke's theorem on linear inequalities. In addition, some crucial steps in our proof are explained in terms of economics, interpreting given matrices as consisting of production processes in von Neumann's model of economic growth.

Our paper runs as follows. Section 2 explains our notation and preliminary results, and Section 3 is devoted to a heuristic story about how to get to an alternative proof. The main proposition is given in Section 4, while Section 5 contains an economic interpretation of some points in our proof. Section 6 gives a numerical example, and the final section closes the paper with concluding remarks.

2. Notation and preliminary results

We first explain our notation, which more or less follows that of Jansen [14] and of Norde [18]. Let m and n be natural numbers, N_m the set of natural numbers $\{1, 2, \ldots, m\}$, $\mathbb{R}^m$ be the m-dimensional real Euclidean space, and $\mathbb{R}^m_+$ be the nonnegative orthant of $\mathbb{R}^m$. The i-th unit vector of $\mathbb{R}^m$ is denoted by e_i, and similarly the one in $\mathbb{R}^n$. The symbol $\mathbf{e}$ stands for the vector whose elements are all unity either in $\mathbb{R}^m$ or in $\mathbb{R}^n$. The inner-product of two vectors, v and w in $\mathbb{R}^m$, is denoted by $v' \cdot w$, with a prime to a vector indicating transposition, or simply by vw, and the premultiplication of a vector $p \in \mathbb{R}^m$ with an $m \times n$ matrix A is written as $p' \cdot A$. The postmultiplication of a vector $q \in \mathbb{R}^n$ with the matrix A is written as $A \cdot q$. To make expressions shorter, the transposition $(')$ and/or the inner product symbol $(\cdot)$ are normally left out as pA or Aq, except when the contrast between column and row vectors is desirable, or ample space is available in a line. The symbol S^m stands for the $(m-1)$-simplex, i.e., $S^m \equiv \{x \mid x \in \mathbb{R}^m_+, \sum_{j=1}^m x_j = 1\}$. In vector comparison, the inequality $x \geq y$ means that the left-hand-side is not less than the right-hand-side in each elementwise comparison; $x > y$ means that $x \geq y$ and $x \neq y$; $x \gg y$ signifies that a strict inequality holds in each elementwise comparison.

Bimatrix games and equilibrium points are defined next. Let A and B be two $m \times n$ matrices. The two-person game in normal form (S^m, S^n, E_A, E_B) with

$$E_A(p, q) \equiv pAq \quad \text{and} \quad E_B(p, q) \equiv pBq$$

for each $p \in S^m$, $q \in S^n$, is called the $m \times n$ bimatrix game corresponding to the ordered pair of matrices A and B and this game is denoted by (A, B). The matrices

A and B are called the payoff matrices of player I and player II, respectively. Similarly, $E_A(p,q)$ and $E_B(p,q)$ are called the payoff functions. The class of all $m \times n$ bimatrix games is denoted by $\mathbb{M}^2_{m\times n}$. A bimatrix game of the form $(A,-A)$ is also called a matrix game and is also denoted simply by A.

A pair of vectors $(p^*,q^*) \in D \equiv S^m \times S^n$ is called an equilibrium point of the bimatrix game (A,B) if

$$p^* \cdot A \cdot q^* = \max_{p\in S^m} p \cdot A \cdot q^* \quad \text{and} \quad p^* \cdot B \cdot q^* = \max_{q\in S^n} p^* \cdot B \cdot q.$$

The set of all equilibrium points of (A,B), which is nonempty by a theorem of Nash [17], will be denoted by $EQ(A,B)$.

We also define the *carriers* of a vector and the sets of pure *best* strategies as:

$$\begin{aligned}
\mathbf{C}(p) &\equiv \{i \mid p_i > 0\};\\
\mathbf{C}(q) &\equiv \{j \mid q_j > 0\}, \text{ and}\\
\mathbf{B}_1(q) &\equiv \{i \mid e_iAq = \max_{k\in N_m} e_kAq\};\\
\mathbf{B}_2(p) &\equiv \{j \mid pBe_j = \max_{l\in N_n} pBe_l\}.
\end{aligned}$$

Note that two sets $\mathbf{B}_1(q)$ and $\mathbf{B}_2(p)$ are never empty. It is clear that $(p,q) \in EQ(A,B)$ if and only if $\mathbf{C}(p) \subset \mathbf{B}_1(q)$ and $\mathbf{C}(q) \subset \mathbf{B}_2(p)$. A quasi-strict equilibrium is obtained when $\mathbf{C}(p) = \mathbf{B}_1(q)$ and $\mathbf{C}(q) = \mathbf{B}_2(p)$.

We first state what we call Tucker's key theorem [23, Theorem 3, p. 11], which is slightly more general than the key theorem described in Good [9, p. 6].

Theorem 1. (Tucker's key theorem) *For a given real $m \times n$ matrix A, the two inequality problems, (i) $Aq \geq 0$ for $q \in S^n$, and (ii) $p' \cdot A \leq 0$ for $p \in S^m$, possess a pair of solutions $q^* \in S^n$ and $p^* \in S^m$ such that*

$$Aq + p^* \gg 0 \quad \textit{and} \quad q^{*\prime} - p^{*\prime} \cdot A \gg 0.$$

For the proof of theorem, the reader is referred to Tucker [23, Theorem 3, p. 11] or Good [9, Theorem of Alternative (Sharpened), p. 11]. Fujimoto [8] contains a simple proof of Tucker's theorem using a minimization problem with a constraint. (Tucker's method of proof can handle the field of rationals, while Good's cannot. Thus Good made an apology that he assumed the field of reals 'to gain much in clarity and comprehension' [9, p. 7].)

We need here an extension of Stiemke's theorem [21], which can be derived from Tucker's key theorem with no trouble.

Theorem 2. (a generalization of Stiemke's theorem) *For a given real $m \times n$ matrix A, the inequality system $Aq \geq 0$ has a strictly positive solution $q \gg 0$ if and only if the inequality system $p' \cdot A < 0$ has no nonnegative solution $p \geq 0$.*

Our Theorem 2 is actually Corollary 3A in Tucker [23, p. 11], where the reader can find a proof. The above extension of Stiemke's theorem [21] is of a similar nature to an extension of Gordan's theorem [10] due to Ville [24], which was used by von Neumann-Morgenstern [25] to prove their minimax theorem. (A historical sketch about this, together with an English translation of Ville's paper, is given in Ben-El-Mechaiekh-Dimand [5].)

Now we prove the existence of a saddle point, which is equivalent to the minimax theorem for zero-sum noncooperative games with a little more property concerning the full use of eligible strategies.

THEOREM 3. (the minimax theorem in a sharper form) *For a given real $m \times n$ matrix A, there exists a pair of $q^* \in S^n$ and $p^* \in S^m$ such that*

$$p^{*\prime} \cdot A \cdot q \le p^{*\prime} \cdot A \cdot q^* \le p' \cdot A \cdot q^* \text{ for any } q \in S^n \text{ and } p \in S^m, \text{ and}$$
$$p_i^* > 0 \text{ when } (A \cdot q^*)_i = v \text{ for } i = 1, \dots, m;$$
$$q_j^* > 0 \text{ when } (p^{*\prime} \cdot A)_j = v \text{ for } j = 1, \dots, n,$$

where $v \equiv p^{*\prime} \cdot A \cdot q^*$, i.e., the value of the game.

Proof. By the minimax theorem, we know the existence of a pair, $q^* \in S^n$ and $p^* \in S^m$ which has the saddle point property described in Theorem 3. Suppose the value of game is $v = 0$, then there can be no $q \in S^n$ such that $Aq \gg 0$, nor $p \in S^m$ such that $-p' \cdot A \gg 0$. In this case, our Theorem 1, i.e., Tucker's key theorem guarantees the existence of a solution pair as required in Theorem 3. Next, suppose $v \neq 0$. We can subtract this value v from each entry of A, that is, $a_{ij} - v$, and create a new matrix game whose value of game is zero. Then we can use what has just been proved, and show the existence of a required pair for this transformed game with $v = 0$. It is evident that this pair can also work as a solution for the original game. ∎

Theorem 3 can be found in few textbooks. One exception is Owen's book [19, p. 21], in which Theorem II.4.4 asserts that either player II has an optimal strategy p^* with $p_m^* > 0$, or player I has an optimal strategy q^* with $(A \cdot q^*)_m > v$. Though this result is concerned with only one particular index m for p, we can combine these strategies for $i = 1$ to m, and normalize the final vector, obtaining our result here for p^*. The property of q^* can be ascertained in a similar way. While Theorem 3 has an important meaning in terms of economics as is explained below in Section 5, it has been somewhat in oblivion among math teachers.

3. Solitaire matrix games or unimatrix games: a heuristic story

Our heuristic story starts with a *solitaire* matrix game, or a *unimatrix* game. Let A be an $m \times n$ real matrix, and there is a single player who tries to find out two vectors $p \in S^m$ and $q \in S^n$ sequentially to win. The rule is like this. First the player chooses a $p \in S^m$ to obtain a vector pA, and recognizes the set of entries which attain the *minimum* value, i.e.,

$$\mathbf{Min}(p) \equiv \{j \mid pAe_j = \min_{l \in N_n} pAe_l\}.$$

Then in the next step, the player is allowed to choose $q \in S^n$ whose carrier is included in $\mathbf{Min}(p)$. Finally the player gets

$$\mathbf{B}(q) \equiv \{i \mid e_i Aq = \max_{k \in N_m} e_k Aq\}.$$

If $\mathbf{C}(p) = \mathbf{B}(q)$, i..e., a sort of quasi-strict equilibrium pair of choices, the player is declared to have won. This is a simple solitaire game, but actually is a disguised form of a minimax game between two players with one additional element of *quasi-strictness*. More concretely, suppose that $m = 3$, $n = 2$, and

$$A \equiv \begin{pmatrix} -4 & 2 \\ 2 & -1 \\ -1 & -1 \end{pmatrix}.$$

When the player chooses at the beginning $p = (1, 2, 0)'$, $p' \cdot A = (0, 0)$, thus both columns are eligible. When the player selects $q = (1, 2)'$, then $A \cdot q = (0, 0, -3)'$. Hence $\mathbf{C}(p) = \mathbf{B}(q) = \{1, 2\}$, making the player the winner. The reader notices that winning choices are in fact those solutions described in Tucker's theorem on inequalities.

Hence, the above solitaire game is mathematically simple, and can be treated within college algebra, with its central piece being Tucker's theorem. Now, we change the rules of the above game in only one place as

$$\mathbf{Max}(p) \equiv \{j \mid pAe_j = \max_{l \in N_n} pAe_l\}.$$

That is, in the second step, the player is allowed to choose $q \in S^n$ whose carrier is included in $\mathbf{Max}(p)$ in place of $\mathbf{Min}(p)$. In the case of above numerical example, we can easily find two pairs of solutions, either $p = (1, 0, 0)'$ and $q = (0, 1)$ with $\mathbf{C}(p) = \mathbf{B}(q) = \{1\}$, or $p = (0, 1, 0)'$ and $q = (1, 0)$ with $\mathbf{C}(p) = \mathbf{B}(q) = \{2\}$. The problem is that we cannot prove the existence of a winning strategy with the help of college algebra; we have to depend on Norde [18]. We can employ Norde's proof because the latter solitaire game can be viewed as a special bimatrix game in which two payoff matrices are the same A, while the former game as a bimatrix game where two matrices are A and $-A$.

Now we proceed to explain how to discover an easier method to prove the existence of quasi-strict equilibrium for bimatrix games. We have seen that the existence can be shown in a simple manner for bimatrix games when two matrices are A and $-A$, i.e., a zero-sum game, thanks to Tucker's theorem and its derivative, a sharper form of the well known minimax theorem. In an equilibrium for a zero-sum game, there exist *two* constraints on the choice of the strategies p and q through A:

$$\begin{aligned} \mathbf{C}(p) \subset \mathbf{B}_1(q) &\equiv \{i \mid e_i Aq = \max_{k \in N_m} e_k Aq\}; \\ \mathbf{C}(q) \subset \mathbf{B}_2(p) &\equiv \{j \mid pBe_j = \max_{l \in N_n} p(-A)e_l\} \\ &= \{j \mid pBe_j = \min_{l \in N_n} pAe_l\} \end{aligned}$$

Suppose that the value of this game is zero, then these constraints entail the requirements at an equilibrium

$$Aq^* \geq 0;$$
$$p^* A \leq 0.$$

This reminds us of the inequalities appearing in Tucker's theorem. Thus, it is possible to apply Theorem 3 above, a sharper form of the minimax theorem. For our problem, i.e., the existence of a quasi-strict equilibrium, we notice that we had better modify a mapping of choosing strategies from $D \equiv S^m \times S^n$ into itself so that we may have more constraints on the choice of strategies and Theorem 2 can be employed to guarantee the required conditions $\mathbf{C}(p) = \mathbf{B}_1(q)$ and $\mathbf{C}(q) = \mathbf{B}_2(p)$, and a fixed point theorem can still be applied after the modification.

4. Our proof of the existence of a quasi-strict equilibrium

We state our proposition.

Proposition. *For a bimatrix game $(A, B) \in \mathbb{M}^2_{m\times n}$, there exists at least one quasi-strict equilibrium.*

Proof. For a pair of vectors $(p, q) \in D \equiv S^m \times S^n$, we define two more symbols,

$$\mathbf{Z}(p; q) \equiv \{i \mid p_i = 0\} \cap \mathbf{B}_1(q); \quad \mathbf{Z}(q; p) \equiv \{j \mid q_j = 0\} \cap \mathbf{B}_2(p).$$

All we have to show is that there exists a Nash equilibrium $(p^*, q^*) \in D$ in which the sets $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$ are empty. As a routine procedure, we consider a multivalued mapping f from D into itself as follows: for $(p^\circ, q^\circ) \in D$,

$$f(p^\circ, q^\circ) \equiv (\{p \in S^m \mid pAq^\circ = \max_{u\in S^m} uAq^\circ\}, \quad \{q \in S^n \mid p^\circ Bq = \max_{v\in S^n} p^\circ Bv\}).$$

By Berge's maximum theorem [3, p. 116], [22, p. 235], this map f is upper semicontinuous, and the image set $f(p^\circ, q^\circ)$ is convex. Therefore, Kakutani's fixed point theorem [15] assures us of the existence of Nash equilibrium $EQ(A, B)$.

Althought there may be infinitely many Nash equilibria, we have only a finite number of types $\mathbf{C}(p^*)$, $\mathbf{C}(q^*)$, $\mathbf{B}_1(q^*)$, and $\mathbf{B}_2(p^*)$ for $(p^*, q^*) \in EQ(A, B)$. Further we define symbols

$$m' \equiv \#(\mathbf{B}_1(q^*)),\ n' \equiv \#(\mathbf{B}_2(p^*)),\ u^* \equiv p^* Aq^*, \text{ and } v^* \equiv p^* Bq^*,$$

where $\#()$ signifies the number of elements in a given set. Now we take up a particular pair $\mathbf{B}_1(q^*)$ and $\mathbf{B}_2(p^*)$, corresponding to a Nash equilibrium (p^*, q^*), together with two matrices $A(p^*, q^*)$ and $B(p^*, q^*)$, which are composed of the rows of A and B in $\mathbf{B}_1(q^*)$ and the columns in $\mathbf{B}_2(p^*)$, respectively. Further we form the new matrices as

$$A^\circ(p^*, q^*) \equiv A(p^*, q^*) - u^* E \text{ and } B^\circ(p^*, q^*) \equiv B(p^*, q^*) - v^* E,$$

where E is the $m' \times n'$ matrix whose elements are all unity. That is, we subtract u^* from each entry of the matrix $A(p^*, q^*)$, and v^* from $B(p^*, q^*)$.

Now we examine whether there exists a $\overline{q} \in \mathbb{R}_+^{2n'}$ such that

$$(1) \qquad (A^\circ(p^*, q^*), B^\circ(p^*, q^*))\, \overline{q} < 0.$$

Note that the dimension of $\overline{q}$ is $2n'$, and the two matrices, $A^\circ(p^*, q^*)$ and $B^\circ(p^*, q^*)$, are juxtaposed horizontally to create an $m' \times 2n'$ matrix. It is evident that an elementwise strict inequality in (1) can take place only in an index which is in $\mathbf{Z}(p^*; q^*)$. Therefore when $\mathbf{Z}(p^*; q^*)$ is empty, there can be no $\overline{q}$ which satisfies the inequality (1). Suppose there is only one index $i \in \mathbf{Z}(p^*; q^*)$, i.e., $(A^\circ(p^*, q^*), B^\circ(p^*, q^*)) \cdot \overline{q})_i < u^*$. In this case, we modify the map f globally by reducing the codomain D to $D' \equiv \{S^m \cap \{p \in \mathbb{R}^m \mid p_i \geq \varepsilon\}\} \times S^n$, where $(1/m) > \varepsilon > 0$. As the codomain shrinks, we modify the multivalued map f as follows.

Let a point in an image of $f(p^\circ, q^\circ)$ be (p', q'). If $(p', q') \in D'$, then no change is required; otherwise we set $p'_i = \varepsilon$ with the other entries of p decreased *proportionately* so that the modified vector p' can stay in D'. When there are two or more strict inequalities involved, we modify f in a similar manner, i.e., those entries are unchanged or set at ε depending upon whether they are not less than or less than ε, and the remaining ones decreased proportionately. Next we do the same examination on the existence of a $\overline{p} \in S^{2m'}$ such that

$$(2) \qquad \overline{p}' \cdot \begin{pmatrix} A^\circ(p^*, q^*) \\ B^\circ(p^*, q^*) \end{pmatrix} < 0.$$

This time, the dimension of $\overline{p}$ is $2m'$, and the two matrices, $A^\circ(p^*, q^*)$ and $B^\circ(p^*, q^*)$, are juxtaposed vertically to create an $2m' \times n'$ matrix. If there is one $\overline{p}$ with the maximum number of inequalities in (2), we reduce, this time, the codomain on the S^n side in a similar way as is explained for $\overline{q}$, i.e., when the j-th elementwise comparison shows a strict inequality in (2), we set $q_j \geq \delta$, where $(1/n) > \delta > 0$, thus reducing the codomain still further. It should be noted that the reduction of the codomain might produce a new Nash equilibrium; in this case, we continue to conduct the same possible modification of the map f. This operation ends within a finite number of trials, because there exists only a finite number of types $\mathbf{C}(p^*)$, $\mathbf{C}(q^*)$, $\mathbf{B}_1(q^*)$, and $\mathbf{B}_2(p^*)$ for a whole set of possible equilibria, either original or created ones. In plain English, our modification of the map f is to remove those Nash equilibria which are not quasi-strict from the set of fixed point of the modified f. This can be observed from what follows in this proof, and a numerical example in Section 6.

Since the above modification of f is conducted globally, i.e., not piece by piece, or neighbourhood by neighbourhood, the map after modification is still upper semicontinuous, and image sets always remain convex thanks to linearity involved; thus we can use Kakutani's fixed point theorem to secure the existence of at least one fixed point (p^*, q^*). If $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$ are both empty, this pair (p^*, q^*) gives a quasi-strict equilibrium. Suppose that neither $\mathbf{Z}(p^*; q^*)$ nor $\mathbf{Z}(q^*; p^*)$ is empty. Because of our modification of the map f, there exists no $\overline{q} \in S^{2n'}$ nor $\overline{p} \in S^{2m'}$ for which the above inequalities (1) and (2) hold, respectively. By Theorem 2 in Section 2, there exist $p^{\dagger} \gg 0$ and $q^{\dagger} \gg 0$ such that $p^{\dagger} \in S^{m'}$ and $q^{\dagger} \in S^{n'}$,

$$p^{\dagger\prime} \cdot (A^{\circ}(p^*, q^*), B^{\circ}(p^*, q^*)) \geq 0 \text{ and}$$
$$\begin{pmatrix} A^{\circ}(p^*, q^*) \\ B^{\circ}(p^*, q^*) \end{pmatrix} \cdot q^{\dagger} \leq 0.$$

Since $p^{\dagger} \gg 0$ and $q^{\dagger} \gg 0$, the above inequality should hold as equalities, i.e.,

$$p^{\dagger\prime} \cdot (A^{\circ}(p^*, q^*), B^{\circ}(p^*, q^*)) = 0 \text{ and}$$
$$\begin{pmatrix} A^{\circ}(p^*, q^*) \\ B^{\circ}(p^*, q^*) \end{pmatrix} \cdot q^{\dagger} = 0.$$

Now we create a vector $p^{\dagger *} \in S^m$ from $p^{\dagger}$ by setting zero to its entries outside of $\mathbf{B}_1(q^*)$, and create a vector $q^{\dagger *} \in S^m$ from $q^{\dagger}$ in the same manner by setting zero to its entries outside of $\mathbf{B}_2(p^*)$. It is easy then to recognize that a pair of vectors

$$((p^* + \eta \cdot p^{\dagger *}) / \left\| p^* + \eta \cdot p^{\dagger *} \right\|, \quad (q^* + \eta \cdot q^{\dagger *}) / \left\| q^* + \eta \cdot q^{\dagger *} \right\|),$$

where η is a positive scalar so small that no disturbance is made to the best choice set $\mathbf{B}_1(q^*)$ and $\mathbf{B}_2(p^*)$ and $\|\cdot\|$ stands for the sum norm, is indeed a quasi-strict equilibrium.

The other cases in which either of the two, $\mathbf{Z}(p^*; q^*)$ and $\mathbf{Z}(q^*; p^*)$, is empty, can similarly be dealt with. ∎

5. Economic interpretation

We had better give a brief economic interpretation to our modification of the mapping used in the previous section so that the reader can firmly remember the point. A von Neumann growth model in von Neumann [26] is adopted here. The matrix A is understood as a set of production processes with its columns as production processes, while the rows representing goods and services (commodities for short, below); negative entries mean inputs, and positive ones outputs. Thus, there are m commodities, and n production processes. As player I chooses $p \in S^m$, player II tries to maximize pAq by selecting $q \in S^n$. When we regard p as a price vector, the product pA shows a row-vector of profits of available processes. The symbol $\mathbf{B}_2(p)$ stands for the set of processes which realize the maximum profits under a price vector p. In a symmetrical way, and not a dual way, when player II chooses $q \in S^n$, player I tries to maximize pAq by selecting $p \in S^m$. Let us regard q as an activity level vector, the product Aq shows a column-vector of net outputs of various commodities. Then the symbol $\mathbf{B}_1(q)$ is the set of commodities which enjoy the maximum net outputs under an activity level vector q. Now we can see the meaning of the condition (1), i.e., the existence of $\overline{q} \in S^{2n'}$ such that the inequality (1) holds for modifying the map f in the preceding section. The matrices $A(p^*, q^*)$ and $B(p^*, q^*)$ under a Nash equilibrium (p^*, q^*) is composed of those processes and commodities which realize the maximum by respective criteria. The inequality (1) tells us that there is an activity vector operating upon those optimal processes, and producing less than the optimal levels in one or more commodities, that is, supplies less than the maximum quantities possible. Our modification should designate those commodities in undersupply as having a positive price at $\varepsilon > 0$, and this in a global sense.

One important implication brought by the existence of a quasi-strict equilibrium in terms of economics is that this gives an assurance of equality among those who have a stake in one of the production processes with the maximum profits. All the eligible production processes with the maximum benefit can actually be employed with a positive probability. This sense of equality may be of vital importance to the players who participate in certain types of economic games.

The reader is referred to Howe [12] for a simple and elegant proof of the existence of growth equilibrium in a von Neumann model; his proof uses Tucker's key theorem.

6. A numerical example

It is helpful to include a numerical example borrowed from Norde [18, p. 37]. We intentionally use some symbols appearing in our proof above for easier comparisons. Let the given two matrices be:

$$A \equiv \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B \equiv \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

A pair of vectors, $p^* = (1/2, 1/2, 0)'$ and $q^* = (1/2, 1/2, 0)'$, are a Nash equilibrium, but not quasi-strict, because $\mathbf{C}(p^*) = \mathbf{C}(q^*) = \{1, 2\}$, while $\mathbf{B}_1(q^*) = \mathbf{B}_2(p^*) = \{1, 2, 3\}$, as one can easily calculate. We can also calculate to have $u^* = v^* = 1/2$. In this Nash equilibrium, we can find a vector $\overline{q} = (2, 1, 1, 3, 1, 0)'/8$, which will yields

$$\begin{aligned}&(A^\circ(p^*, q^*), B^\circ(p^*, q^*)) \cdot 8 \cdot \overline{q} \\ &= \begin{pmatrix} -3/2 & 3/2 & 1/2 & 1/2 & -3/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 & 3/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.\end{aligned}$$

Thus, p_1^* and p_3^* should be made positive. On the other hand, we can find a vector

$\overline{p} = (1, 3, 0, 0, 0, 0)/4$ such that

$$4 \cdot \overline{p}' \cdot \begin{pmatrix} A^\circ(p^*, q^*) \\ B^\circ(p^*, q^*) \end{pmatrix} = (\,1, \quad 3, \quad 0, \quad 0, \quad 0, \quad 0\,) \begin{pmatrix} -3/2 & 3/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -3/2 & -1/2 \\ -1/2 & 3/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}$$

$$= (\,0, \quad 0, \quad -1\,)\,.$$

Thus, q_3^* should be made positive. Actually we know, from Fig. 1 of Norde [18, p. 37], a quasi-strict equilibrium is given, e.g., as $p^{\dagger *} = (1/3, 0, 2/3)'$ and $q^{\dagger *} = (0, 0, 1)'$.

7. Concluding remarks

5.1. Use of fixed point theorems

Fixed point theorems may be too advanced for students in upper secondary schools and in junior levels of universities. Their meanings or contents are, however, easily grasped, and quite 'exciting' for many math oriented students. When they are successfully utilized in proving seemingly unrelated propositions, the sensation may get 'spine-tingling'. We math teachers often ask students to try to discover proofs different from those presented in class or in textbooks. In so doing, we may occasionally suggest the use of one of fixed point theorems. Through these exercises, students get to know when fixed point theorems are applicable, or why they cannot be employed in certain cases.

5.2. Tucker's theorem as a fixed point theorem

Tucker's theorem can be restated as a fixed point theorem. Let A be a given $m \times n$ real matrix, and we consider a multivalued mapping f from $D \equiv S^m \times S^n$ into itself defined as:

$$\begin{aligned} &f : (x^\circ, y^\circ) \in D \\ &\quad \to (\mathrm{relint}\{x \mid x'Ay^\circ = \max_{x \in S^m} \cdot xAy^\circ\}, \mathrm{relint}\{y \mid x^{\circ\prime}Ay = \min_{y \in S^n} \cdot x^{\circ\prime}Ay\}). \end{aligned}$$

Here, the symbol relint stands for the relative interior of the set which follows it. Tucker's theorem asserts that the above map f has a fixed point in D. Norde's theorem [18] on the existence of quasi-strict equilibrium for bimatrix games can also be rephrased as a fixed point theorem of this type.

5.3. Rational numbers

So far we have considered real numbers only. On reflection, however, we can find equilibrium strategies at the end by solving a system of simultaneous linear equations, whether it is just determined or underdetermined. Thus, thanks to Cramer's rule, it is clear that we can have equilibrium strategies consisting of rational numbers so long as given data are rational numbers. This remains true when we have a pair of solution vectors within a relative interior of an equilibrium set, though there are uncountable solutions around it made of irrational numbers. Gordan [10] already had in mind the solvability in a filed other than reals. Tucker [23, p. 5] mentioned the applicability of his method in any ordered field. Tucker's claim is in fact carried out in a more powerful way by Bartl [1, 2], which was expounded in a broader context by Jaćimović [13]. It is well known that the Fourier-Motzkin elimination works within rationals. In spite of these facts, it is interesting to note that rational bimatrix games can produce rational solutions though we may depend upon fixed point theorems to prove the existence of solutions.

Thus, when looked at as a fixed point theorem, we can deal with A with its all entries consisting of rational numbers, and the guaranteed existence of a fixed point is within the field of rational numbers. This feat is made possible, helped by the linearity brought in through the use of matrices and linear operations by them.

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