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SUBSTITUTIONS IN DIFFERENTIAL EQUATIONS AS A GEOMETRY OF THEIR SYMMETRIES

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Abstract. We enlighten a geometrical background for familiar substitutions in certain ordinary differential equations. We explain how the existence of a symmetry of a differential equation provides a change of coordinates (i.e., a substitution) in which the initial equation becomes in quadrature form.

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1. Introduction

The simplest differential equation is one of the form

$$\frac{dx}{dt} = h(t), \quad x(t_0) = x_0,$$

where $h : I \to \mathbb{R}$ is a continuous function defined on an interval $I \subseteq \mathbb{R}$. Its only solution is obtained by a direct integration:

$$x(t) = x_0 + \int_{t_0}^{t} h(\tau)\, d\tau.$$

The next very simple form is

$$\frac{dx}{dt} = h(x), \quad x(t_0) = x_0, \tag{1}$$

where $h(x_0) \neq 0$. It follows from the assumptions that $x(t)$ has the differentiable inverse in some neighbourhood of x_0, so we can transform (1) into

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{h(x)}, \quad t(x_0) = t_0,$$

which is again solvable by integrating.

We accept the following definition.

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Definition 1. An ordinary differential equation is *in quadrature form* if it is of the first order,

$$\frac{dx}{dt} = f(t,x) \tag{2}$$

and if the right-hand side, i.e. the function $f(t,x)$, depends only on t or on x.

It is convenient to transform any differential equation $\frac{dx}{dt} = f(t,x)$ to the one in quadrature form; in some examples this can be done by a suitable choice of new variables, i.e. by a substitution. In this note we will try to lighten the background of certain substitutions from a geometrical point of view, i.e. to explain how some substitutions naturally arise from a symmetry of a differential equation.

Let us recall some examples first.

Example 2. (Homogeneous equation) Consider the equation

$$\frac{dx}{dt} = g\left(\frac{x}{t}\right).$$

Using the substitution

$$s(t,x) = \log t, \quad y(t,x) = \frac{x}{t}$$

one transforms it to quadrature form. The substitution $s(t,x) = t$, $y(t,x) = \frac{x}{t}$ transforms it to the equations with separable variables:

$$\frac{dx}{dt} = a(x)b(t).$$

Example 3. (Generalized homogeneous equation) It is an equation of the form

$$\frac{dx}{dt} = \frac{x}{t}\, g\left(\frac{x^\alpha}{t^\beta}\right).$$

The substitution

$$s(t,x) = \log t, \quad y(t,x) = \frac{x^\alpha}{t^\beta}$$

transforms it to quadrature form and the substitution

$$s(t,x) = t, \quad y(t,x) = \frac{x}{t}$$

transforms it to the equations with separable variables.

Example 4. (Linear equation) The equation

$$\frac{dx}{dt} = p(t)x(t) + q(t) \tag{3}$$

can be transformed into the one in quadrature form using the substitution

$$s = t, \quad y = xe^{-\int p(t)\,dt}.$$

2. Substitutions

Let us first precise some terminology. Let $\mathcal{U}$ be a domain (a connected open subset of $\mathbb{R}^n$ or of a manifold M) and let $F = F(t,x)$ be a vector field defined on $\mathcal{U}$ which may also depend on $t \in I \subseteq \mathbb{R}$, meaning

$$F : I \times \mathcal{U} \to TM, \quad F(t,x) \in T_xM, \quad \text{for all} \quad x \in \mathcal{U},\ t \in I.$$

If $M = \mathbb{R}^n$ we have the identification $T_xM \cong \mathbb{R}^n$, so

$$F : I \times \mathcal{U} \to \mathbb{R}^n.$$

Consider the differential equation

$$\frac{dx}{dt} = F(t, x(t)). \tag{4}$$

We have the following notions:

- the domain $\mathcal{U}$ is called the *phase space* of the equation (4)
- the product $I \times \mathcal{U}$ is called the *extended phase space* of the equation (4)
- the image of any solution $x : I \to \mathcal{U}$ of the equation (4) is called the *phase curve* of the equation (4).

A substitution in the autonomous differential equation

$$\frac{dx}{dt} = F(x(t)) \tag{5}$$

on a domain $\mathcal{U}$ is a change of local coordinates, i.e. a diffeomorphism defined on $\mathcal{U}$ which hopefully simplifies the form of (5). More precisely, let

$$\phi : \mathcal{U} \to \mathcal{V}$$

be a diffeomorphism between domains $\mathcal{U}$ and $\mathcal{V}$. Let ϕ_*F denotes the push-forward of F by ϕ:

$$\phi_*F(y) := d\phi_{\phi^{-1}(y)}(F(\phi^{-1}(y)),$$

where $d\phi_{\phi^{-1}(y)}$ denotes the derivative of ϕ at the point $\phi^{-1}(y)$.

PROPOSITION 5. *The diffeomorphism ϕ maps the phase curves of the equation* (5) *to the phase curves of the equation*

$$\frac{dy}{dt} = \phi_*F(y(t)).$$

Proof. By differentiation we check that $y(t) := \phi(x(t))$ is a phase curve of a vector field ϕ_*F. ■

Let us now consider the non-autonomous case (4). The most important non-autonomous equation that we study in this note is the following simple equation:

$$\frac{dx}{dt} = f(t, x), \tag{6}$$

where f is a function of two variables defined at some domain in $\mathbb{R}^2$. It is often more elegant to study a non-autonomous case as an autonomous equation in the extended phase space. This means that we treat $(t,x) \in I \times \mathcal{U}$ as a new function of τ and instead of the equation (4) we consider the following autonomous equation in the extended phase space:

$$\begin{cases} \dfrac{dt}{d\tau} = g(t,x) \\ \dfrac{dx}{d\tau} = \dfrac{dx}{dt}\dfrac{dt}{d\tau} = F(t,x) \cdot g(t,x), \end{cases} \tag{7}$$

for a suitable choice of $g(t,x)$ (desirably of a constant sign, so that $\tau \mapsto t$ is a bijection).

The simplest example of making an autonomous equation out of the non-autonomous one is the choice $g(t,x) = 1$; now (4) can also be written as

$$\frac{d\tilde{x}}{d\tau} = \widetilde{F}(\tilde{x}), \quad \text{where} \quad \tilde{x} = (t,x) \in I \times \mathcal{U}, \quad \widetilde{F}(\tilde{x}) := (1, F(\tilde{x})).$$

The solutions of the equation (4) are the projections of the solutions of the equation (7) to the phase space $\mathcal{U}$.

The substitution is now a diffeomorphism

$$\phi : I \times \mathcal{U} \to J \times \mathcal{V},$$

where $\mathcal{U}, \mathcal{V} \subseteq M$ (or $\mathbb{R}^n$) and $I, J \subseteq \mathbb{R}$.

3. Symmetries

Definition 6. Let F be a vector field on a domain $\mathcal{U}$. We say that a diffeomorphism $\varphi : \mathcal{U} \to \mathcal{U}$ is *a symmetry* of the differential equation (5) if $\varphi_* F = F$. In the non-autonomous case (4) a symmetry is a diffeomorphism of the extended phase space $I \times \mathcal{U}$ which is a symmetry for the autonomous equation (7).

Remark 7. In the non-autonomous case (4), the symmetry $\varphi : I \times \mathcal{U} \to I \times \mathcal{U}$ is a notion that depends on the choice of the function g in (7).

Proposition 8. *The symmetry φ of the autonomous system* (5) *maps the phase curves of given system to the phase curves of the same system. The same is true for the non-autonomous system* (4).

Proof. The assertion for (5) follows directly from Proposition 5.

Regarding the system (4), from the autonomous case we conclude that $\varphi : I \times \mathcal{U} \to I \times \mathcal{U}$ maps the phase curves of the extended system (7) to themselves. The phase curves of the initial system are solutions of (4). Since

$$\varphi_*(g(t,x), F(t,x) \cdot g(t,x)) = (g(t,x), F(t,x) \cdot g(t,x)),$$

we see that the projections of the phase curves are solutions to

$$\frac{dx}{d\tau} = F(t,x) \cdot g(t,x).$$

But this is the reparametrized curve $\frac{dx}{dt} = F(t,x)$, since $\frac{dt}{d\tau} = g(t,x)$. ■

Definition 9. Let φ_ε be a symmetry of the equation (5) for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. We say that φ_ϵ is a *one-parameter local Lie group* if the following holds:

- φ_0 is the identity map
- $\varphi_s \circ \varphi_t = \varphi_{s+t}$
- for every $x \in \mathcal{U}$, the map $\varepsilon \mapsto \varphi_\varepsilon(x)$ is smooth.

If φ_ε is defined for all $\varepsilon \in \mathbb{R}$ so that the above conditions holds, we have the action of the Lie group $(\mathbb{R}, +)$ on $\mathcal{U}$. Recall that the action of the group $(G, \circ)$ on a set $\mathcal{U}$ is a mapping

$$G \times \mathcal{U} \to \mathcal{U}, \quad (g, x) \mapsto g \cdot x$$

such that

$$e \cdot x = x, \quad f \cdot (g \cdot x) = (f \circ g) \cdot x$$

(where e is the neutral in G and $\circ$ is the group operation). Recall also that the *orbit* of a point $x \in \mathcal{U}$ is defined as the set

$$G \cdot x := \{g \cdot x \mid g \in G\}.$$

Example 10. The one-parameter Lie group $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$\varphi_\varepsilon : (t, x) \mapsto (t, x + \varepsilon)$$

is a symmetry for the non-autonomous equation in quadrature form

$$\frac{dx}{dt} = f(t).$$

Remark 11. Example 10 is very simple, actually, one does not need to notice any symmetry in order to *solve* an equation which is in quadrature form. However, this simple example is important since it notices a symmetry that an equation in quadrature form possesses – namely the symmetry with respect to the translation along x-axis.[1] The orbits of this action are vertical lines in the plane. The purpose of this paper is to show how to find a substitution which transforms the orbits of a given (more complex) symmetry (of a more complex differential equation) to the straight lines (vertical or horizontal ones), and thus transforms the more complicated equation to the simple equation in Example 10. We will come back to this later, in Proposition 16 and Theorem 17.

The following example illustrates the previous remark.

[1] On the other hand, the one-parameter Lie group $\varphi_\varepsilon : (t, x) \to (t + \varepsilon, x)$ is a symmetry for the equation $\frac{dx}{dt} = f(x)$.

EXAMPLE 12. (HOMOGENEOUS LINEAR EQUATION) The equation[2]

$$\frac{dx}{dt} = p(t) \cdot x, \quad x(0) = x_0 > 0 \tag{8}$$

is invariant with respect to the homothety

$$x \mapsto \lambda x.$$

Of course, it is solvable as an equation with separated variables. The equation

$$\frac{dx}{x} = p(t)dt$$

is easy to solve, since the left side is the derivative of the function $y(t) = \log x(t)$. The appearance of the logarithm here is not coincidental, from symmetries point of view, since:

$$\log : (\mathbb{R}^+, \cdot, 1) \to (\mathbb{R}, +, 0)$$

is a group homomorphism, so it transforms the action of the group of homotheties to the action of the group of translations from Example 10. Therefore the substitution $y = \log x$ transforms the equation (8) to the equation of Example 10, i.e. to the one in quadrature form. This is also done with two different substitutions in Example 2 and Example 3. We will see later that this is actually a method.

EXAMPLE 13. Let $f(t, x)$ be homogeneous function:

$$f(e^\varepsilon t, e^\varepsilon x) = f(t, x).$$

Then the one-parameter Lie group $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$:

$$\varphi_\varepsilon : (t, x) \to (e^\varepsilon t, e^\varepsilon x)$$

is a symmetry for the autonomous system

$$\begin{cases} \dfrac{dt}{d\tau} = t \\ \dfrac{dx}{d\tau} = tf(t, x) \end{cases} \tag{9}$$

which is an extension (in sense of (7)) of the non-autonomous equation

$$\frac{dx}{dt} = f(t, x).$$

This example is a generalization of Example 12.

EXAMPLE 14. Let $f(t, x)$ be a quasi-homogeneous function:

$$f(e^{\alpha\varepsilon} t, e^{\beta\varepsilon} x) = e^{(\beta-\alpha)\varepsilon} f(t, x).$$

Then the one-parameter Lie group

$$\varphi_\varepsilon : (t, x) \to (e^{\alpha\varepsilon} t, e^{\beta\varepsilon} x)$$

is a symmetry for the the autonomous system (9).

[2]The assumption $x_0 > 0$ does not diminish the generality.

Example 15. The one-parameter Lie group

$$\varphi_\varepsilon : (t,x) \mapsto \left(t, x + \varepsilon e^{\int p(t)\,dt}\right)$$

is a symmetry for the system

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = p(t)x + q(t). \end{cases}$$

This is an extended system of the linear equation from Example 4.

Propositon 16. *Let f be a continuous function of two variables defined at some domain in $\mathbb{R}^2$. If there exists an one parameter local Lie group φ_ε of symmetries of the equation* (6) *and a local diffeomorphism ϕ (defined on some neighbourhood of (t_0, x_0)) such that*

$$\phi \circ \varphi_\varepsilon \circ \phi^{-1} : (y,s) \mapsto (y+\varepsilon, s),$$

then ϕ transforms (6) *into quadrature form.*

Proof. Denote the new coordinates by $(s,y) := \phi(t,x)$ and

$$\psi_\varepsilon := \phi \circ \varphi_\varepsilon \circ \phi^{-1}.$$

We have

$$\psi_\varepsilon : (s,y) \mapsto (s, y+\varepsilon).$$

Let $\tilde{f} = (g, fg)$ be a vector field on $\mathbb{R}^2$ such that

$$(\varphi_\varepsilon)_* \tilde{f} = \tilde{f}$$

(which exists by the definition of a symmetry). Note that ψ_ε is a symmetry for vector field $\tilde{g} := \phi_* \tilde{f}$, since

$$(\psi_\varepsilon)_* \tilde{g} = \phi_* (\varphi_\varepsilon)_* \phi_*^{-1} \phi_* \tilde{f} = \phi_* (\varphi_\varepsilon)_* \tilde{f} = \phi_* \tilde{f} = \tilde{g}.$$

But since $d\psi_\varepsilon = \mathrm{Id}$, and ψ_ε is a symmetry for $\tilde{g}$, we have $g(s, y+\varepsilon) = g(s,y)$, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Therefore, in some neighbourhood of $(s_0, y_0) = \phi(t_0, x_0)$, the function g does not depend on y, meaning that in the new coordinates (s,y) the equation (6) is in quadrature form. ∎

Theorem 17. *Let φ_ε be a symmetry of* (6) *such that $\frac{d\varphi_\varepsilon}{d\varepsilon}\big|_{\varepsilon=0}(t_0, x_0) \neq 0$. Then there is a change of coordinates defined in a neighbourhood of (t_0, x_0) that transforms* (6) *into a quadrature form.*

Proof. By Proposition 16, we need to find the diffeomorphism ϕ with

$$\phi \circ \varphi_\varepsilon \circ \phi^{-1} : (t,x) \mapsto (t, x+\varepsilon).$$

This is equivalent to the condition

$$\phi_*\left(\frac{\partial\varphi_\varepsilon}{\partial\varepsilon}\right)=\frac{\partial}{\partial x}.$$

The rest of the proof is similar to the proof of the Rectification theorem (see [1] for the proof of the Rectification theorem).

We can assume that $(t_0,x_0)=(0,0)$ and

$$\frac{\partial\varphi_\varepsilon}{\partial\varepsilon}\bigg|_{\varepsilon=0}(t_0,x_0)=\frac{\partial}{\partial x}. \tag{10}$$

Indeed, if this is not a case, we can apply a simple affine change of coordinates (this is always possible since the condition (10) concerns the vector field $\frac{\partial\varphi_\varepsilon}{\partial\varepsilon}$ only at one point, so this is just a transformation that sends one given vector to another one). Consider a smooth mapping ψ defined in some neighbourhood of $(0,0)$ in $\mathbb{R}^2$:

$$\psi:(t,x)\mapsto\varphi_x(t,0).$$

Since $\varphi_0=\mathrm{Id}$, we have $\psi|_{\{x=0\}}=\mathrm{Id}_{\{x=0\}}$, therefore

$$\frac{\partial\psi}{\partial t}(0,0)=d\psi_{(0,0)}\left(\frac{\partial}{\partial t}\right)=\frac{\partial}{\partial t}.$$

Since

$$\frac{\partial\psi}{\partial x}(0,0)=\frac{\partial\varphi_x}{\partial x}\bigg|_{x=0}(0,0)=\frac{\partial}{\partial x},$$

we have

$$d\psi(0,0)=\mathrm{Id}\,.$$

By the Inverse function theorem there exists a neighbourhood $\mathcal{U}$ of (t_0,x_0) and $\mathcal{V}$ of $\psi(t_0,x_0)$ such that

$$\psi|_{\mathcal{U}}:\mathcal{U}\to\mathcal{V}$$

is a diffeomorphism. By the construction of ψ we have

$$\psi_*\left(\frac{\partial}{\partial x}\right)=\frac{\partial\psi}{\partial x}=\frac{\partial\varphi_\varepsilon}{\partial\varepsilon}.$$

Therefore, if we denote by $\phi:=\psi^{-1}$, we have

$$\phi_*\left(\frac{\partial\varphi_\varepsilon}{\partial\varepsilon}\right)=\frac{\partial}{\partial x}. \quad \blacksquare$$

Remark 18. Theorem 17 assures that (under the given assumptions) one can always transform the equation (6) to the one in quadrature form, without providing an explicit way of doing it. However, by analysing the proof of Theorem 17 we can also recover a method.

Note that we choose a new coordinate y to be an orbit of the symmetry φ_ε, by transforming the vector field $\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}$ into $\frac{\partial}{\partial y}$. In order to make an orbit of φ_ε be the coordinate line $s = \text{const}$, i.e.

$$s(\varphi_\varepsilon(t,x)) = \text{const}. \tag{11}$$

we differentiate (11) with respect to ε and obtain that the function s satisfies

$$\nabla s(t,x) \cdot \left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = 0. \tag{12}$$

On the other hand, we want to have new coordinates such that

$$y(\varphi_\varepsilon(t,x)) = y_0 + \varepsilon \tag{13}$$

(since the symmetry in the new coordinates has the form $(s,y) \mapsto (s, y+\varepsilon)$). By differentiating (13) with respect to ε we get

$$\nabla y(t,x) \cdot \left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = 1. \tag{14}$$

Remark 19. The existence of a symmetry in a differential equation also leads to the existence and uniqueness of its solution, without the assumptions from Picard or Peano theorem. Namely, the equation in quadrature form (with a given initial condition) has a unique solution defined on an interval, whenever the function (6) is integrable, and not necessarily Lipschitz or continuous.

Note that the Lipschitz condition from Picard theorem or continuity condition from Peano theorem are analytical by nature, while the symmetry existence assumption is either algebraic (if we consider F in the equation (4) as an algebraic term that possesses a symmetry - see also Example 12) or geometric (if we consider F as a vector field which is invariant with respect to a Lie group action - as in Theorem 17). This is the illustration of three points of view on the problem of solvability of a differential equation.

Example 20. Let us provide the substitution for a homogeneous equation (Example 2) using the previous method. The one-parameter Lie group of symmetries of the extended equation (see Example 13) is given by

$$\varphi_\varepsilon : (t,x) \mapsto (e^\varepsilon t, e^\varepsilon x),$$

so

$$\left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} (t,x) = (t,x).$$

Hence we are looking for the new coordinates (s,y) such that

$$\nabla s(t,x) \cdot (t,x) = 0, \quad \nabla y(t,x) \cdot (t,x) = 1.$$

We see that the substitution

$$s = \frac{x}{t}, \quad y = \log t$$

satisfies the above condition, since

$$\frac{\partial s}{\partial t} \cdot t + \frac{\partial s}{\partial x} \cdot x = -\frac{x}{t^2} \cdot t + \frac{1}{t} \cdot x = 0,$$
$$\frac{\partial y}{\partial t} \cdot t + \frac{\partial y}{\partial x} \cdot x = \frac{1}{t} \cdot t + 0 \cdot x = 1.$$

In Example 2 we exchanged the roles of s and y to have the usual substitution.[3]

Remark 21. The condition (12) ensures that s is constant along the orbits of φ_ε, implying that φ_ε acts only on y. The condition (14) provides that φ_ε is a translation along y-direction. We can replace the condition (14) by the condition:

$$\nabla y(t,x) \cdot \left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \neq 0 \tag{15}$$

and still obtain local coordinates (by the Inverse function theorem). By a suitable choice of a new coordinate $\tilde{y} = \alpha(y)$ we can obtain

$$\nabla \tilde{y} \cdot \left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = \alpha'(y) \nabla y \cdot \left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = 1,$$

so in the coordinates $(s, \tilde{y})$ we can obtain the equation in quadrature form:

$$\frac{d\tilde{y}}{ds} = \beta(s).$$

But this is equivalent to the equation

$$\frac{dy}{ds} = \frac{\beta(s)}{\alpha'(y)}$$

which is an equation with separated variables. This is sometimes more convenient. This is how the substitution

$$s(t,x) = \frac{x}{t}, \quad y(t,x) = t$$

in Example 2 transforms the homogeneous equation to the one with separated variables.

Example 22. Let us derive the substitution in a generalized homogeneous equation (Example 3) using the symmetry from Example 14. Namely,

$$\left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = (\alpha t, \beta x)$$

and the substitution

$$s(t,x) = \frac{x^\alpha}{t^\beta}, \quad y(t,x) = \frac{1}{\alpha} \log t$$

[3] Of course in both cases we obtain the quadrature form; the only difference is whether $h(t,x)$ in (2) depends on t or on x.

satisfies the conditions (12), (14), hence transforms the initial equation to quadrature form.[4] Indeed:

$$\begin{aligned}
&\frac{\partial s}{\partial t}\cdot\alpha t+\frac{\partial s}{\partial x}\cdot\beta x=-\beta\frac{x^{\alpha}}{t^{\beta+1}}\cdot\alpha t+\alpha\frac{x^{\alpha-1}}{t^{\beta}}\cdot\beta x=0,\\
&\frac{\partial y}{\partial t}\cdot\alpha t+\frac{\partial y}{\partial x}\cdot\beta x=\frac{1}{\alpha t}\cdot\alpha t+0\cdot\beta x=1.
\end{aligned}$$

On the other hand, the substitution

$$s(t,x)=\frac{x^{\alpha}}{t^{\beta}},\quad y(t,x)=t$$

satisfies the conditions (12), (15), hence transforms it to the one with separated variables.

Example 23. The substitution in a linear equation (see Example 4) can be derived from the symmetry from Example 15:

$$\left.\frac{\partial\varphi_{\varepsilon}}{\partial\varepsilon}\right|_{\varepsilon=0}=\left(0,e^{\int p(t)\,dt}\right).$$

If we set

$$s(t,x)=t,\quad y(t,x)=xe^{-\int p(t)\,dt},$$

we have that s and y satisfy (12) and (14) since:

$$\begin{aligned}
&\frac{\partial s}{\partial t}\cdot 0+\frac{\partial s}{\partial x}\cdot e^{\int p(t)\,dt}=1\cdot 0+0\cdot e^{\int p(t)\,dt}=0,\\
&\frac{\partial y}{\partial t}\cdot 0+\frac{\partial y}{\partial x}\cdot e^{\int p(t)\,dt}=-xe^{-\int p(t)dt}p(t)\cdot 0+e^{-\int p(t)dt}\cdot e^{\int p(t)dt}=1.
\end{aligned}$$

The linear equation (3) becomes

$$\frac{dy}{ds}=e^{-\int p(s)\,ds}q(s).$$

4. Lowering the dimension or the order using symmetries

In the case of an equation in $\mathbb{R}^n$ (or on a manifold) we can use symmetries to reduce the equation to the one in $\mathbb{R}^{n-1}$, or even in a space of lower dimension.

Example 24. (Motion in a central force field) The second Newton law gives the differential equation

$$m\ddot{\boldsymbol{r}}=F(\boldsymbol{r}).$$

Since $\boldsymbol{r}=(x,y,z)\in\mathbb{R}^3$, and this is a second order equation, we actually deal with a system whose phase space is of dimension six.[5] We say that the force field is central if it is of the form

$$F(\boldsymbol{r})=f(r)\boldsymbol{r},\quad r=\|\boldsymbol{r}\|. \tag{16}$$

[4]The substitution $s(t,x)=\frac{x^{\alpha}}{t^{\beta}}$, $y(t,x)=\log t$ also transforms a generalized equation to quadrature from, although the condition (14) is not fulfilled.

[5]Using the substitution $(x_1,x_2,x_3,x_4,x_5,x_6):=(\boldsymbol{r},\dot{\boldsymbol{r}})$.

The condition (16) is equivalent to F being invariant with respect to the group of rotations $SO(3)$.

A central field is always conservative: if we set

$$U(\boldsymbol{r}) = U(r) := -\int r f(r)\, dr,$$

we have

$$-\nabla U = -\frac{dU}{dr}\nabla r = r\, f(r)\, \frac{1}{r}\, \boldsymbol{r} = f(r)\, \boldsymbol{r} = F(\boldsymbol{r}).$$

Therefore the Law of conservation of energy holds:

$$\frac{m\|\dot{\boldsymbol{r}}\|^2}{2} + U(\boldsymbol{r}) = E,$$

for some constant $E \in \mathbb{R}$.

The angular momentum, $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ is also constant in a motion with a central force field. Indeed:

$$\frac{d}{dt}\left(\boldsymbol{r} \times \dot{\boldsymbol{r}}\right) = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} = \boldsymbol{0} + \boldsymbol{r} \times \frac{1}{m} f(r)\boldsymbol{r} = \boldsymbol{0}.$$

Note that here we used the fact that the system is $SO(3)$-invariant, i.e. that it is of the form (16). We conclude that a position vector $\boldsymbol{r}$ satisfies

$$\frac{m\|\dot{\boldsymbol{r}}\|^2}{2} + U(\boldsymbol{r}) = E, \quad \boldsymbol{r} \times \dot{\boldsymbol{r}} = \vec{\mu},$$

for some constant $\vec{\mu} \in \mathbb{R}^3$. For a generic choice of $E \in \mathbb{R}$, $\vec{\mu} \in \mathbb{R}^3$ the set

$$\mathcal{S} := \left\{ (\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid m\|\boldsymbol{v}\|^2 + U(\boldsymbol{u}) = E,\, \boldsymbol{u} \times \boldsymbol{v} = \vec{\mu} \right\}$$

is a two dimensional manifold, i.e. a surface. Therefore we lowered the order the dimension of the phase space from six to two. The roots of these two conservation laws lie in the fact that a central force field possesses the spherical symmetry, meaning that the potential energy and the force field are invariant under rotations around the origin (see [1, 4] for more details).

Theorem 25. *Let φ_ε be a symmetry of the system* (4) *and let*

$$\left.\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} (t_0, x_0) \neq 0.$$

Then there exist a neighbourhood $I_1 \times \mathcal{U}_1$ of (t_0, x_0) in $I \times \mathcal{U}$ and local coordinates

$$(s, y) = (s, y_1, \dots, y_n)$$

in $I_1 \times \mathcal{U}_1$ such that the equation (4) *has the form*

$$\frac{dy}{ds} = G(s, y_1, \dots, y_{n-1}).$$

Since the vector field G does not depend on y_n, we reduce the initial equation to the one in $\mathbb{R}^{n-1}$ (at least locally). Let

$$G(s, y_1, \ldots, y_{n-1}) = (g_1(s, y_1, \ldots, y_{n-1}), \ldots, g_n(s, y_1, \ldots, y_{n-1})).$$

If we are able to solve the new equation

$$\begin{cases} y_1'(s) = g_1(s, y_1, \ldots, y_{n-1}) \\ \quad \vdots \\ y_{n-1}'(s) = g_{n-1}(s, y_1, \ldots, y_{n-1}), \end{cases}$$

we get the solution of the initial equation by solving:

$$y_n'(s) = g_n(s, y_1(s), \ldots, y_{n-1}(s))$$

which is in quadrature form.

Proof of Theorem 25. The proof is basically the same as the proof of Theorem 17. We choose coordinates $(s, y) = \phi(t, x)$ such that

$$\psi_\varepsilon := \phi \circ \varphi_\varepsilon \circ \phi^{-1} \text{ and } G := \phi_* F$$

satisfy

$$\psi_\varepsilon : (s, y_1, \ldots, y_n) \mapsto (s, y_1, \ldots, y_n + \varepsilon), \quad (\psi_\varepsilon)_* G = G.$$

This implies

$$G(s, y_1, \ldots, y_n + \varepsilon) = (s, y_1, \ldots, y_n),$$

i.e. G does not depend on y_n.

The construction of the diffeomorphism ϕ goes as following. First we map $\frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}$ to $\frac{\partial}{\partial x_n}$. Then we define

$$\psi : (t, x_1, \ldots, x_n) \to \varphi_{x_n}(t, x_1, \ldots, x_{n-1}, 0).$$

Using the Inverse function theorem, we define ϕ as ψ^{-1} (locally). ∎

In practice, as in the case $n = 1$, we can find the new coordinates $(s, y_1, \ldots, y_n)$ from the condition:

$$\begin{aligned} \nabla s(t, x_1, \ldots, x_n) \cdot \frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} &= 0 \\ \nabla y_1(t, x_1, \ldots, x_n) \cdot \frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} &= 0 \\ &\vdots \\ \nabla y_{n-1}(t, x_1, \ldots, x_n) \cdot \frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} &= 0 \\ \nabla y_n(t, x_1, \ldots, x_n) \cdot \frac{\partial \varphi_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} &= 1. \end{aligned}$$

From the above construction we can also see a geometrical meaning of some substitutions which reduce the order of the equation:

$$x^{(n)} = F(t, x, x', \dots, x^{(n-1)}) \tag{17}$$

to $n-1$, since every equation (17) can be transformed to (4).

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